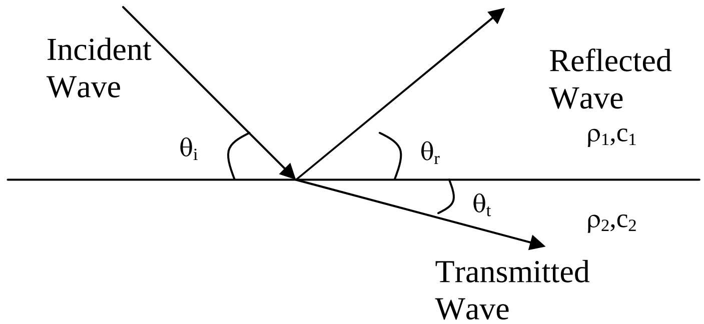
Boundary Losses

Let's revisit Snell's Law and investigate what happens to a sound wave incident upon a boundary.

Using the figure below, we will try to determine how much of the sound energy of an incident wave is actually reflected at the boundary. According to Snell's Law,

$$\frac{\cos\theta_i}{c_1} = \frac{\cos\theta_t}{c_2}$$



Additionally, we expect that the incident angle and the reflected angle are the same. This follows logically from Snell's Law as well since the speed of sound for the incident and reflected waves are the same.

$$\frac{\cos\theta_i}{c_1} = \frac{\cos\theta_r}{c_1} \Rightarrow \theta_i = \theta_r$$

Reflection Coefficient

The reflection coefficient expresses the ratio of the intensity of the reflected wave to the intensity of the incident wave ($I_r = RI_i$). In all cases we are referring to the time average of the acoustic intensities and the rms acoustic pressures and particle velocities. The equation for the reflection coefficient would be:

$$R = \frac{I_r}{I_i} = \frac{\dfrac{p_r^2}{\rho_1 c_1}}{\dfrac{p_i^2}{\rho_1 c_1}} = \frac{p_r^2}{p_i^2}$$

To figure out how much energy is reflected, we must examine the following boundary conditions:

1) The pressure at the boundary is continuous.
2) The normal component of the velocity must also be continuous at the boundary.

To say that a property is continuous means that it is the same on both sides of the boundary. Let's look at the first condition concerning the pressure. To state this condition in equation format:

$$p_i + p_r = p_t$$

Since both the incident and reflected wave are on the same side of the boundary, their rms acoustic pressures added together must equal the rms acoustic pressure of the transmitted wave.

To satisfy the second condition, the component of the velocity normal to the boundary must also be continuous at the boundary. An equation for this would be:

$$|u_i|\sin\theta_i - |u_r|\sin\theta_r = |u_t|\sin\theta_t$$

The negative sign in the reflected term is because it is moving in the opposite direction as the other two velocities.

We can relate the rms particle velocity to the rms acoustic pressure using the following relationship:

$$p_a = zu = \rho c u$$

where z=ρc is the acoustic impedance of the medium. Thus:

$$\frac{p_i}{\rho_1 c_1}\sin\theta_i - \frac{p_r}{\rho_1 c_1}\sin\theta_r = \frac{p_t}{\rho_2 c_2}\sin\theta_t$$

If we substitute in the value of transmitted acoustic pressure from the pressure boundary condition we have,

$$\frac{p_i}{\rho_1 c_1}\sin\theta_i - \frac{p_r}{\rho_1 c_1}\sin\theta_r = \frac{p_i + p_r}{\rho_2 c_2}\sin\theta_t = \frac{p_i}{\rho_2 c_2}\sin\theta_t + \frac{p_r}{\rho_2 c_2}\sin\theta_t$$

Remembering that the incident and reflected angles are the same, we will rearrange to bring terms with reflected pressure and incident pressure on opposite sides of the equation.

$$\frac{p_i}{\rho_1 c_1}\sin\theta_i - \frac{p_i}{\rho_2 c_2}\sin\theta_t = \frac{p_r}{\rho_1 c_1}\sin\theta_i + \frac{p_r}{\rho_2 c_2}\sin\theta_t$$

Rearranging,

$$p_i\left(\rho_2 c_2 \sin\theta_i - \rho_1 c_1 \sin\theta_t\right) = p_r\left(\rho_2 c_2 \sin\theta_i + \rho_1 c_1 \sin\theta_t\right)$$

Or,

$$\frac{p_r}{p_i} = \frac{\left(\rho_2 c_2 \sin\theta_i - \rho_1 c_1 \sin\theta_t\right)}{\left(\rho_2 c_2 \sin\theta_i + \rho_1 c_1 \sin\theta_t\right)} = \frac{\left(\frac{\rho_2}{\rho_1}\sin\theta_i - \frac{c_1}{c_2}\sin\theta_t\right)}{\left(\frac{\rho_2}{\rho_1}\sin\theta_i + \frac{c_1}{c_2}\sin\theta_t\right)}$$

Using this result, we can easily establish an expression for the Reflection Coefficient, R.

$$R\left(\theta_i, \theta_t\right) = \frac{p_r^2}{p_i^2} = \left[\frac{m\sin\theta_i - n\sin\theta_t}{m\sin\theta_i + n\sin\theta_t}\right]^2$$

$$\text{where } m = \frac{\rho_2}{\rho_1} \;\;\&\;\; n = \frac{c_1}{c_2}$$

Notice that the subscripts are reversed in the equations for m and n. From this equation we can see that the Reflection Coefficient is dependent upon not only the mediums on each side of the boundary, but also the angle of incidence and the transmitted angle of the wave.

Further, we can express θ_t in terms of θ_i using Snell's Law and some trigonometric identities.

$$\frac{\cos\theta_i}{c_1} = \frac{\cos\theta_t}{c_2}$$

$$\cos\theta_t = \frac{c_2}{c_1}\cos\theta_i = \frac{\cos\theta_i}{n}$$

$$\sin\theta_t = \sqrt{1-\cos^2\theta_t} = \sqrt{1-\frac{\cos^2\theta_i}{n^2}}$$

A more useful expression for R then becomes:

$$R\left(\theta_i\right) = \left[\frac{m\sin\theta_i - \sqrt{n^2-\cos^2\theta_i}}{m\sin\theta_i + \sqrt{n^2-\cos^2\theta_i}}\right]^2$$

where m and n are as expressed earlier.

Normal Incidence

A useful case to study is when the incident wave arrives at an angle of 90° or normal to the boundary surface. Substituting θ_i = 90°, we get the following for the reflection coefficient:

$$R = \left(\frac{m-n}{m+n} \right)^2$$

dB Loss

A logical definition for the loss at a boundary is to subtract the reflected level from the incident level in dB. Applying the definition of the decibel level and the rules for subtraction,

$$dB_{loss} = L_{in} - L_{ref} = 10\log\left(\frac{I_{in}}{I_o}\right) - 10\log\left(\frac{I_{ref}}{I_o}\right) = 10\log\left(\frac{I_{in}}{I_{ref}}\right) = -10\log\left(\frac{I_{ref}}{I_{in}}\right) = -10\log\left(R\right)$$

Total Reflection

One special case is when there is total reflection (R=1). This occurs when the incident angle is less than a special angle called the critical angle. For there to be a critical angle, the speed of sound in the incident medium MUST BE less than the speed of sound in the second medium or:

$$\frac{c_1}{c_2} < 1$$

If this condition exists, the critical incident angle can be calculated using Snell's Law and letting the transmitted angle go to its minimum possible value of zero:

$$\theta_c = \cos^{-1}\left(\frac{c_1}{c_2}\right)$$

Transmission Coefficient

We will define the Transmission Coefficient in a manner consistent with the Reflection Coefficient.

$$T = \frac{I_t}{I_i} = \frac{\frac{p_t^2}{\rho_2 c_2}}{\frac{p_i^2}{\rho_1 c_1}} = \frac{\rho_1 c_1}{\rho_2 c_2}\frac{p_t^2}{p_i^2} = \frac{n}{m}\frac{p_t^2}{p_i^2}$$

The Transmission Coefficient can be easily derived if we take a look at the rate at which energy of the wave crosses the boundary. Since the energy of the incident wave must be conserved, it must equal the energy in the reflected plus transmitted wave. To express this in terms of an equation:

$$
\begin{aligned}
&I_i = I_t + I_r \quad \text{or,} \\
&1 = \frac{I_t}{I_i} + \frac{I_r}{I_i} \\
&1 = T + R \\
&\text{thus,} \\
&T(\theta_i) = 1 - R(\theta_i)
\end{aligned}
$$

Rather than establishing a separate equation for the transmission coefficient, we will generally first calculate the reflection coefficient using the equation above and then solve for the transmission coefficient by subtracting the reflection coefficient from one.

dB loss on transmission across a boundary would be defined similar to that for the dB loss on reflection.

$$dB_{loss} = L_{in} - L_{trans} = 10\log\left(\frac{I_{in}}{I_o}\right) - 10\log\left(\frac{I_{trans}}{I_o}\right) = 10\log\left(\frac{I_{in}}{I_{trans}}\right) = -10\log\left(\frac{I_{trans}}{I_{in}}\right) = -10\log(T)$$

Angle of Intromission

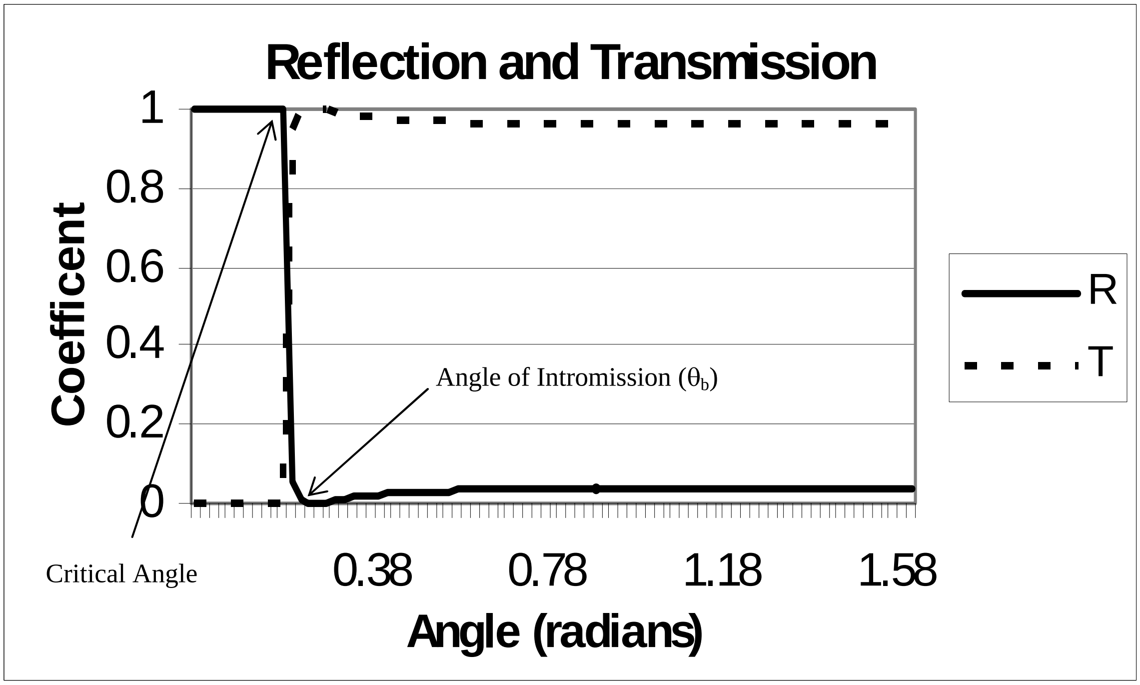
One special case for the Transmission Coefficient is when the Transmission Coefficient equals one ($T(\theta_i)=1$) and there is complete transmission of the incident wave and none of the energy is reflected. This occurs only at one angle (if it occurs at all) and that angle is referred to as the angle of intromission, θ_b. Using the equation for Reflection Coefficient when R = 0, and solving for the angle, we find:

$$\theta_b = \cos^{-1}\left[\sqrt{\frac{(m^2 - n^2)}{(m^2 - 1)}}\right]$$

(Note that the quantity $\frac{(m^2 - n^2)}{(m^2 - 1)}$ must be positive and less than 1 for the angle to exist. This is a rare case for most acoustics problems.)

Example

An example to illustrate how each of the coefficients vary as a function of angle is shown. For this example we will use m = 0.65, n = 0.98 and plot the Reflection Coefficient and Transmission Coefficient as a function of the angle of incidence (θ_i).



If we solve for the critical angle and angle of intromission we find:

$$\theta_c = \cos^{-1}(n)$$
$$\theta_c = 0.20 \text{ radians or,}$$
$$\theta_c = 11.5°$$

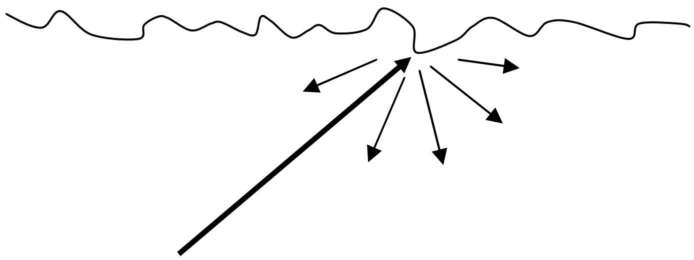
and

$$\theta_b = \cos^{-1}\left[\sqrt{\frac{(m^2 - n^2)}{(m^2 - 1)}}\right]$$
$$\theta_b = 0.26 \text{ radians or,}$$
$$\theta_b = 15.2°$$

as seen on the plot above. Also of interest to note is that the Reflection Coefficient is equal to 1 below the critical angle, θ_c, but the Transmission Coefficient is equal to 1 **only** at the angle of intromission.

Reflection from a Rough Surface

This entire discussion has assumed reflection from a sound ray bouncing off a smooth surface. This is called "specular reflection." Often the boundary is not smooth as in the case of a coral filled or rocky ocean floor, or a wind blown wave filled surface. In this case sound comes of the surface at various angles and the result is referred to as scattering.

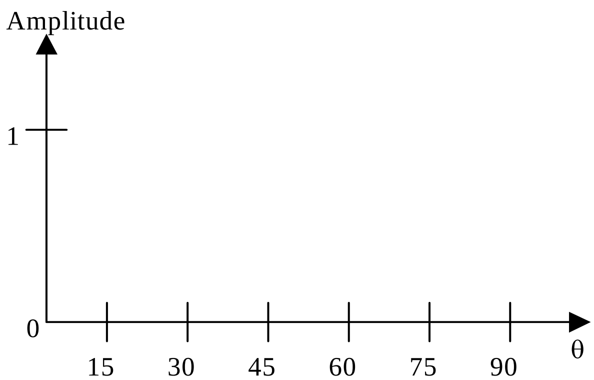


Some of the energy comes back in the direction toward the source of the incident sound and is called backscattering. In the operation of an active sonar system this backscattering results in the reception of unwanted sound which tends to mask the target echo. This unwanted sound is called "surface reverberation."

Problems

1. Given m = 0.5 and n = 0.7, determine:

   a) The angle of intromission if it exists.
   b) The <u>critical angle</u> if it exists.
   c) <u>Sketch</u> a plot of the reflection (dashed line) and transmission coefficients (solid line) as functions of angle from $0^o \le \theta \le 90^o$. Also <u>compute</u> R(90^o) and T(90^o).



   d) Using your graph above, if I_i = 0.2 W/m^2, determine I_r and I_t if:
      i) $\theta_1 \le \theta_c$?
      ii) $\theta_1 = \theta_b$?
      iii) $\theta_1 = 90^o$?

2. An SH-60F produces noise with an intensity of 750 KW/m^2 in a hover just above a glassy smooth sea. Given c_1 = 343 m/s, ρ_1=1.2 kg/m^3, c_2 = 1500 m/s, and ρ_2 = 1000 kg/m^3, determine the level of sound transmitted underwater in dB re 1 μPa (strikes sea surface θ_1 = 90^o).

3. If the transmitter is positioned at an angle where the reflection coefficient is 0.57, determine the intensity of a sound wave immediately below the surface of the sand if the incident intensity is 75 W/m^2.

4. For a sound wave in water incident onto a specially coated material,

$$m = 0.85$$
$$n = 0.95$$

Sketch a plot of R = I_r/I_i and T = I_t/I_i as a function of θ from 0^o to 90^o. (note that T = 1 – I)

5. Given the following data for the sediments in the Arctic Ocean bottom and for sea water near the bottom:

	Density	Sound Speed
Sea Water	1050 kg/m^3	1520 m/s
Artic Bottom	1300 kg/m^3	1440 m/s

a) For a plane acoustic wave incident on the bottom from the water, is there a critical angle? If so, calculate it.
b) Is there an angle of intromission? If so, calculate it.
c) Express the reflection loss in dB (assume normal incidence). The db loss would be found from 10 log (R).
d) Sketch a plot of $R = I_r/I_i$ and $T = I_t/I_i$ as a function of θ from 0^o to 90^o. (note that T = 1 – I)

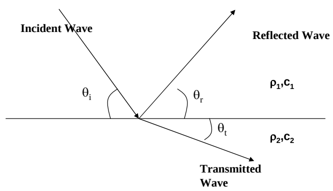
6. A plane sound wave is incident normally from air onto a smoth ocean surface. Given the following data:

	Density	Sound Speed
Air	1.20 kg/m^3	350 m/s
Artic Bottom	1000 kg/m^3	1500 m/s

a) If the intensity of sound in air is 10^{-2} W/m^2 incident normal to the air-water interface, what is the intensity of the sound in the water just below the surface?
b) What is the level in dB re 1 μPa below the surface?

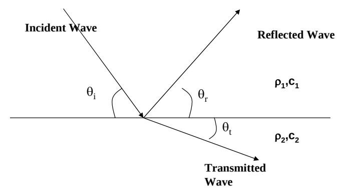
Transmission and Reflection

$$\theta_i = \theta_r$$



$$\frac{\cos\theta_i}{c_1} = \frac{\cos\theta_t}{c_2}$$

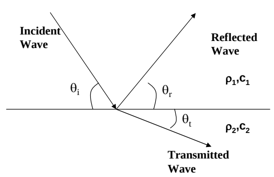
Reflection Coefficient



$$R = \frac{I_r}{I_i} = \frac{\frac{p_r^2}{\rho_1 c_1}}{\frac{p_i^2}{\rho_1 c_1}} = \frac{p_r^2}{p_i^2}$$

Boundary Conditions

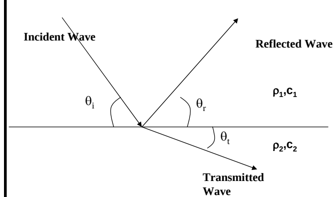
- **The pressure at the boundary is continuous.**
- **The normal component of the velocity must also be continuous at the boundary.**



$$p_i + p_r = p_t$$

$$|u_i|\sin\theta_i - |u_r|\sin\theta_r = |u_t|\sin\theta_t$$

Reflection Coefficient



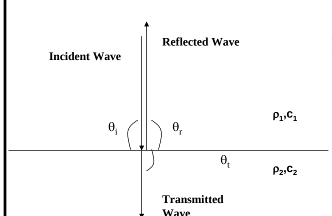
$$R(\theta_i, \theta_t) = \frac{p_r^2}{p_i^2} = \left[\frac{m\sin\theta_i - n\sin\theta_t}{m\sin\theta_i + n\sin\theta_t}\right]^2$$

$$\text{where } m = \frac{\rho_2}{\rho_1} \ \ \& \ \ n = \frac{c_1}{c_2}$$

Snell's Law

$$R(\theta_i) = \left[\frac{m\sin\theta_i - \sqrt{n^2 - \cos^2\theta_i}}{m\sin\theta_i + \sqrt{n^2 - \cos^2\theta_i}}\right]^2$$

Normal Incidence

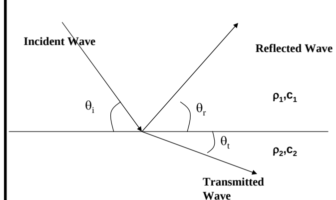


$$R(\theta_i) = \left[\frac{m\sin\theta_i - \sqrt{n^2 - \cos^2\theta_i}}{m\sin\theta_i + \sqrt{n^2 - \cos^2\theta_i}}\right]^2$$

$$R = \left(\frac{m-n}{m+n}\right)^2$$

$$m = \frac{\rho_2}{\rho_1} \ \ \& \ \ n = \frac{c_1}{c_2}$$

Critical Angle

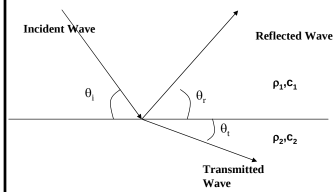


$$\frac{c_1}{c_2} < 1$$

$$\theta_c = \cos^{-1}\left(\frac{c_1}{c_2}\right)$$

Incident Angles less than the critical angle cannot have a transmitted wave

Transmission Coefficient



$$T = \frac{I_t}{I_i} = \frac{\frac{p_t^2}{\rho_2 c_2}}{\frac{p_i^2}{\rho_1 c_1}} = \frac{\rho_1 c_1}{\rho_2 c_2}\frac{p_t^2}{p_i^2} = \frac{n}{m}\frac{p_t^2}{p_i^2}$$

$$I_i = I_t + I_r \quad \text{or,}$$

$$1 = \frac{I_t}{I_i} + \frac{I_r}{I_i}$$

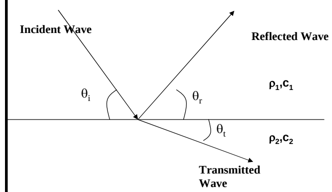
$$1 = T + R$$

thus,

$$T(\theta_i) = 1 - R(\theta_i)$$

$$m = \frac{\rho_2}{\rho_1} \ \& \ n = \frac{c_1}{c_2}$$

Angle of Intromission



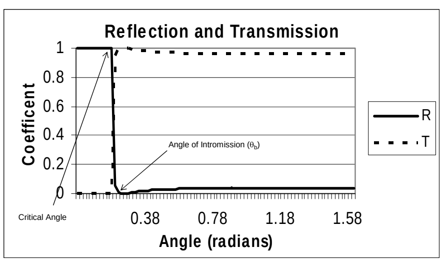
$$T = 1$$

$$R(\theta_i) = \left[\frac{m\sin\theta_i - \sqrt{n^2 - \cos^2\theta_i}}{m\sin\theta_i + \sqrt{n^2 - \cos^2\theta_i}}\right]^2 = 0$$

$$\theta_b = \cos^{-1}\left[\sqrt{\frac{(m^2 - n^2)}{(m^2 - 1)}}\right]$$

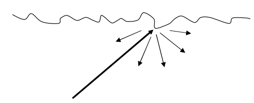
$$m = \frac{\rho_2}{\rho_1} \ \& \ n = \frac{c_1}{c_2}$$

Example: m = 0.65, n = 0.98



Find the critical angle and the angle of intromission

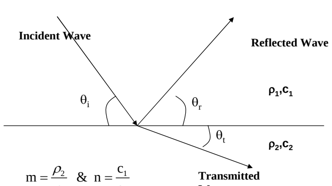
Non-specular Reflection



Backscattering
Reverberation

Summary

$$\theta_i = \theta_r$$



$$m = \frac{\rho_2}{\rho_1} \ \& \ n = \frac{c_1}{c_2}$$

$$\frac{\cos\theta_i}{c_1} = \frac{\cos\theta_t}{c_2}$$

$$R(\theta_i) = \left[\frac{m\sin\theta_i - \sqrt{n^2 - \cos^2\theta_i}}{m\sin\theta_i + \sqrt{n^2 - \cos^2\theta_i}}\right]^2$$

$$R = \left(\frac{m-n}{m+n}\right)^2$$

$$1 = T + R$$

$$\theta_c = \cos^{-1}\left(\frac{c_1}{c_2}\right)$$

$$\theta_b = \cos^{-1}\left[\sqrt{\frac{(m^2 - n^2)}{(m^2 - 1)}}\right]$$

Backup Slides

$$p_a = zu = \rho cu$$

$$\frac{p_i}{\rho_1 c_1}\sin\theta_i - \frac{p_r}{\rho_1 c_1}\sin\theta_r = \frac{p_t}{\rho_2 c_2}\sin\theta_t$$

$$\frac{p_i}{\rho_1 c_1}\sin\theta_i - \frac{p_r}{\rho_1 c_1}\sin\theta_r = \frac{p_i + p_r}{\rho_2 c_2}\sin\theta_t = \frac{p_i}{\rho_2 c_2}\sin\theta_t + \frac{p_r}{\rho_2 c_2}\sin\theta_t$$

$$\frac{p_i}{\rho_1 c_1}\sin\theta_i - \frac{p_i}{\rho_2 c_2}\sin\theta_t = \frac{p_r}{\rho_1 c_1}\sin\theta_i + \frac{p_r}{\rho_2 c_2}\sin\theta_t$$

$$p_i\left(\rho_2 c_2 \sin\theta_i - \rho_1 c_1 \sin\theta_t\right) = p_r\left(\rho_2 c_2 \sin\theta_i + \rho_1 c_1 \sin\theta_t\right)$$

$$\frac{p_r}{p_i} = \frac{\left(\rho_2 c_2 \sin\theta_i - \rho_1 c_1 \sin\theta_t\right)}{\left(\rho_2 c_2 \sin\theta_i + \rho_1 c_1 \sin\theta_t\right)} = \frac{\left(\frac{\rho_2}{\rho_1}\sin\theta_i - \frac{c_1}{c_2}\sin\theta_t\right)}{\left(\frac{\rho_2}{\rho_1}\sin\theta_i + \frac{c_1}{c_2}\sin\theta_t\right)}$$

$$\frac{\cos\theta_i}{c_1} = \frac{\cos\theta_t}{c_2}$$

$$\cos\theta_t = \frac{c_2}{c_1}\cos\theta_i = \frac{\cos\theta_i}{n}$$

$$\sin\theta_t = \sqrt{1-\cos^2\theta_t} = \sqrt{1-\frac{\cos^2\theta_i}{n^2}}$$

$$R(\theta_i) = \left[\frac{m\sin\theta_i - \sqrt{n^2-\cos^2\theta_i}}{m\sin\theta_i + \sqrt{n^2-\cos^2\theta_i}}\right]^2 = 0$$

$$m\sin\theta_i - \sqrt{n^2-\cos^2\theta_i} = 0$$

$$m^2\sin^2\theta_i - n^2 + \cos^2\theta_i = m^2\left(1-\cos^2\theta_i\right) - n^2 + \cos^2\theta_i = 0$$

$$m^2 - m^2\cos^2\theta_i - n^2 + \cos^2\theta_i = m^2 - n^2 + \left(1-m^2\right)\cos^2\theta_i = 0$$

$$m^2 - n^2 = \left(m^2 - 1\right)\cos^2\theta_i$$

$$\cos^2\theta_i = \frac{m^2-n^2}{m^2-1}$$