Beam Pattern Function for Two Element Array

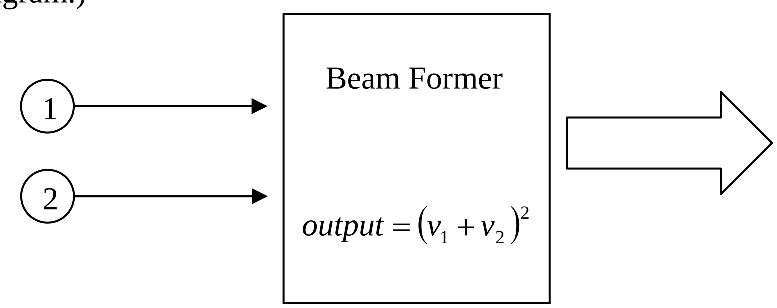
If we had a single hydrophone, with an omni-directional response, sounds would appear to come from all directions. In other words, we could not determine what direction a sound came from. If we could somehow limit the direction our system would listen, we could possibly determine the bearing a sound came from and maybe increase the ratio of the signal power received to the noise power received. (Increase SNR which is a good thing.)

One way to do this is to use more than one hydrophone. What if we use two hydrophones connected at a distance d apart from each other. Recall from our previous studies that the hydrophone converts the mechanical sound signal to an electrical signal or voltage. We can mathematically describe this process by introducing a quantity M, the transducer sensitivity constant. M is used to convert the mechanical pressure quantity to an electrical signal, where:

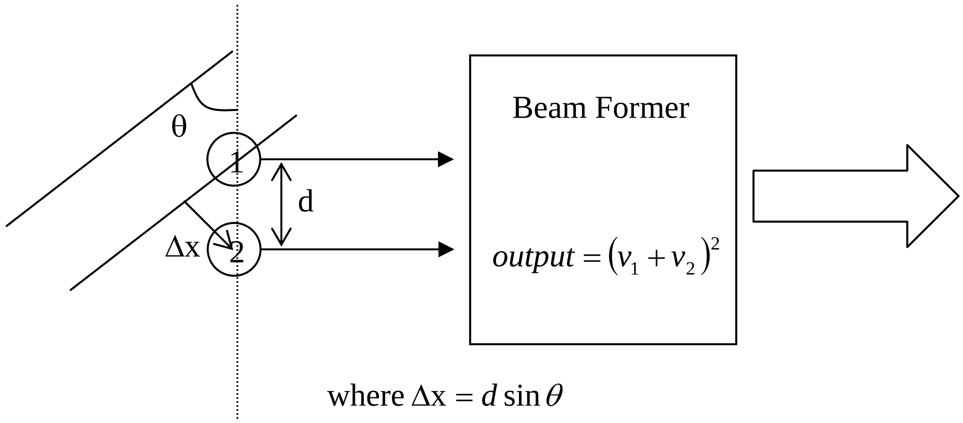
$$v(t) = M * p(t)$$

Now let's look at the arrangement of the two hydrophones and how their output is used.

First examine the diagram for a basic two-hydrophone array, sonar system. The outputs of each hydrophone are combined in a beam former (they are added together), then the quantity squared to find the amount of power in the signal and noise incident on the hydrophones. (See the following diagram.)



If a sound wave is incident upon the two hydrophones at some angle other than perpendicular to the axis of the two hydrophones, the sound wave will have to travel some distance Δx further to reach the second hydrophone. (See diagram below.)



where $\Delta x = d\sin\theta$

A phase factor, δ, can be inserted in the one-dimensional wave equation to describe the pressure of the wave as it is incident upon each hydrophone where:

$$\begin{aligned} v_1 &= Mp_1(t) = Mp_{max}\cos(k(0)+\omega t) \\ v_2 &= Mp_2(t) = Mp_{max}\cos(k(-\Delta x)+\omega t) \\ v_1 + v_2 &= Mp_{max}\left[\cos(\omega t)+\cos(-\delta+\omega t)\right] \\ \text{where }\ \delta &= k\Delta x = kd\sin\theta \end{aligned}$$

When the output is then squared it is actually measuring is the power of the incoming signal (or a signal proportional to the rate of sound energy incident on the hydrophones.)

$$Power = \frac{V^2}{R} = \frac{output^2}{R}$$

$$P = \frac{(Mp_{\max})^2}{R}\left[\cos(\omega t)+\cos(-\delta+\omega t)\right]^2$$

If we then display the time-averaged power derived from the equation above, we get:

$$\langle P\rangle = \frac{\langle Mp_{\max}\rangle^2}{R}\left\langle \cos^2\omega t + \cos^2(\omega t-\delta) + 2\cos\omega t\cos(\omega t-\delta)\right\rangle$$

$$\langle P\rangle = \frac{(Mp_{\max})^2}{R}\left\langle \cos^2\omega t + \cos^2(\omega t-\delta) + 2\left[\frac{1}{2}\{\cos(2\omega t-\delta)+\cos\delta\}\right]\right\rangle$$

$$\langle P\rangle = \frac{(Mp_{\max})^2}{R}\left[\frac{1}{2}+\frac{1}{2}+\langle\cos(2\omega t-\delta)\rangle+\langle\cos\delta\rangle\right]$$

$$\langle P\rangle = \frac{(Mp_{\max})^2}{R}\left[1+\cos\delta\right]$$

So depending on the value of δ (which is equal to kdsin(θ)), the time averaged power will be:

$$0 \le \langle P \rangle \le \frac{2(Mp_{\max})^2}{R}$$

Two-dimensional Beam Pattern

Why have we calculated the time-averaged power? Since the value for δ depends on the angle of the incoming sound wave from the array axis, the power received depends on the angle at which the sound ray is incident on the array. We can describe this angular dependence with one equation to relate the actual power received to the time averaged power on the axis (where θ=0° and the power is a maximum.) This ratio is the two-dimensional beam pattern function of the array, b(θ) where:

$$b(\theta) = \frac{\langle P(\theta) \rangle}{\langle P(\theta = 0°) \rangle} = \frac{\frac{(Mp_{\max})^2}{R}(1 + \cos\delta)}{\frac{(Mp_{\max})^2}{R}(1 + \cos 0°)}$$

$$b(\theta) = \frac{\frac{(Mp_{\max})^2}{R}(1 + \cos(kd\sin\theta))}{2\frac{(Mp_{\max})^2}{R}} = \frac{(1 + \cos(kd\sin\theta))}{2}$$

using a trigonometric identity that $1 + \cos\theta = 2\left(\cos^2\left(\frac{\theta}{2}\right)\right)$:

$$b(\theta) = \left[\cos^2\left(\frac{kd\sin\theta}{2}\right)\right]$$

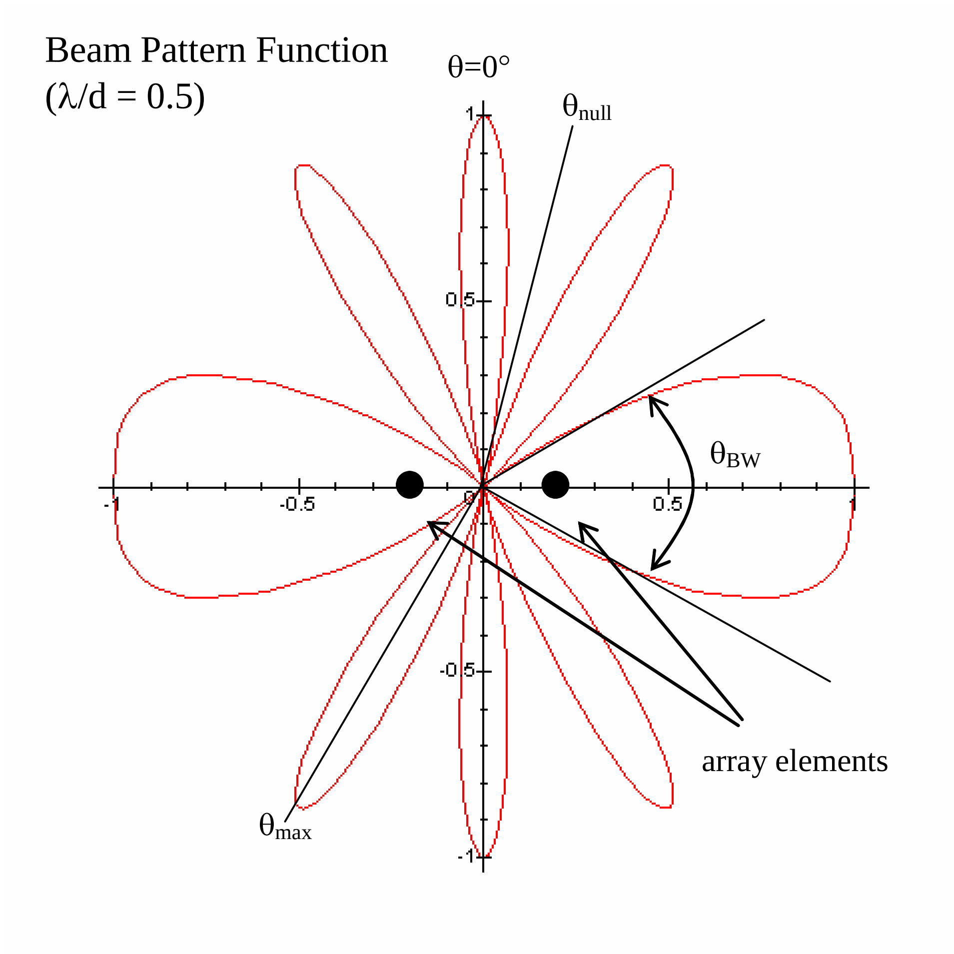
or

$$b(\theta) = \left[\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)\right]$$

The beam pattern function determines the magnitude of the received power at every angle, to the maximum received power, thus the beam pattern function will vary (as a function of angle) between 0 and 1.

$$0 \le b(\theta) \le 1$$

The key now is to determine what important parameters we can determine from the beam pattern function. Below is a polar plot of the beam pattern function for a two element array where the separation in elements is equal to twice the wavelength.



Beam Pattern Function
(λ/d = 0.5)
θ=0°
θnull
θBW
θmax
array elements
1
0.5
-0.5
-1
0

Maximum Power Angles (θ_{max})

Any angle where b(θ) = 1. Using our previously derived formula for b(θ), there can be many angles where this occurs. From b(θ):

$$b(\theta_{\max}) = 1 = \cos^2\left[\frac{\pi d \sin\theta_{\max}}{\lambda}\right]$$

$$\cos\left[\frac{\pi d \sin\theta_{\max}}{\lambda}\right] = \pm 1$$

$$\frac{\pi d \sin\theta_{\max}}{\lambda} = n\pi \quad \text{where n} = 0,1,2,3,...$$

$$\sin\theta_{\max} = \frac{n\lambda}{d}$$

$$\theta_{\max} = \sin^{-1}\left[\frac{n\lambda}{d}\right]$$

Below are listed the max power angles for various ratios of λ/d:

λ/d	θ_{max} (between 0° and 90°)
2.0	0°
1.0	0°,90°
0.5	0°, 30°, 90°
0.333	0°, 19.5°, 41.8°, 90°
0.25	0°, 14.5°, 30°, 48.6°, 90°

Notice that the lower the ratio of λ/d, the higher the number of maximum power angles.

Null Angles (θ_{null})

The angles where the beam pattern function is equal to zero. If any sound ray arrives at any of the null angles, little or no power from the incoming sound ray is received because of destructive interference between the signals received by each of the separate elements in the array. We calculate the null angle by setting the beam pattern function equal to zero as shown below.

$$b(\theta_{null}) = 0 = \cos^2\left[\frac{\pi d \sin\theta_{null}}{\lambda}\right]$$

$$\cos\left[\frac{\pi d \sin\theta_{null}}{\lambda}\right] = 0$$

$$\frac{\pi d \sin\theta_{null}}{\lambda} = n\frac{\pi}{2} \quad \text{where n} = 1,3,5,7,...$$

$$\sin\theta_{\text{null}} = \frac{n\lambda}{2d}$$

$$\theta_{null} = \sin^{-1}\left[\frac{n\lambda}{2d}\right]$$

Below are listed the null angles for various ratios of λ/d:

λ/d	θ_{null} (between 0° and 90°)
2.0	90°
1.0	30°
0.5	14.5°, 48.6°
0.333	9.6°, 30°, 56.4°
0.25	7.2°, 22.0°, 38.7°, 61.0°

Beamwidth (θ_{BW})

The beamwidth of a beam is the angular displacement between the angles where the beam pattern function, $b(\theta)$, is greater than 0.5. If any sound ray arrives at any angle within the beamwidth, the sound ray may be detectable. We assume that if a ray arrives at an angle outside the beamwidth that the signal will not be detectable. Within each beam, at least half of the power of the original wave will be received (not cancelled due to destructive interference between the elements of the array.)

The beamwidth is important because it is proportional to the bearing accuracy of the specific beam.

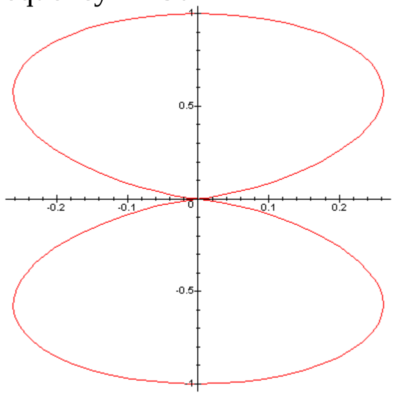
When we detect a sound, we can electronically determine which beam that the sound arrived in but not specifically at what exact bearing in that beam. Thus, the smaller the beam width, the greater the bearing accuracy. It is important to not then that beam width is not only a function of the frequency of the sound but what beam the sound arrives in.

Referring to the diagram on page 13-4, the beams on the "beam" of the array (perpendicular to the array axis) are much narrower than the beams on the array axis (also called the "end-fire" beams.)

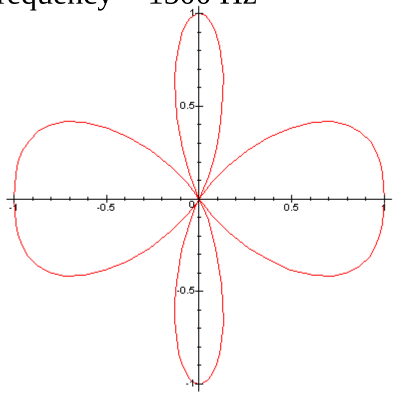
Dependence Of Beam Pattern On Frequency

For most physical arrays, the separation distance between the elements, d, is a fixed distance. Since all of the previous parameters depended on the ratio of λ/d, every one of the parameters will depend on the frequency (and thus the wavelength) of the sound incident on the array. To show the dependence of the beam pattern of a fixed array on frequency, several beam patterns are shown below:

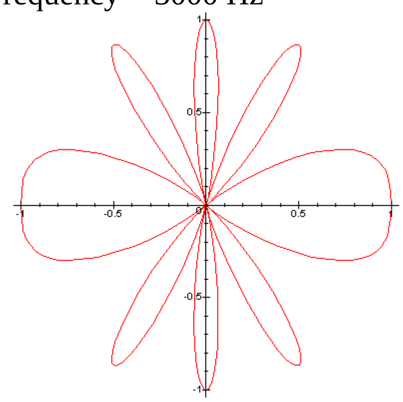
Frequency = 750 Hz



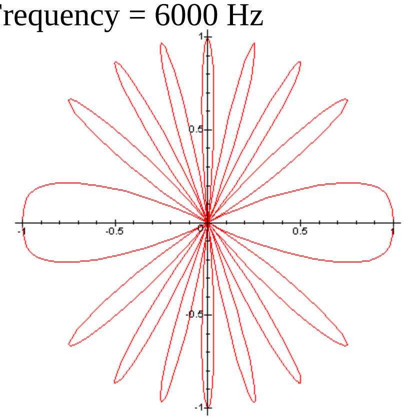
Frequency = 1500 Hz



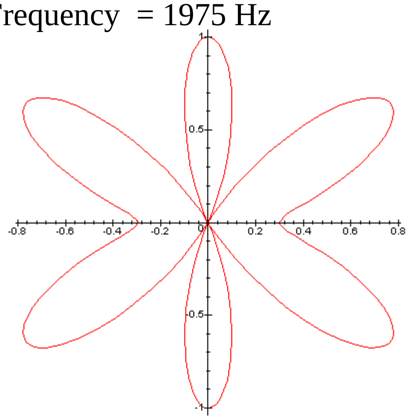
Frequency = 3000 Hz



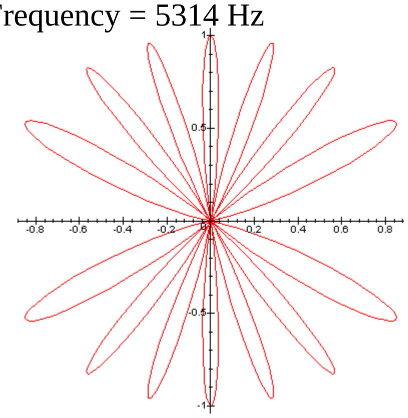
Frequency = 6000 Hz



Frequency = 1975 Hz

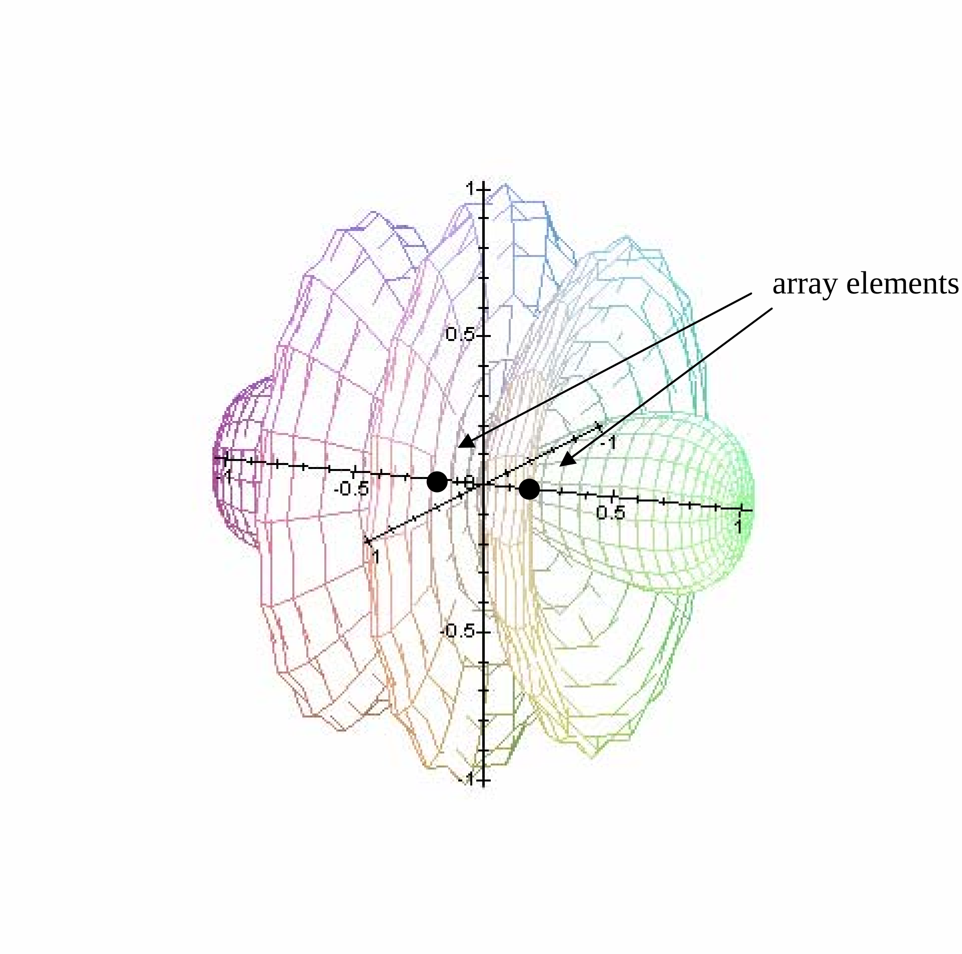


Frequency = 5314 Hz



Three-Dimensional Beam Pattern

Lastly, we must remember that we live in a three-dimensional world. So why did we spend so much time exploring the two-dimensional beam pattern? The beam pattern is independent of the angle ϕ in a three-dimensional environment. An example of a three-dimensional beam pattern is shown below.

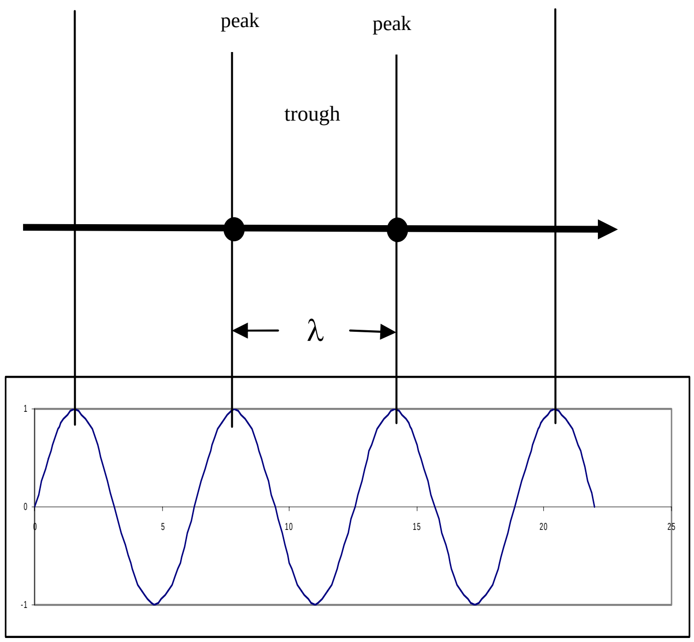


The only difference between the two-dimensional beam patterns we previously derived and the three-dimensional beam pattern shown above is that the three-dimensional beam pattern is the two-dimensional pattern rotated about the array axis. In the example above, the elements lie on the x-axis as shown.

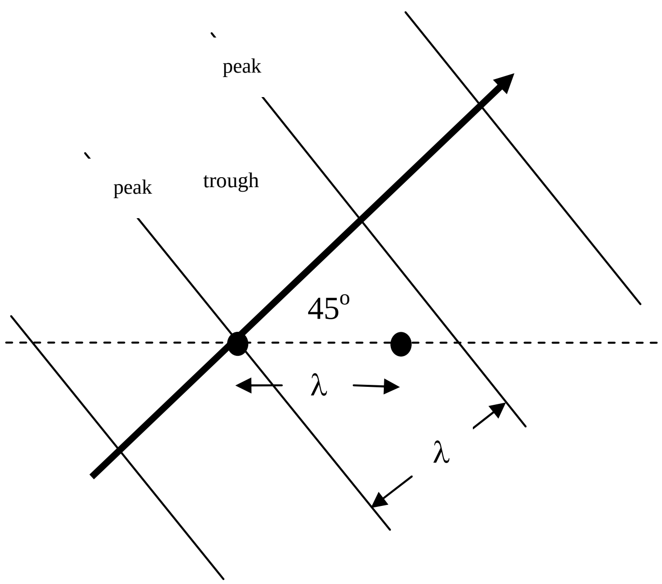
Problems:

1. You have a two element array as shown in the sketches below. The separation between the elements is as indicated. Each point element is omni-directional and calibrated to give 0.001 volt per Pascal. Find the total voltage generated from the array for a traveling wave $p(x) = p_o \cos\left(\frac{2\pi}{\lambda}x - \omega t\right)$ (with maximum amplitude $p_o = 1$ Pa) in each of the following situations. The time is at the instant shown in the sketch

a)



b)



c) Repeat b) for an angle of 30^o. (Draw your own sketch)
d) Repeat a) for the case where one of the elements is moved to the trough. (draw your own sketch)

2. Given a 2 element array with a 1.0 m spacing between elements, determine the following assuming the frequency is 3000 Hz and c = 1500 m/s.

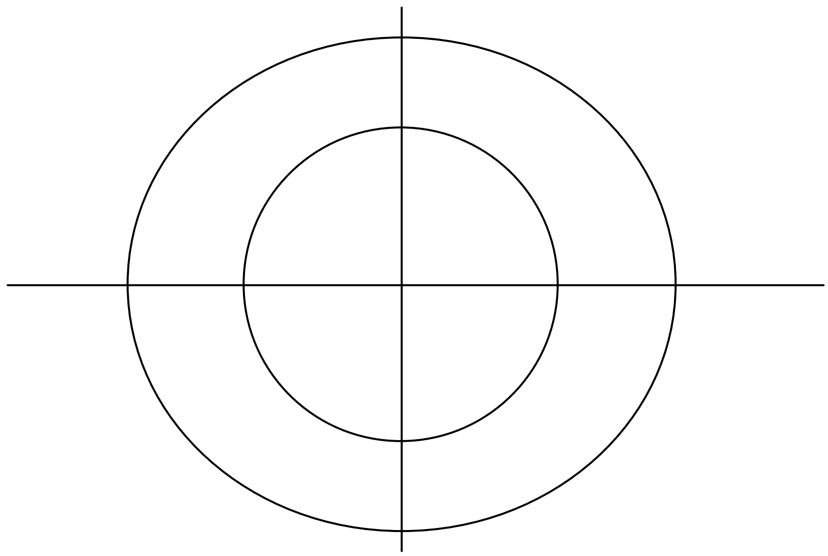
   a) The wavelength of the sound.

   b) The maximum power angles from $0° \leq \theta \leq 90°$.

   c) The null angles from $0° \leq \theta \leq 90°$.

   d) The beam width about 0°
.
   e) The beam width about 30°.

   f) Complete a polar plot of $b(\theta)$.



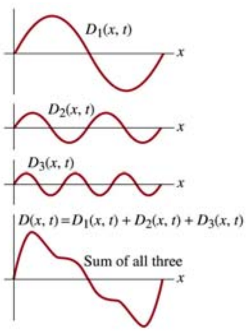
3. The half power beamwidth is defined as:
   a) The angular separation between the first two null angles of an array.
   b) The angular separation between the two "3dB down" angles of the main beam of the array.
   c) The directivity index of the array divided by 2.
   d) The area of the beam pattern of an array where there is no chance of detection.

4. You are given a two element array with identical omni directional hydrophones. Let the spacing between the hydrophones be $\lambda/2$. Calculate the beam width of the main lobe (beam width is the angular separation of the half power points)

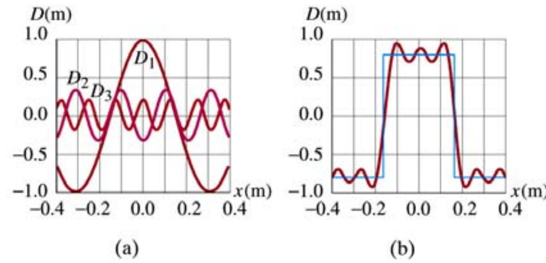
5. An array consisting of two identical elements placed 40 cm apart is receiving sound of a wavelength of 12 cm.
   a) Locate the angles where there are nulls in the beam pattern function.
   b) Locate the angles where there are maxima (or side lobes).
   c) Calculate the value of $b(\theta)$ for a sufficient number of additional angles such that you can plot $b(\theta)$ for $0<\theta<90$. Plot $b(\theta)$ vs θ on polar graph paper.

6. Design a 2 element array with a half –power full beam width of 25 degrees at 15 kHz. The spacing between the two elements is:______________________

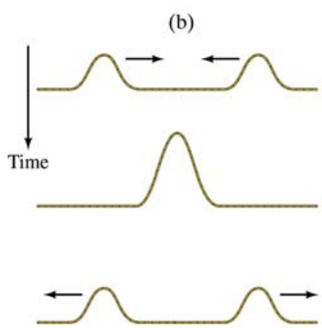
Superposition



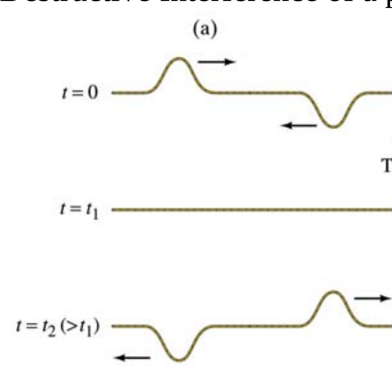
Fourier Series



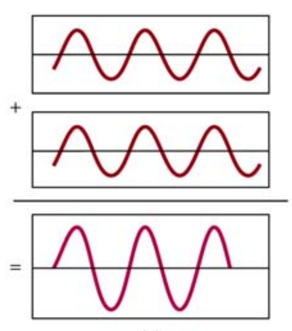
Constructive Interference of a pulse



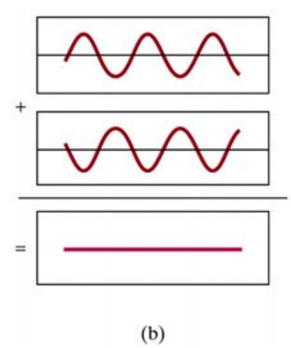
Destructive Interference of a pulse



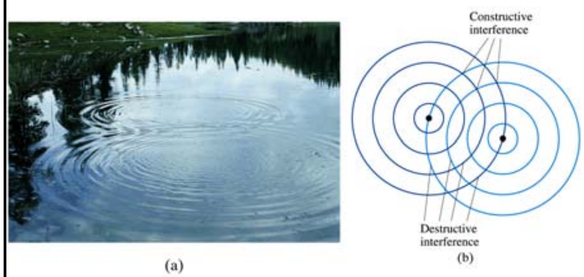
Constructive Interference of Harmonic Waves



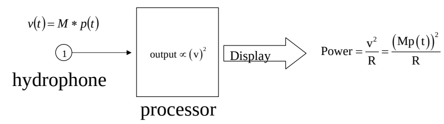
Destructive Interference of Harmonic Waves



2 Dimensional Example

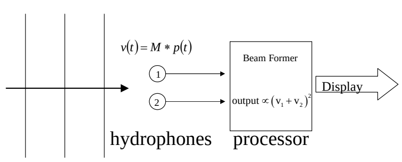


Single Hydrophone



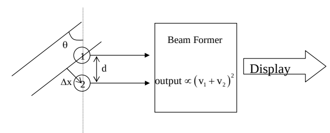
"omni-directional"

Two Hydrophones



Why not output $\propto v_1^2 + v_2^2$???

Incident Wave



$$v_1 = Mp_1(t) = Mp_{max} \cos(k(0) + \omega t)$$
$$v_2 = Mp_2(t) = Mp_{max} \cos(k(-\Delta x) + \omega t)$$
$$\text{output} \propto (v_1 + v_2)^2 = \{Mp_{max}[\cos(\omega t) + \cos(-\delta + \omega t)]\}^2$$
where $\delta = k\Delta x = kd\sin\theta$

Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$
$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$
$$\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$
$$\cos\theta\cos\phi = \frac{1}{2}[\cos(\theta+\phi) + \cos(\theta-\phi)]$$
$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

Power Output from the Processor

$$P = \frac{(Mp_{max})^2}{R}[\cos(\omega t) + \cos(-\delta + \omega t)]^2$$
$$\langle P\rangle = \frac{\langle Mp_{max}\rangle^2}{R}\langle\cos^2\omega t + \cos^2(\omega t - \delta) + 2\cos\omega t\cos(\omega t - \delta)\rangle$$
$$\langle P\rangle = \frac{(Mp_{max})^2}{R}\left\langle\cos^2\omega t + \cos^2(\omega t - \delta) + 2\left[\frac{1}{2}\{\cos(2\omega t - \delta) + \cos\delta\}\right]\right\rangle$$
$$\langle P\rangle = \frac{(Mp_{max})^2}{R}\left[\frac{1}{2} + \frac{1}{2} + \langle\cos(2\omega t - \delta)\rangle + \langle\cos\delta\rangle\right]$$
$$\langle P\rangle = \frac{(Mp_{max})^2}{R}[1 + \cos\delta] \qquad \delta = k\Delta x = kd\sin\theta$$
$$0 \le \langle P\rangle \le \frac{2(Mp_{max})^2}{R}$$

Beam Pattern Function

$$b(\theta)=\frac{\langle P(\theta)\rangle}{\langle P(\theta=0^\circ)\rangle}=\frac{\frac{(Mp_{max})^2}{R}(1+\cos\delta)}{\frac{(Mp_{max})^2}{R}(1+\cos 0^\circ)}$$

$$\delta = k\Delta x = kd\sin\theta$$

$$b(\theta)=\frac{(1+\cos(kd\sin\theta))}{2}$$

Trig identity

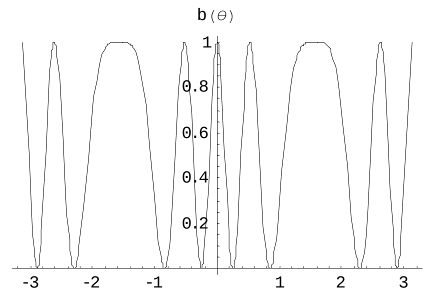
$$1+\cos\theta = 2\left(\cos^2\left(\frac{\theta}{2}\right)\right)$$

$$b(\theta)=\left[\cos^2\left(\frac{kd\sin\theta}{2}\right)\right]$$

or

$$b(\theta)=\left[\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)\right]$$

Example λ = .5 d



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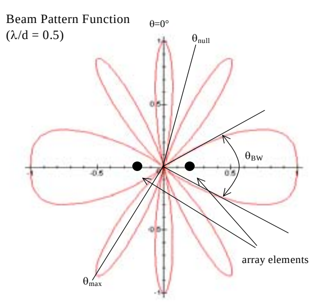


Maximum Power Directions

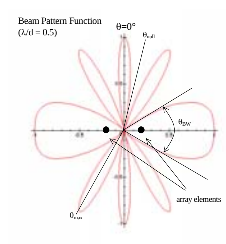
$$b(\theta_{max})=1=\cos^2\left[\frac{\pi d\sin\theta_{max}}{\lambda}\right]$$

$$\cos\left[\frac{\pi d\sin\theta_{max}}{\lambda}\right]=\pm 1$$

$$\frac{\pi d\sin\theta_{max}}{\lambda}=n\pi \quad \text{where } n = 0,1,2,3,...$$

$$\sin\theta_{max}=\frac{n\lambda}{d}$$

$$\theta_{max}=\sin^{-1}\left[\frac{n\lambda}{d}\right]$$



Null Angles

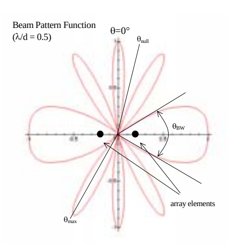
$$b(\theta_{null})=0=\cos^2\left[\frac{\pi d\sin\theta_{null}}{\lambda}\right]$$

$$\cos\left[\frac{\pi d\sin\theta_{null}}{\lambda}\right]=0$$

$$\frac{\pi d\sin\theta_{null}}{\lambda}=n\frac{\pi}{2} \quad \text{where } n = 1,3,5,7,...$$

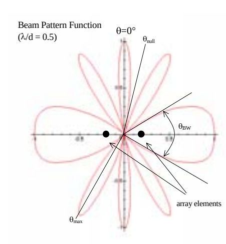
$$\sin\theta_{null}=\frac{n\lambda}{2d}$$

$$\theta_{null}=\sin^{-1}\left[\frac{n\lambda}{2d}\right]$$

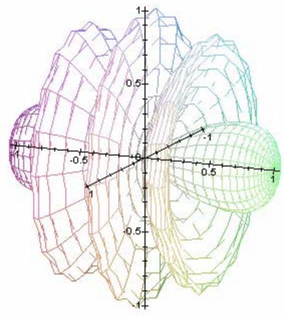


Beam Width

- The beamwidth of a beam is the angular displacement between the angles where the beam pattern function, b(θ), is greater than 0.5.
- 3 dB down points
- The beamwidth is important because it is proportional to the bearing accuracy of the specific beam.



3-d Beam Pattern



Effect of Increasing Frequency