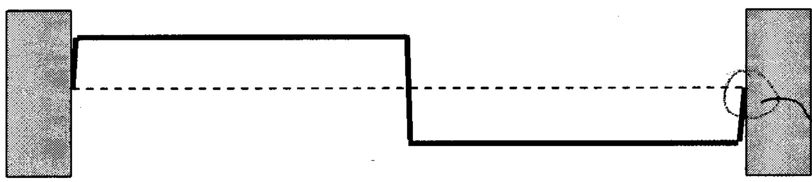
Name____________________________ SP436 Exam II

1. (25) A stretchy string of length, L, is suspended between two fixed posts. A template is created that stretches the string into the initial position shown below:



x=0 x=L/2 x=L

BC: $y(L,t)=0$

$kL = n\pi$

$k_n = \frac{n\pi}{L}$

A reasonable approximation for this initial condition is:

$$y(x,t=0)=\begin{cases}+h & 0<x<\frac{L}{2}\\ -h & \frac{L}{2}<x<L\end{cases}$$

If the template is rapidly removed from this initial rest position, the string vibrates according to the general result:

$$y(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\omega_n t+B_n\sin\omega_n t\right)\sin k_n x$$

a. (15) Find the expression for the coefficients A_n and B_n.

b. (5) If h = 1 cm, find the amplitude of the fundamental mode and the first five overtones (n=1 to n=6). Sketch the non-zero modes.

ρ

$c = \sqrt{\frac{T}{\rho}} = \sqrt{\frac{5N}{.003kg/.314m}} = 22.9\,m/s$

c. (5) If the mass of the string is .0030 kg and the length is 0.314 m and the tension is 5.0 N, How much energy is in the n=2 mode?

$B_n = \frac{2}{\omega_n L}\int_0^L u(x,0)\sin k_n x\,dx$ $\quad u(x,t)=0 \Rightarrow \boxed{B_n = 0}$

$A_n = \frac{2}{L}\int_0^L y(x,0)\sin k_n x\,dx = \frac{2}{L}\left\{\int_0^{L/2} h\sin\left(\frac{n\pi}{L}x\right)dx + \int_{L/2}^{L}(-h)\sin\left(\frac{n\pi}{L}x\right)dx\right\}$

TI-92 $\longrightarrow\ = \frac{2}{\cancel{L}}\left\{\frac{h\cancel{L}}{n\pi}\left(1-2\cos\frac{n\pi}{2}+\cos n\pi\right)\right\} = \boxed{\frac{2h}{n\pi}\left(1-2\cos\frac{n\pi}{2}+\cos n\pi\right)}$

$A_1 = \frac{2h}{\pi}(1-0+(-1)) = 0$

$A_2 = \frac{2h}{2\pi}(1-2(-1)+1) = \frac{4h}{\pi} = \frac{4(.01m)}{\pi} = \boxed{.0127m}$

$A_3 = \frac{2h}{3\pi}(1-0+(-1)) = 0$

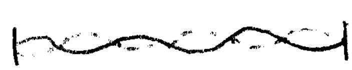
$A_4 = \frac{2h}{4\pi}(1-2(+1)+1) = 0$

$A_5 = \frac{2h}{5\pi}(1-0+(-1)) = 0$

$A_6 = \frac{2h}{6\pi}(1-2(-1)-(-1)) = \frac{8h}{6\pi} = \frac{4(.01m)}{3\pi} = \boxed{.00424m}$

n = 2

n = 6



$E_n = \frac{1}{4}m_s\,\omega_n^2 A_n^2$

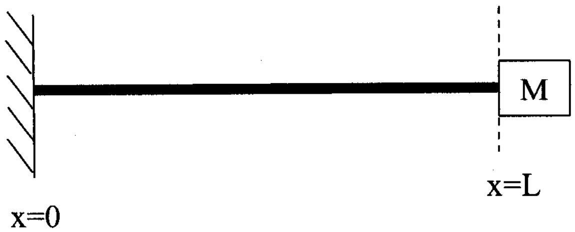
$k_2 = \frac{2\pi}{.314m}$

$\omega_2 = ck_2 = \frac{(22.9m/s)(2\pi)}{.314m} = 458\,\frac{r}{s}$

$E_2 = \frac{1}{4}(.003kg)(458/s)^2(.0127m)^2 = \boxed{.0253\ J}$

Name______________________________ SP436 Exam II

2. (25) A long, Aluminum bar is fixed at one end (x=0) and loaded with a large mass, M, at the other end (x=L). The bar has cross sectional area, S.



The bar vibrates in the longitudinal mode subject to the wave equation,

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2}$$

where $c^2 = \frac{Y}{\rho}$. We showed that the complex harmonic solution of the wave equation is,

$$\tilde{\xi}(x,t) = \tilde{A}e^{j(\omega t - kx)} + \tilde{B}e^{j(\omega t + kx)}.$$

a. (10) What are the boundary conditions for this system.

$$\boxed{\begin{aligned} \xi(x=0) &= 0 \\ -SY\frac{\partial \xi}{\partial x}\Big|_{x=L} &= M\frac{\partial^2 \xi}{\partial t^2}\Big|_{x=L} \end{aligned}}$$

b. (5) Use your boundary condition at x=0 to show that: $\tilde{\xi}(x,t) = -j2\tilde{A}e^{j\omega t}\sin kx$

$\xi(x=0) = \tilde{A}e^{j(\omega t)} + \tilde{B}e^{j(\omega t)} = 0 \Rightarrow \tilde{A} = -\tilde{B}$; $\xi(x,t) = \tilde{A}e^{j\omega t}(e^{-jkx} - e^{jkx}) = -2j\tilde{A}e^{j\omega t}\sin kx$ Q.E.D

c. (5) Combine with your boundary condition at x=L to show that: $\cot kL = \frac{M\omega^2}{SYk}$

d. (5) Since $Y = \rho c^2 = \frac{m_{bar}}{SL}c^2$, show this can be rewritten: $\cot kL = \frac{M}{m_{bar}}kL$

$$-SY\frac{\partial \xi}{\partial x}\Big|_{x=L} = M\frac{\partial^2 \xi}{\partial t^2}\Big|_{x=L}$$

$$+SY(+2j\tilde{A}e^{j\omega t}k\cos kL) = M(+2j\tilde{A}(j\omega)^2 e^{j\omega t}\sin kL)$$

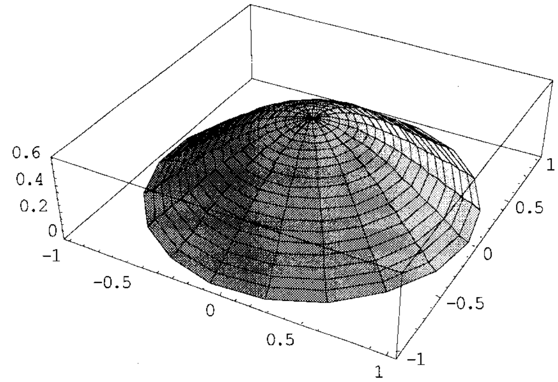
$$SYk\cos kL = M\omega^2\sin kL$$

$$\cot kL = \frac{M\omega^2}{SYk} \quad \text{Q.E.D}$$

$$\cot kL = \frac{M\omega^2}{S\left(\frac{M_{bar}}{SL}\right)c^2 k} = \frac{M}{m_{bar}}\frac{\omega}{c}L = \frac{M}{m_{bar}}kL \quad \text{Q.E.D.}$$

Name______________________________ SP436 Exam II

3. (25) An elastic membrane is stretched and clamped on a rigid circular frame with uniform tension. The fundamental frequency shown below is 70 Hz. The radius of the membrane is 0.10 m and the area density is 0.20 kg/m^2.



$$f_{01} = \frac{j_{01} c}{2\pi a}$$

a. (5) What is the phase speed of the standing wave on the membrane?

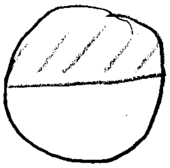
$$c = \frac{2\pi f_{01} a}{j_{01}} = \frac{2\pi(70\text{Hz})(.1\text{m})}{2.4} = \boxed{18.3\ \text{m/s}}$$

b. (5) What is the tension per unit length (assumed to be isotropic) of the elastic membrane?

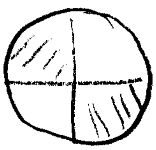
$$T = \rho_s c^2 = (0.2\tfrac{\text{kg}}{\text{m}^2})(18.3\tfrac{\text{m}}{\text{s}})^2 = 67.2\ \text{N/m}$$

c. (10) What are the next 5 lowest frequencies at which this membrane can vibrate?

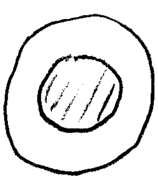
d. (5) Include a rough sketch of the oscillation spatial pattern similar to figure 4.4.1 in your textbook for each frequency.



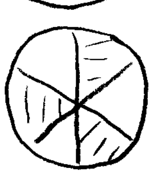
$$f_{1,1} = \frac{j_{11} c}{2\pi a} = (70\text{ Hz})\left(\frac{j_{11}}{j_{01}}\right) = 70\text{Hz}\left(\frac{3.83}{2.4}\right) = \boxed{112\text{ Hz}}$$



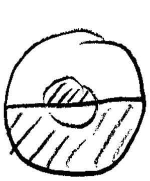
$$f_{2,1} = 70\text{Hz}\left(\frac{j_{21}}{j_{01}}\right) = 70\text{ Hz}\left(\frac{5.14}{2.4}\right) = \boxed{150\text{ Hz}}$$



$$f_{0,2} = 70\text{ Hz}\left(\frac{j_{02}}{j_{01}}\right) = 70\text{ Hz}\left(\frac{5.52}{2.4}\right) = \boxed{161\text{ Hz}}$$



$$f_{3,1} = 70\text{ Hz}\left(\frac{j_{31}}{j_{01}}\right) = 70\text{ Hz}\left(\frac{6.38}{2.4}\right) = \boxed{186\text{ Hz}}$$



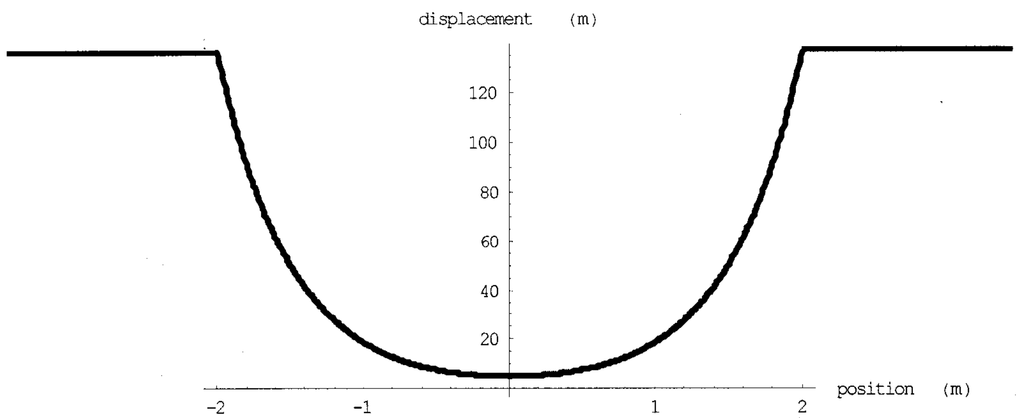
$$f_{1,2} = 70\text{ Hz}\left(\frac{j_{12}}{j_{10}}\right) = 70\text{Hz}\left(\frac{7.02}{2.4}\right) = \boxed{205\text{Hz}}$$

Name______________________________ SP436 Exam II

Extra Credit: (10) The transverse wave equation for a wave traveling on a string is:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}.$$

Show by direct substitution that $y = \cosh\left[k(ct - x)\right]$ is a solution to the wave equation.



$$y = \cosh\left(k(ct - x)\right)$$

$$\frac{\partial y}{\partial x} = -k\sinh\left(k(ct - x)\right) \qquad \frac{\partial y}{\partial t} = kc\sinh\left(k(ct - x)\right)$$

$$\frac{\partial^2 y}{\partial x^2} = (-k)^2\cosh\left(k(ct - x)\right) \qquad \frac{\partial^2 y}{\partial t^2} = k^2c^2\cosh\left(k(ct - x)\right)$$

$$k^2\cosh\left(k(ct - x)\right) = \frac{1}{\cancel{c^2}}\left(k^2\cancel{c^2}\cosh\left(k(ct - x)\right)\right) \checkmark$$

CHECKS