Speed of Sound in the Sea

The speed of a wave propagating through a medium is not a constant. This is especially true for the non-homogeneous medium, the ocean. The speed of sound through water has been found to be mainly a function of three factors. They are **temperature, pressure or depth and salinity**. Because the speed is not constant, sound does not travel along straight paths.

Temperature

In general, for most areas of the ocean, the water temperature decreases from the surface to the bottom, but there are many local variations. Shallow layers see the most variation with time and depth (ie. Surface mixing, solar heating, currents, seasonal variations, etc). In vary deep water, the temperature eventual becomes constant with depth at about 4 C.

Depth

Hydrostatic pressure makes sound velocity increase with depth because of variations in the bulk modulus, *B*. This effect is linear in the first approximation with an increase of 0.017 m/s per meter increase in depth.

Recall in Physics I we showed that pressure varies with depth according to the simple formula,

$$P = P_o + \rho g h$$

Leroy formula (1968) gives a precise hydrostatic pressure:

$$P = \left[1.0052405\left(1 + 5.28\times10^{-3}\sin\phi\right)z + 2.36\times10^{-6}z^2 + 10.196\right]\times10^4 \text{ Pa}$$

ϕ - latitude in degrees

z - depth in meters

(From: Lurton, X. An Introduction to Underwater Acoustics, 1st ed. London, Praxis Publishing LTD, 2002, p37)

Salinity

The change in the mix of pure water and dissolved salts effects sound velocity. Salinity is expressed in practical salinity units (p.s.u.). These unit have the same magnitude as the traditional parts per thousand (‰). Most oceans have a salinity of 35 p.s.u., although salinity can vary locally based on hydrological conditions. Closed seas have a greater difference in their salinity (38 p.s.u. for Mediterranean Sea due to evaporation, 14 p.s.u. for Baltic Sea due to large freshwater input). Salinity varies very little with depth, but there can be stronger variations near river estruaries, melting ice, etc.

Velocity Models

In the 1940's, sound velocity variations and their affect on acoustic propagation were first noticed and studied. It is very difficult to locally measure sound velocity, but easy to measure the parameters that affect it (temperature, salinity, and depth). Several models have been created to predict sound velocity. A good first approximation is that developed by Medwin (1975). It is simple but limited to 1000 meters in depth:

$$c(t,z,S) = 1449.2 + 4.6t - 5.5x10^{-2}t^2 + 2.9x10^{-4}t^3 + \left(1.34 - 10^{-2}t\right)(S-35) + 1.6x10^{-2}z$$

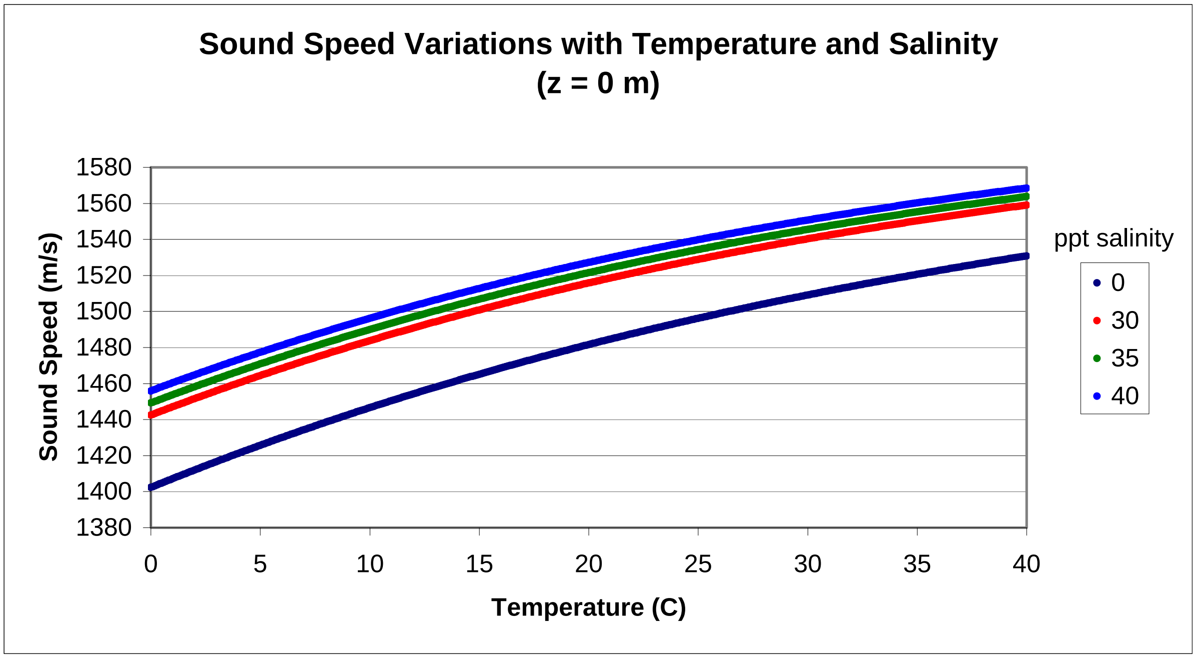
with the following limits:

$0 \le t \le 35°$ C

$0 \le S \le 45$ p.s.u.

$0 \le z \le 1000$ meters

Where c is the speed of sound as a function of temperature, t, depth, z, and salinity, S.



(From: Lurton, X. An Introduction to Underwater Acoustics, 1st ed. London, Praxis Publishing LTD, 2002, p37)

More recent and accurate models have been developed and include Chen and Millero (1977). Their model is endorsed by UNESCO and used as the standardized reference model:

$$c = c_0 + c_1 P + c_2 P^2 + c_3 P^3 + AS + BS^{\frac{3}{2}} + CS^2$$

P = Pressure from Leroy Formula

$$c_o = 1402.388 + 5.03711t - 5.80852x10^{-2} t^2 + 3.3420x10^{-4} t^3 - 1.478x10^{-6} t^4 + 3.1464x10^{-9} t^5$$

$$c_1 = 0.153563 + 6.8982x10^{-4} t - 8.1788x10^{-6} t^2 + 1.3621x10^{-7} t^3 - 6.1185 + 1.3621x10^{-10} t^4$$

$$c_2 = 3.126x10^{-5} - 1.7107x10^{-6} t + 2.5974x10^{-8} t^2 - 2.5335x10^{-10} t^3 + 1.0405x10^{-12} t^4$$

$$c_3 = -9.7729x10^{-9} + 3.8504x10^{-10} t - 2.3643x10^{-12} t^2$$

$$A = A_o + A_1 P + A_2 P^2 + A_3 P^3$$

$$A_1 = 9.4742x10^{-5} - 1.258x10^{-5} t - 6.4885x10^{-8} t^2 + 1.0507x10^{-8} t^3 - 2.0122x10^{-10} t^4$$

$$A_2 = -3.9064x10^{-7} + 9.1041x10^{-9} t - 1.6002x10^{-10} t^2 + 7.988x10^{-12} t^3$$

$$A_3 = 1.1x10^{-10} + 6.649x10^{-12} t - 3.389x10^{-13} t^2$$

$$B = -1.922x10^{-2} - 4.42x10^{-5} t + \left(7.3637x10^{-5} + 1.7945x10^{-7} t\right) P$$

$$C = -7.9836x10^{-6} P + 1.727x10^{-3}$$

Where,

t - temperature (° C)
z - depth (m)
S - salinity (p.s.u.)

As you can see, the speed of propagation has a very complicated dependence on these three factors. Some thumbrules that you can use to relate the dependence of the speed of sound in seawater to each of the factors are:

$$1° \text{ C increase in temperature} \Rightarrow 3 \text{ m/s increase in speed}$$
$$100 \text{ meters of depth} \Rightarrow 1.7 \text{ m/s increase in speed}$$
$$1 \text{ ppt increase in salinity} \Rightarrow 1.3 \text{ m/s increase in speed}$$

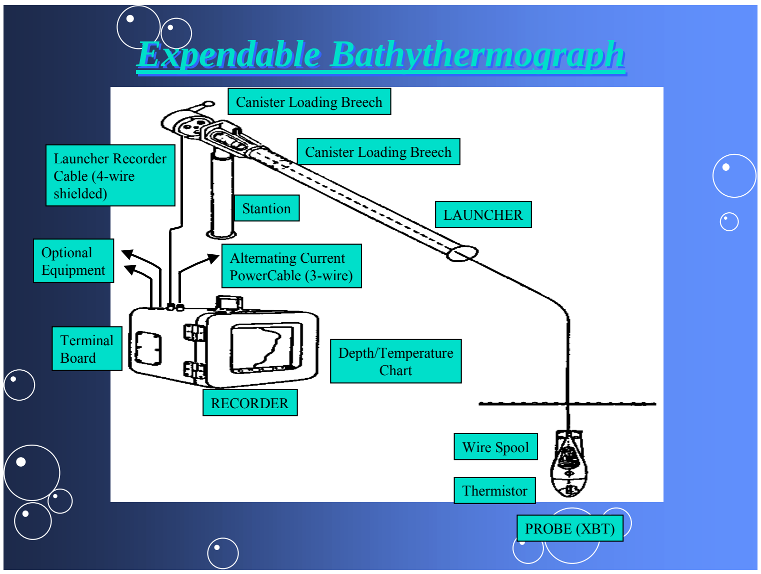
(From: Principles of Naval Weapons Systems, Edited by Joseph B. Hall, CDR, USN, Dubuque, IA: Kendall/Hunt Publishing Co, 2000, p.179)

Seawater contains many inhomogenieties, including bubble layers close to the surface, mineral particles in suspension, and living organisms. These are all potential scatterers of acoustic waves, especially at higher frequencies.

Measuring the Speed of Sound in the Ocean

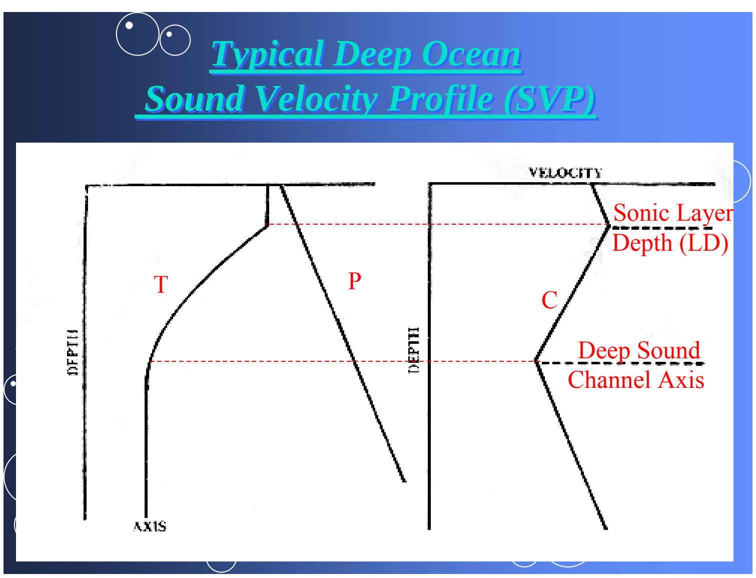
To predict the direction of propagation of a sound wave in the ocean, we must know the speed of sound as a function of position (or depth) in the ocean water. To measure the speed of sound in water, the Navy has developed several tools to measure the temperature of the seawater as a function of depth or the velocity of sound directly.

The most widely used tool is an Expendable BathyThermograph or XBT (picture compliments of ES419). XBTs are launched from submarines, surface ships and even aircraft. These measure the temperature of the water as the device sinks at a known rate and transmits this back to the launching platform. This provides a detailed plot of temperature as a function of depth. Neglecting salinity, the Sound Velocity Profile or SVP can be calculated as a function of depth and temperature (since these cause the greatest variation in the speed of sound in seawater.)



Many modern submarines are often equipped with velocimeters that calculate the speed of sound *in situ*. Other submarines have systems that calculate and record sound speed using temperature and depth measurements from onboard ships instruments.

Expendable Bathythermographs produce graphs of water temperature and sound speed as a function of water depth as seen below. In the next lesson we will examine typical plots in more detail for tactical significance. For now you should familiarize yourself with the basic shape of these typical plots.

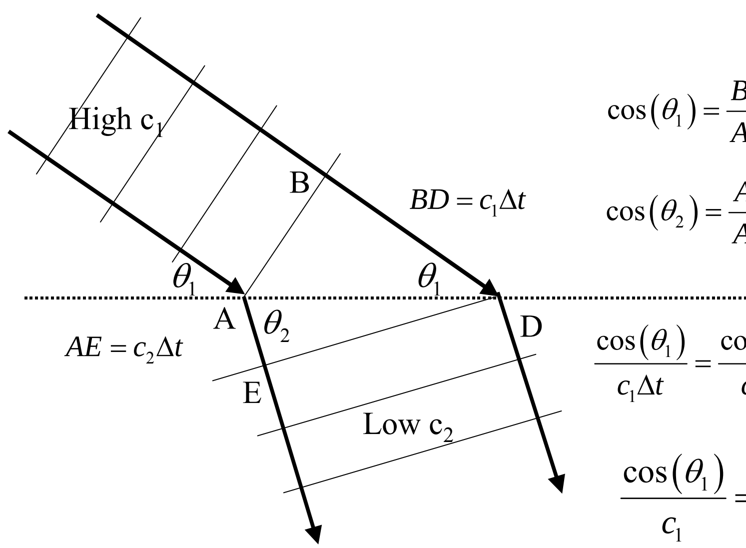


Using a Sound Velocity Profile and Snell's Law

We will now shift from thinking of sound as a wave and using the wave equations to sound as a ray and using Snell's Law. We can look at either the grazing angles, referenced to the horizontal and used when looking at refraction, or incidence angles, referenced to the vertical and used for refraction and backscattering.

In the below sketch, a plane wave is moving towards a boundary beyond which the speed of sound is much slower. As the wavefronts hit the boundary they slow down and bend more normal to the boundary. Specific examination of the wave after the right edge hits the boundary at point A shows that the left side of the wavefront must travel a distance from B to D expressed as the product of the sound speed c_1 and some time interval Δt. In that same time interval the right edge of the wave front moves from A to E expressed as the product of sound speed c_2 and

Refraction



$$\cos(\theta_1) = \frac{BD}{AD} = \frac{c_1\Delta t}{AD}$$

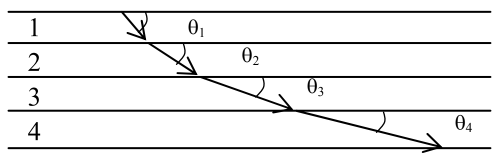
$$\cos(\theta_2) = \frac{AE}{AD} = \frac{c_2\Delta t}{AD}$$

$$\frac{\cos(\theta_1)}{c_1\Delta t} = \frac{\cos(\theta_2)}{c_2\Delta t} = \frac{1}{AD}$$

$$\frac{\cos(\theta_1)}{c_1} = \frac{\cos(\theta_2)}{c_2}$$

some time interval Δt. Using trigonometry we see that the ratio of the cosine of the grazing angle to the speed of sound remains constant across the boundary. This observation is called Snell's Law.

Snell's law and ray theory are well suited for each other. Imagine that a sound ray is transmitted through a series of mediums label 1 through 4 with sequentially increasing sound speed. In each medium, the angle the ray makes with the horizontal, θ, will depend on the angle it has in the previous medium and the speed of sound for each medium. The figure below depicts the relation.



where $c_1 < c_2 < c_3 < c_4$ and $\theta_1 > \theta_2 > \theta_3 > \theta_4$

According to Snell's Law

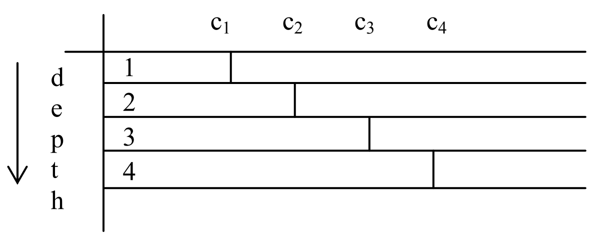
$$\frac{cos(\theta_1)}{c_1} = \frac{cos(\theta_2)}{c_2} = \frac{cos(\theta_3)}{c_3} = ... = \frac{cos(\theta_n)}{c_n} = \text{constant}$$

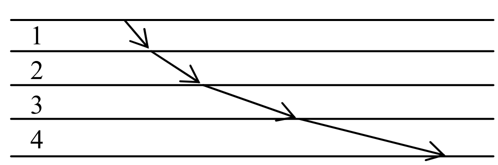
Notice that when a ray is in a layer and horizontal, $\theta = 0°$ and the $\cos(\theta) = 1$. We call the speed of sound when the ray is horizontal, c_o.

Sound Rays Travel in Arcs

Using Snell's Law from above, we can approximate the behavior of a sound ray as it travels through a medium where the speed of sound is changing at a constant rate. Let's take the example where the speed of sound increases as a function of depth as shown.

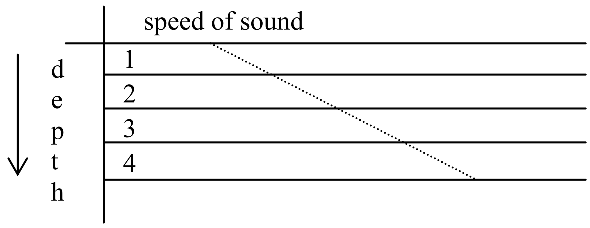
If the speed of sound increased in each layer as shown, a sound ray would travel in a path the same as the one already shown.





(Notice that the sound ray is bending back towards the layers where the sound speed is lower. This can be used later to qualitatively determine the ray path for sound in water.)

More realistically though, the speed of sound changes as a continuous function. If we use a continuous function instead of the step function for the speed of sound vs. depth, the speed of sound as a function of depth can be described by a simple linear equation. This result can be used to find functions for the radius of the path of the sound ray as well as other quantities.



(Korman, M.S. Principles of Underwater Sound and Sonar, the preliminary edition. Dubuque, IA: Kendall/Hunt Publishing Company, 1995, pgs 145-147)

The speed of sound, shown as the dotted line, can be expressed as (c_1 is the surface temperature):

$$c = c_1 + gz$$

where g is the gradient, $g = \frac{\Delta c}{\Delta z}$. From Snell's Law and inserting our relation ship for c, yields:

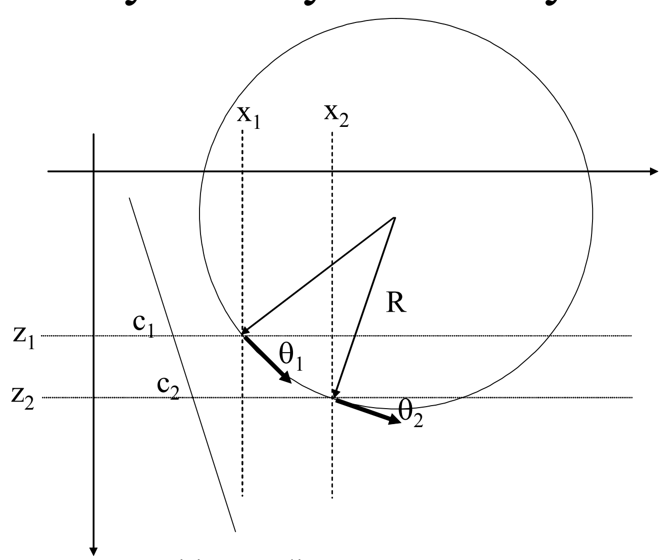
$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c}$$

$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta_2}{c_1 + gz}$$

$$z = R\left(\cos\theta - \cos\theta_1\right)$$

where R is defined as: $R \equiv \frac{c_1}{g\cos\theta_1}$. Soon we will show R is the radius of curvature of the sound ray. θ is always measured clockwise from the horizontal axis.

Ray Theory Geometry



Positive gradient, g

In polar coordinates we know that the slope of a line is

$$\frac{dz}{dx} = \tan\theta$$

From above we see that $dz = -R\sin\theta d\theta$. To find how the ray angle varies with distance x,

$$dx = -R\frac{\sin\theta}{\tan\theta}d\theta = -R\cos\theta d\theta$$

Integrating both sides gives the result that:

$$x - x_1 = -R\left[\sin\theta - \sin\theta_1\right]$$

Integrating both sides of dz gives:

$$z - z_1 = R\left[\cos\theta - \cos\theta_1\right]$$

Rearranging these two equations:

$$x - x_1 - R\sin\theta_1 = -R\sin\theta$$

$$z - z_1 + R\cos\theta_1 = R\cos\theta$$

Or

$$x - x_p = -R\sin\theta$$

$$z - z_p = R\cos\theta$$

With

$$x_p = x_1 - R\sin\theta_1$$

$$z_p = z_1 + R\cos\theta_1$$

Squaring the top two equations and adding the results gives the equation of a circle,

$$\left(x - x_p\right)^2 + \left(z - z_p\right)^2 = R^2$$

Specifically, the circle has radius, $R \equiv \frac{c_1}{g\cos\theta_1}$, and is centered at the point (x_p,z_p). Thus we have shown that a sound ray in a layer of constant sound speed will travel along the arc of a circle.

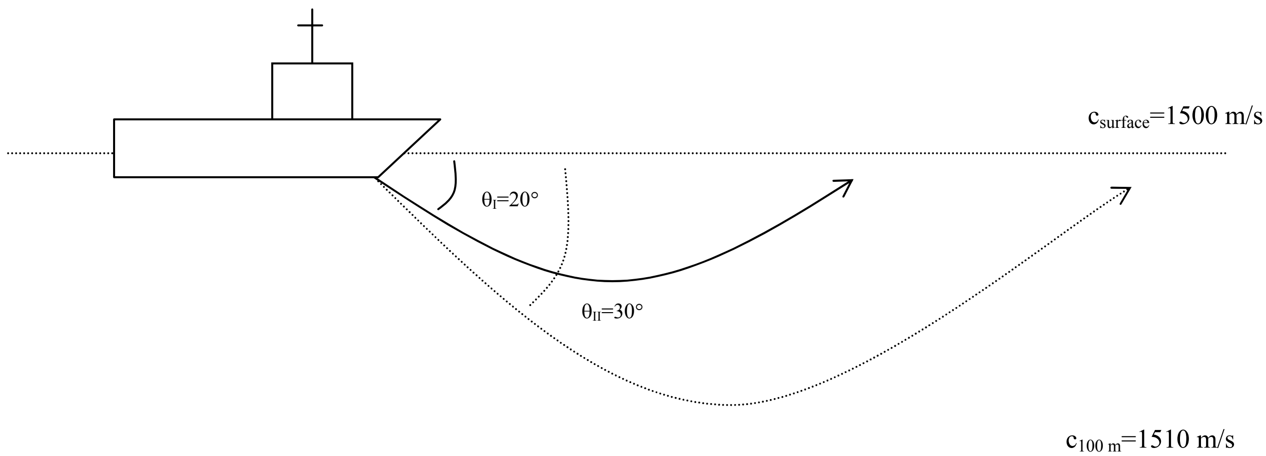
To sum up the results then:

$R = \dfrac{c_n}{g\,cos\,\theta_n}$	Radius of arc of the circle
$g = \dfrac{\Delta c}{\Delta z}$	Gradient (How fast the speed of sound changes per meter change in depth.)
$\dfrac{c_n}{cos\,\theta_n} = \text{constant}$	Snell's Law
$\Delta z = z_2 - z_1 = R(cos\,\theta_2 - cos\,\theta_1)$	Vertical Displacement
$\Delta x = x_2 - x_1 = -R(sin\,\theta_2 - sin\,\theta_1)$	Horizontal Displacement
$t_n = \left\lvert \dfrac{1}{g_n} \ln\left(\dfrac{\tan\left(\dfrac{\theta_n}{2}\right)}{\tan\left(\dfrac{\theta_{n-1}}{2}\right)} \right) \right\rvert$	Time to travel in layer n
$s_n = \dfrac{c_{n-1}}{g\cos\theta_{n-1}}(\theta - \theta_{n-1})$	Curvilinear Path Length

These equations will only work for one specific sound ray emanating from a source in an environment with a constant gradient. The last two equations in the table are presented without proof, but are useful results from many standard sources.

Example 1

Let's look at the following example.



Sound leaves the ship at two different angles, θ_1 and θ_2. Note the path travelled by each ray is different and if we calculate the parameters R, Δx and Δz, each of these will be different for each ray.

For both rays, the gradient, g, is a constant. This is calculated as such:

$$g = \frac{\Delta c}{\Delta z} = \frac{(1500 - 1510)\text{m/s}}{(0 - 100)\text{m}}$$
$$g = 0.1\,\text{sec}^{-1}$$

We must now calculate the radius of curvature, R of each ray separately:

$$R_I = \frac{c}{g\cos\theta_1} = \frac{1500\text{ m/s}}{(0.1\,\text{sec}^{-1})(\cos 20°)}$$
$$R_I = 16{,}000\text{ meters}$$

and

$$\text{R}_{\text{II}} = \frac{\text{c}}{\text{g}\cos\theta_{\text{II}}} = \frac{1500\text{ m/s}}{(0.1\text{ sec}^{-1})(\cos 30°)}$$
$$\text{R}_{\text{II}} = 17{,}300\text{ meters}$$

The skip distance, X, is the distance between successive places where the sound ray stikes the surface. The easiest way to calculate this is to calculate the displacement, Δx, from where the sound strikes the surface first to where the sound has leveled off or gone horizontal ($\theta_2 = 0°$). Thus:

$$X = 2\Delta x = -2R(\sin 0° - \sin\theta)$$
$$X = 2R\sin\theta$$

where θ is the angle of reflection from the surface. So for each ray:

$$X_I = 2(16{,}000\ \text{m})\sin 20^\circ$$
$$X_I = 11{,}000\ \text{m}$$

and

$$X_{II} = 2(17{,}300\ \text{m})\sin 30^\circ$$
$$X_{II} = 17{,}300\ \text{m}$$

The results of the calculations for each ray are significantly different from each other and show how the ray paths depend on the initial angle of the ray.

We can do the same for the depth the rays to. The maximum depth excursion of the ray below its starting depth occurs when the ray goes horizontal again ($\theta_2 = 0°$) or:

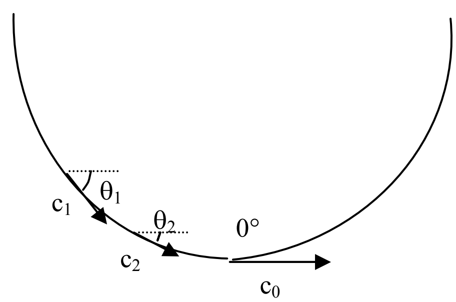
$$\Delta z_{\max} = R(\cos 0^\circ - \cos\theta)$$
$$\Delta z_{\max} = R(1 - \cos\theta)$$

so for each ray:

$$\Delta z_{\max,I} = 965\ \text{m}\ (3170\ \text{feet})$$
$$\Delta z_{\max,II} = 2320\ \text{m}\ (7600\ \text{feet})$$

Example 2

Also try the following example problem.[1]



Use the figure above and the following information to answer the questions.

a. If $\theta_2 = 30°$, $c_2 = 1299$ m/s, $c_1 = 964$ m/s, what is θ_1?
   Using Snell's Law we have:

$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta_2}{c_2}$$
$$\cos\theta_1 = c_1\frac{\cos\theta_2}{c_2}$$
$$\theta_1 = \cos^{-1}\left[\frac{964}{1299}\cos 30^\circ\right] = 50^\circ$$

[1] From: Korman, M. S. Principles of Underwater Sound and Sonar, the preliminary edition, p. 144.

b. Determine c_0.

Again using Snell's law and the $\theta_0 = 0°$

$$\frac{c_2}{\cos\theta_2} = c_0$$

$$c_0 = \frac{1299\ m/s}{\cos 30°} = 1500\ \text{m/s}$$

c. What is the gradient if $\Delta z = 3000$ m between points "1" and "0"?

$$g = \frac{\Delta c}{\Delta z} = \frac{c_0 - c_1}{\Delta z} = \frac{1500m/s - 964m/s}{3000m}$$

$$= .18\ \text{s}^{-1}$$

d. What is the radius of the sound ray path?

$$R = \frac{c_1}{g\cos\theta_1}$$

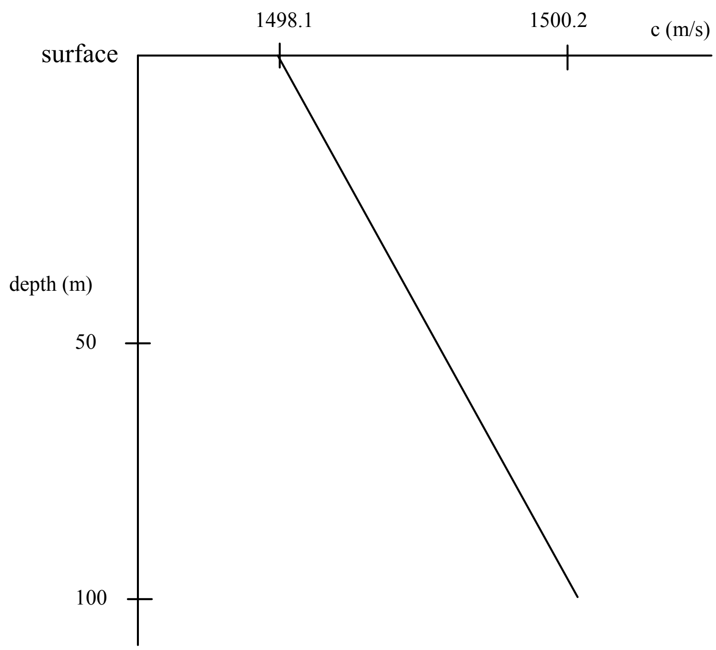
$$R = \frac{964\ \text{m/s}}{\left(0.18\ \text{s}^{-1}\right)\left(\cos(50°)\right)}$$

$$R = 8330\ \text{meters}$$

Problems

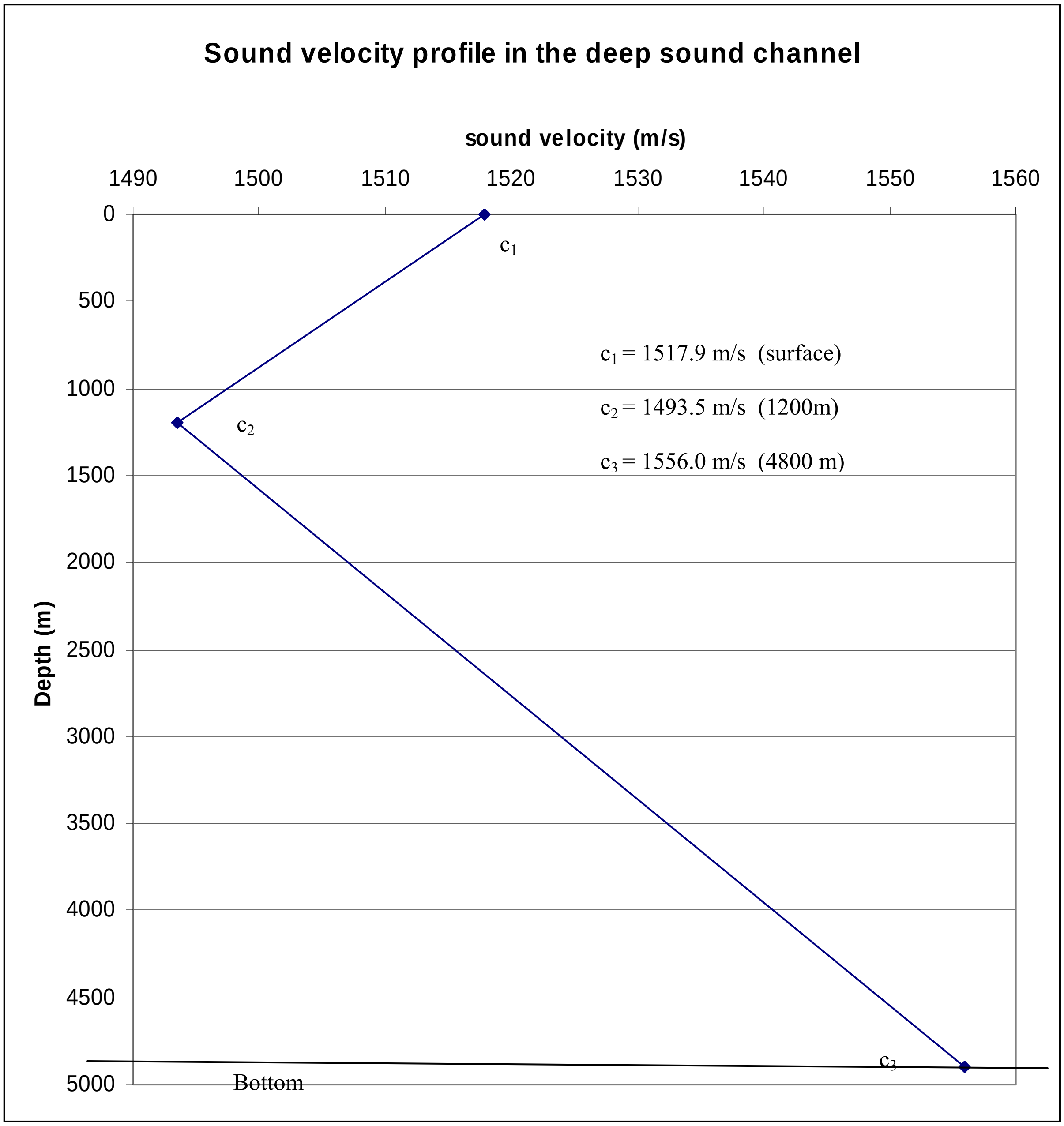
1. A submerged submarine is at 10 meters.
   a) Use Medwin's Equation to determine the speed of sound in the water if the salinity is 35.0 ppt and the seawater injection temperature is 20.0°C.
   b) If the submarine in the problem above submerges to 200. meters and the seawater injection temperature goes down to 5.00°C, what is the new sound speed?
   c) Determine the average gradient between the two depths in the problems above.
   d) Is this a positive or negative sound gradient?
   e) Sketch the SVP and sketch the approximate of several sound rays emanating from the sub when it is at a depth of 10.0 meters.
   f) If sound radiates from the sub at a depth of 10.0 m at an initial angle of 15° with respect to the horizontal, determine the angle of depression of the sound when it has reached a depth of 500. meters (assume the gradient is constant.)
   g) Determine the Radius of Curvature of the sound ray.
   h) Determine the horizontal displacement of the sound ray as it goes from 10.0 meters to 500. meters.

2. Use the following SVP to complete the next problems:



   a) Calculate the gradient of the SVP.
   b) If a sound ray exits horizontally from a sub that is at 50.0 m, what will be its grazing angle when it hits the surface of the ocean?
   c) If a ray reflects off the surface of the ocean at an angle of 2.15° (assume the surface is perfectly flat), what will be the skip distance of the sound ray?
   d) This is an example of:
      i) a positive gradient
      ii) a negative gradient
   e) If a sub is at 50.0 m, what is the largest angle below the horizontal where the ray will **not** reach 100. m?
   f) What is the skip distance of the limiting ray?

Use the following SVP for the remaining problems:



3. Compute the sound speed gradients for $0 < z < 1200$ m and $1200 < z < 4800$.

4. A ray starts at 1525 m with a grazing angle if 15 degrees (recall a positive grazing angle is pointed below the horizontal).
   a) What are the sound speeds at depths of 1525 m and 2440 m?
   b) Does the ray curve upward or downward?
   c) What is the grazing angle at 2440 m?
   d) At what sound speed will the ray become horizontal (a horizontal ray has a grazing angle of 0 degrees)?

5. A sound source is at a depth of 1200 m.
   a) At what angle with respect to the horizontal does the ray have to make at 1200 m so that that when it reaches the surface the grazing angle is 0 degrees? This is called the surface limiting ray.
   b) What angle with respect to the horizontal does a ray have to make at 1200 m so that when it reaches the bottom at 4900 m, the angle is 0 degrees? This is called the bottom limiting ray.
   c) At what depth below 1200 m is the sound speed equal to that at the surface?
   d) At what angle with respect to the horizontal does a ray have to make at 1200 m so that when it reaches the the depth found in c), the grazing angle is 0 degrees? This is called the lower limiting ray
   e) Compute the radius of the surface limiting ray.
   f) Compute the radius of the bottom limiting ray.
   g) Compute the radius of the lower limiting ray.
   h) Compute the horizontal distance that the bottom limiting ray travels from the source until it grazes the bottom.
   i) Compute the horizontal distance that the surface limiting ray travels from the source until it just grazes the surface.

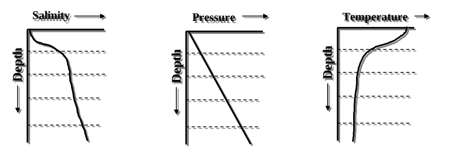
6. A ray leaving a sound source at 1200 m points downward with an angle of 30 degrees with respect to the surface.
   a) How far will it travel horizontally until its angle with the horizontal is 25 degrees?
   b) At what depth does the ray in a) make an angle of 25 degrees with respect to the horizontal.

Speed of Sound in Water

Medium Effects: Elasticity and Density

Variable Effects of:

Salinity **Pressure** **Temperature**



Speed of Sound Factors

- Temperature
- Pressure or Depth
- Salinity

1° C increase in temperature $\Rightarrow$ 3 m/s increase in speed

100 meters of depth $\Rightarrow$ 1.7 m/s increase in speed

1 ppt increase in salinity $\Rightarrow$ 1.3 m/s increase in speed

Temperature, Pressure, and Salinity

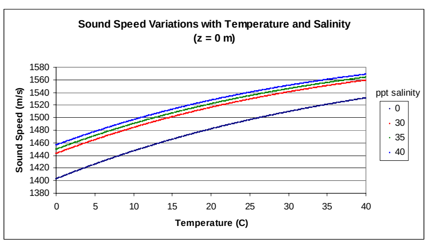
$$c(t,z,S) = 1449.2 + 4.6t - 5.5\text{x}10^{-2}t^2 + 2.9\text{x}10^{-4}t^3 + \left(1.34 - 10^{-2}t\right)\left(S-35\right) + 1.6\text{x}10^{-2}z$$

with the following limits:

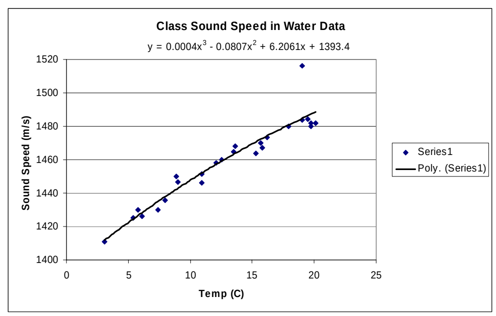
$0 \le t \le 35°$ C

$0 \le S \le 45$ p.s.u.

$0 \le z \le 1000$ meters



Class Sound Speed Data



More Curve Fitting

$$c = c_0 + c_1P + c_2P^2 + c_3P^3 + AS + BS^{\frac{3}{2}} + CS^2$$

Chen and Millero

P = Pressure from Leroy Formula

$$c_o = 1402.388 + 5.03711t - 5.80852\text{x}10^{-2}t^2 + 3.3420\text{x}10^{-4}t^3 - 1.478\text{x}10^{-6}t^4 + 3.1464\text{x}10^{-9}t^5$$

$$c_1 = 0.153563 + 6.8982\text{x}10^{-4}t - 8.1788\text{x}10^{-6}t^2 + 1.3621\text{x}10^{-7}t^3 - 6.1185 + 1.3621\text{x}10^{-10}t^4$$

$$c_2 = 3.126\text{x}10^{-5} - 1.7107\text{x}10^{-6}t + 2.5974\text{x}10^{-8}t^2 - 2.5335\text{x}10^{-10}t^3 + 1.0405\text{x}10^{-12}t^4$$

$$c_3 = -9.7729\text{x}10^{-9} + 3.8504\text{x}10^{-10}t - 2.3643\text{x}10^{-12}t^2$$

$$A = A_o + A_1P + A_2P^2 + A_3P^3$$

$$A_1 = 9.4742\text{x}10^{-5} - 1.258\text{x}10^{-5}t - 6.4885\text{x}10^{-8}t^2 + 1.0507\text{x}10^{-8}t^3 - 2.0122\text{x}10^{-10}t^4$$

$$A_2 = -3.9064\text{x}10^{-7} + 9.1041\text{x}10^{-9}t - 1.6002\text{x}10^{-10}t^2 + 7.988\text{x}10^{-12}t^3$$

$$A_3 = 1.1\text{x}10^{-10} + 6.649\text{x}10^{-12}t - 3.389\text{x}10^{-13}t^2$$

$$B = -1.922\text{x}10^{-2} - 4.42\text{x}10^{-5}t + \left(7.3637\text{x}10^{-5} + 1.7945\text{x}10^{-7}t\right)P$$

$$C = -7.9836\text{x}10^{-6}P + 1.727\text{x}10^{-3}$$

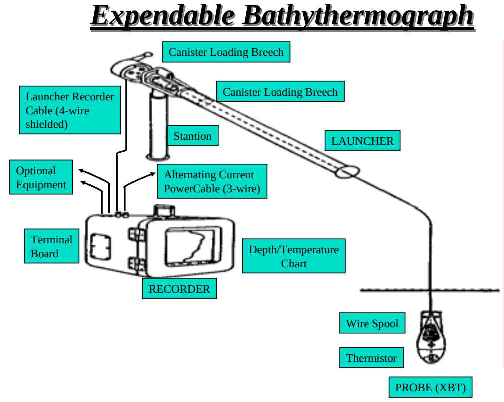
$$P = \left[1.0052405\left(1 + 5.28\times10^{-3}\sin\phi\right)z + 2.36\times10^{-6}z^2 + 10.196\right]\times10^4 \text{ Pa}$$

Leroy

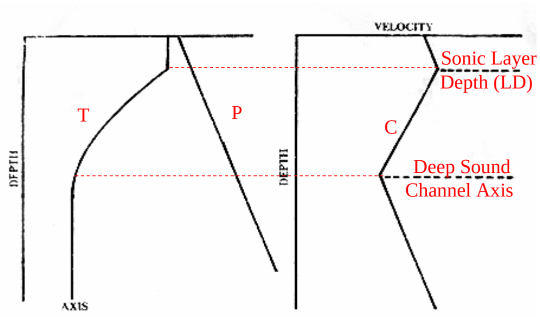
ϕ - latitude in degrees

z - depth in meters

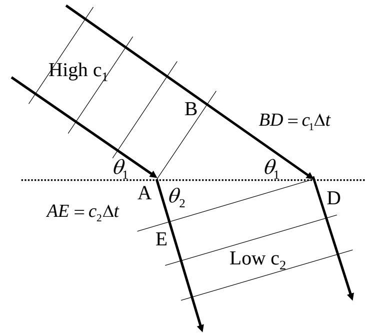
Expendable Bathythermograph



Typical Deep Ocean Sound Velocity Profile (SVP)



Refraction



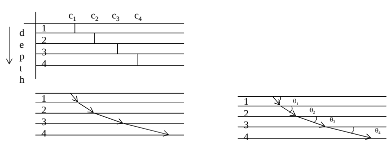
$$\cos(\theta_1) = \frac{BD}{AD} = \frac{c_1\Delta t}{AD}$$

$$\cos(\theta_2) = \frac{AE}{AD} = \frac{c_2\Delta t}{AD}$$

$$\frac{\cos(\theta_1)}{c_1\Delta t} = \frac{\cos(\theta_2)}{c_2\Delta t} = \frac{1}{AD}$$

$$\frac{\cos(\theta_1)}{c_1} = \frac{\cos(\theta_2)}{c_2}$$

Multiple Boundary Layers



where $c_1 < c_2 < c_3 < c_4$ and $\theta_1 > \theta_2 > \theta_3 > \theta_4$

$$\frac{\cos(\theta_1)}{c_1} = \frac{\cos(\theta_2)}{c_2} = \frac{\cos(\theta_3)}{c_3} = ... = \frac{\cos(\theta_n)}{c_n} = \text{constant}$$

Simple Ray Theory

$$\text{gradient} = g = \frac{\Delta c}{\Delta z} = \frac{c - c_1}{z - z_1}$$

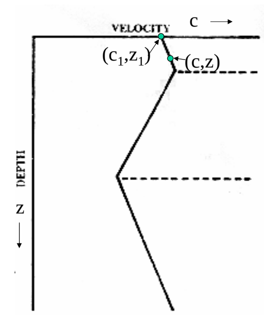
$$c = c_1 + gz$$

Snell's Law

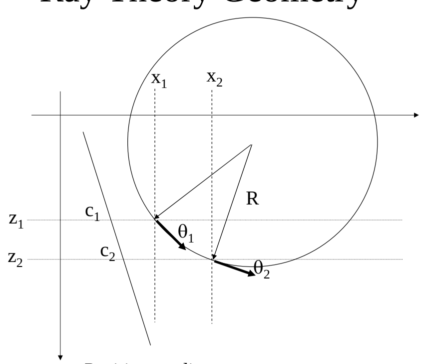
$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c}$$

$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c_1 + gz}$$

$$z = R(\cos\theta - \cos\theta_1) \qquad R = \frac{c_1}{g\cos\theta_1}$$

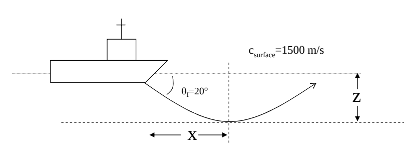


Ray Theory Geometry



Positive gradient, g

The **z (Depth)** and x (Range) Directions



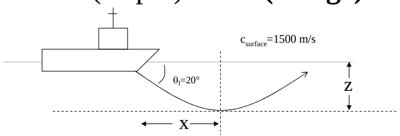
$$R = \frac{c_1}{g\cos\theta_1}$$

$$z = R(\cos\theta - \cos\theta_1) \qquad dz = -R\sin\theta d\theta$$

$$\int_{z_1}^{z} dz = -R\int_{\theta_1}^{\theta} \sin\theta d\theta$$

$$\boxed{z - z_1 = R(\cos\theta - \cos\theta_1)}$$

The z (Depth) and **x (Range)** Directions



$$R = \frac{c_1}{g\cos\theta_1}$$

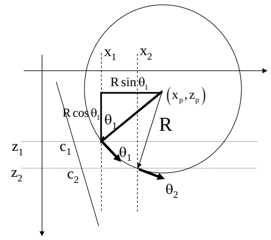
$$z = R\left(\cos\theta - \cos\theta_1\right) \qquad dz = -R\sin\theta d\theta$$

$$\tan\theta = \frac{dz}{dx}$$

$$\int_{x_1}^{x} dx = \int_{z_1}^{z} \frac{dz}{\tan\theta} = -R\int_{\theta_1}^{\theta} \frac{\sin\theta d\theta}{\tan\theta} = -R\int_{\theta_1}^{\theta} \cos\theta d\theta$$

$$\boxed{x - x_1 = -R\left(\sin\theta - \sin\theta_1\right)}$$

Why is R = Radius?



Positive gradient, g

$$x - x_1 = -R\left(\sin\theta - \sin\theta_1\right)$$
$$z - z_1 = R\left(\cos\theta - \cos\theta_1\right)$$

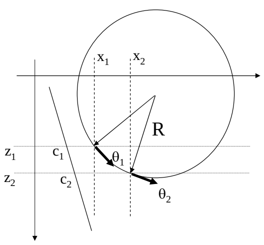
$$x - x_p = -R\sin\theta$$
$$z - z_p = R\cos\theta$$

$$x_p = x_1 + R\sin\theta_1$$
$$z_p = z_1 - R\cos\theta_1$$

$$\left(x - x_p\right)^2 + \left(z - z_p\right)^2 = \left(-R\right)^2\sin^2\theta + R^2\cos^2\theta$$

$$\left(x - x_p\right)^2 + \left(z - z_p\right)^2 = R^2$$

Summary



Positive gradient, g

$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c}$$

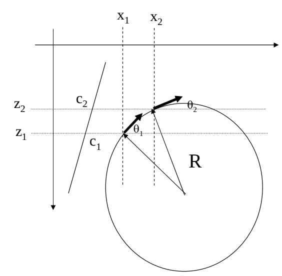
$$x - x_1 = -R\left(\sin\theta - \sin\theta_1\right)$$

$$z - z_1 = R\left(\cos\theta - \cos\theta_1\right)$$

$$R = \frac{c_1}{g\cos\theta_1}$$

$$g = \frac{\Delta c}{\Delta z} = \frac{c - c_1}{z - z_1}$$

Negative Gradient



Negative gradient, g

$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c}$$

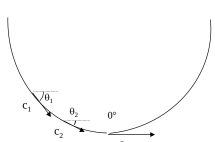
$$x - x_1 = -R\left(\sin\theta - \sin\theta_1\right)$$

$$z - z_1 = R\left(\cos\theta - \cos\theta_1\right)$$

$$R = \frac{c_1}{g\cos\theta_1}$$

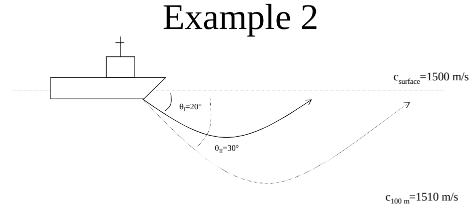
$$g = \frac{\Delta c}{\Delta z} = \frac{c - c_1}{z - z_1}$$

Example 1



- Given: c_1 = 964 m/s, c_2 = 1299 m/s, θ_2 = 30°
  Δz(between 1 and 0) = 3000m
- Find: θ_1, c_o, g (between pt 1 and 0), R

Example 2



- Find gradient, g
- Find Radius of Curvature, R, for each ray.
- Skip distance – i.e. the distance until the ray hits the surface again
- Max depth reached by each ray

Backups

$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c}$$

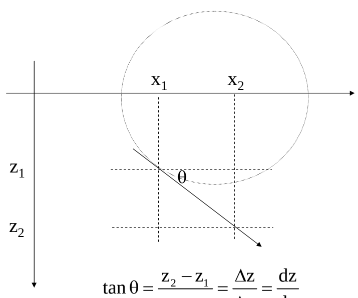
$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c_1 + gz}$$

$$c_1\cos\theta_1 + gz\cos\theta_1 = c_1\cos\theta$$

$$z = \frac{c_1}{g\cos\theta_1}\left(\cos\theta - \cos\theta_1\right)$$

$$\boxed{z = R\left(\cos\theta - \cos\theta_1\right)} \qquad \boxed{R = \frac{c_1}{g\cos\theta_1}}$$

Slope = tanθ



$$\tan\theta = \frac{z_2 - z_1}{x_2 - x_1} = \frac{\Delta z}{\Delta x} = \frac{dz}{dx}$$