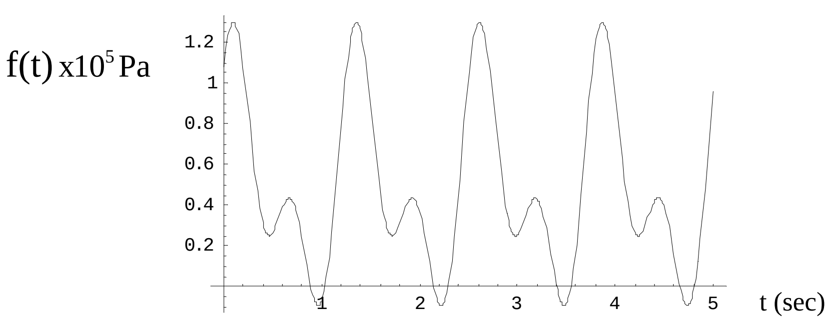
Fourier Analysis

In our Mathematics classes, we have been taught that complicated functions can often be represented as a long series of terms whose sum closely approximates the actual function. Taylor series is one very powerful application of this idea. In the case of Taylor series, the function is approximated by a constant value of the function at a particular point added to successive derivatives evaluated at that same point and multiplied by specific constants or coefficients. Another type of series is the Fourier series. Here specific constants are multiplied by sine and cosine terms to generate the series that approximates the function.

As an example, consider the following series of five terms that represent the oscillating pressure sensed by a hypothetical detector as a sound passes by:

$$f(t) = \left\{ \frac{1}{2} + \frac{1}{3}\cos[5\,t] + \frac{1}{4}\cos[10\,t] + \frac{1}{3}\sin[5\,t] + \frac{1}{4}\sin[10\,t] \right\} x10^5\,\text{Pa}$$

Notice some things about this series. The first term is a constant, sometimes called the "DC" term using an analogy to electrical voltages and currents. The second and third are cosine terms. The angular frequency of the second term is 5 rad/sec and the amplitude is $1/3\,x10^5\,\text{Pa}$. The third term has twice the angular frequency so it oscillates twice as fast, but has and amplitude of only $1/4\,x10^5\,\text{Pa}$. The fourth and fifth terms have the same frequency and amplitude as the second and third but are shifted in phase by 90 degrees relative to the cosines. When plotted for 5 sec, this series looks like this:



Most often in experimental acoustics, we have a detector to receive a signal like this one and it is our purpose to work backwards and determine the frequencies and the amplitudes of the tones (terms in the series) that make up the periodic signal. The method of finding these tones is called "Fourier Analysis." Finding the frequencies is simply a matter of determining the overall period of the repeating signal. The fundamental frequency, or frequency of the first sine or cosine term in the series (in Hertz), is simply the reciprocal of that frequency. Higher frequency terms are just multiples or harmonics of the fundamental frequency. Generally the frequency is given in rad/sec instead of Hz.

Finding the coefficients or amplitude of each term occurs using a very clever bit of mathematics discovered by Fourier. This method is sometimes called "Fourier's Hammer" because it is used to hammer out each of the coefficients (amplitudes) in the series. We'll study this method in some detail below.

In fact, many sounds are combinations of discrete frequency components that we hear as one sound. In class, we will use spectrum analyzers and digital oscilloscopes which use digital signal processing algorithms to find the magnitude (proportional to the Fourier Series Coefficient) and frequency of each component of a signal.

Calculating Coefficients

Starting with a ***periodic function*** (such as a sound wave), we can breakdown this function into separate frequency components by using Fourier Analysis. Note that we must KNOW the period of the wave and BE ABLE TO DEFINE the function, f(t), over that period to be able to use Fourier Analysis. Often the function will be zero, a constant, or a straight line with constant slope. Whatever it is, we must be able to write a math expression (or a good approximation) for the function over the entire period.

First let us be very specific about the frequency in rad/sec. Once we have identified the period over witch the function repeats, the angular frequency is:

$$\omega = \frac{2\pi}{T}$$

In the example plot of the periodic function above, the period is approximately 1.25 sec by inspection of the time scale. This is consistent with the equation we plotted since

$$\omega = \frac{2\pi}{T} = \frac{2\pi\text{rad}}{1.25\,\text{sec}} \approx 5\text{rad}/\text{sec}.$$

Other terms in the Fourier series will have frequencies that are multiples of 5 rad/sec, e.g. 10 rad/sec, 15 rad/sec, 20 rad/sec,…..

Calculating the amplitudes is somewhat more complicated. First consider the equation we plotted above (where I have dropped the units and constant $10^{5)}$:

$$f(t) = \frac{1}{2} + \frac{1}{3}\cos[5\,t] + \frac{1}{4}\cos[10\,t] + \frac{1}{3}\sin[5\,t] + \frac{1}{4}\sin[10\,t]$$

Even though we know the amplitude of the first cosine term is 1/3, let's try to develop a method to unmask it. First, multiply each term by cos(5t).

$$f(t)\cos(5t) = \frac{1}{2}\cos(5t) + \frac{1}{3}\cos[5\,t]\cos(5t) + \frac{1}{4}\cos[10\,t]\cos(5t) + \frac{1}{3}\sin[5\,t]\cos(5t) + \frac{1}{4}\sin[10\,t]\cos(5t)$$

Next, we find the time average of each term in the series using the normal definition for the time average of a function. This is a reasonable approach because we are looking for a representative value for the amplitude averaged over at least one cycle, not an instantaneous value.

$$\langle f(t)\rangle = \frac{1}{T}\int_0^T f(t)dt$$

The result looks complicated and long but will quickly simplify.

$$\frac{1}{T}\int_0^T f(t)\cos(5t)dt = \frac{1}{T}\int_0^T \frac{1}{2}\cos(5t)\,dt + \frac{1}{T}\int_0^T \frac{1}{3}\cos[5\,t]\cos(5t)\,dt + \frac{1}{T}\int_0^T \frac{1}{4}\cos[10\,t]\cos(5t)\,dt$$

$$+\frac{1}{T}\int_0^T \frac{1}{3}\sin[5\,t]\cos(5t)dt + \frac{1}{T}\frac{1}{4}\int_0^T \sin[10\,t]\cos(5t)dt$$

A quick inspection of the left side of the equal sign reveals that most of the terms integrate to zero. In fact all but one term are zero since,

$$\int_0^T \sin n\omega t \cos m\omega t dt = \int_0^T \sin n\omega t \sin m\omega t dt = \int_0^T \cos n\omega t \cos m\omega t dt = 0$$

unless m=n. In that case, (sine would be identical)

$$\frac{1}{T}\int_0^T \cos n\omega t \cos m\omega t dt = \frac{1}{T}\int_0^T \cos^2 n\omega t dt = \frac{1}{2}$$

This leaves us with the following:

$$\frac{1}{T}\int_0^T f(t)\cos(5t)dt = 0 + \frac{1}{T}\int_0^T \frac{1}{3}\cos^2[5\,t]dt + 0 + 0 + 0 = \frac{1}{3}\left(\frac{1}{2}\right)$$

Rearranging slightly shows that the coefficient we were trying to find, i.e. the 1/3, must be calculated as follows:

$$\frac{1}{3} = \frac{2}{T}\int_0^T f(t)\cos(5t)dt = a_1$$

The name we will give to this coefficient is a_1. We arbitrarily decide to call all the coefficients for cosine terms "a" and for sine terms "b." The subscript tells us which harmonic of the fundamental frequency the coefficient is associated with. In this case, n=1 is the fundamental term.

Hopefully you see that this approach can be used to find **any** coefficient (any value of a_n or b_n). All we have to do is multiply the series by either cosnωt or sinnωt and time average the result. Since most of the terms average to zero, the result can be summarized in the following set of rules. In truth, finding Fourier coefficients can be a very mechanical procedure that you can perform simply by learning these rules.

Let us start with any time varying signal, f(t). If f(t) is periodic over the interval $0 \le t \le T$, it can be broken down into a series of frequency components (coefficients) where:

$$\omega = \frac{2\pi}{T}$$

the coefficients are calculated by:

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt \quad \text{for } n = 0,\ 1,\ 2,\ 3,\ ....$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt \quad \text{for } n = 1,\ 2,\ 3,\ ....$$

Note that n goes from 0 to ∞ for a_n but n goes from 1 to ∞ for b_n. That is because there is no b_0 term. The sin of ($n\omega t$) where n=0 is always 0, thus b_0 is always 0.

The coefficents a_0, a_1, a_2, …, and b_1, b_2, b_3, … are the Fourier coefficients of the function, f(t). Now the original function f(t), can be described as the summation of many different sine and cosine functions.

$$f(t) = \frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t) + a_3\cos(3\omega t) + \cdots$$
$$+ b_1\sin(\omega t) + b_2\sin(2\omega t) + b_3\sin(3\omega t) + \cdots$$

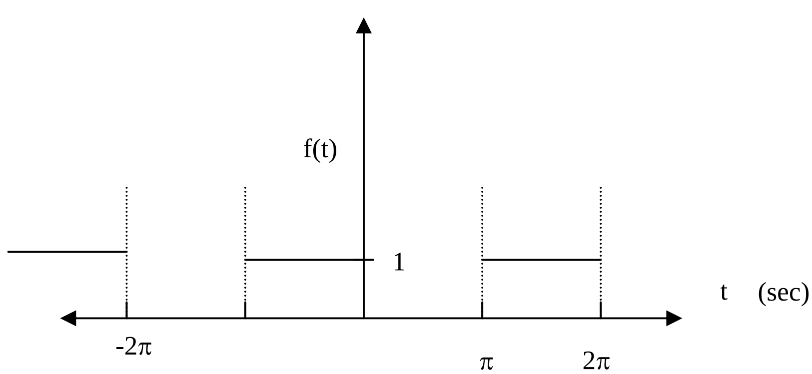
or,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right]$$

Example

Given the periodic function : $f(t) = \begin{cases} 0 & \text{when } 0 < t < \pi \\ 1 & \text{when } \pi < t < 2\pi \end{cases}$

which repeats every 2π seconds. A sketch of the function would look like:



The function can be expanded into a series of sine and cosine terms that when added together, replicate the original function. It is our job to find the coefficients of those terms.

First we must identify the period of the repeating function. Hopefully it is obvious that T = 2π seconds. From this we find the angular frequency, ω.

$$\omega = \frac{2\pi}{T} = \frac{2\pi \text{rad}}{2\pi \sec} = 1 \text{ rad}/\sec.$$

This is a convenient result since the angular frequency of harmonic terms is just nω = n rad/sec.

The coefficients are then found as follows. Notice that we break the integral up into 2 pieces where the function has two different constant values, zero and one.

$$a_n = \frac{2}{T}\int_0^{2\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi}\int_0^{\pi} 0*\cos(nt)\,dt + \frac{1}{\pi}\int_{\pi}^{2\pi} 1*\cos(nt)\,dt$$

$$a_n = \frac{1}{\pi}\int_{\pi}^{2\pi} \cos(nt)\,dt = \frac{1}{n\pi}\sin(nx)\Big|_{\pi}^{2\pi}$$

$$a_n = \frac{1}{n\pi}\left[\sin(n*2\pi) - \sin(n*\pi)\right] = 0$$

and

$$b_n = \frac{2}{T}\int_0^{2\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_0^{\pi} 0*\sin(nt)\,dt + \frac{1}{\pi}\int_{\pi}^{2\pi} 1\sin(nt)\,dt$$

$$b_n = \frac{1}{\pi}\int_{\pi}^{2\pi} \sin(nt)\,dt = -\frac{1}{n\pi}\cos(nx)\Big|_{\pi}^{2\pi}$$

$$b_n = -\frac{1}{n\pi}\left[\cos(n*2\pi) - \cos(n*\pi)\right]$$

$$b_n = -\frac{1}{n\pi}\left(1 - \cos(n\pi)\right) \quad \text{for n = odd numbers otherwise } b_n = 0$$

$$b_n = -\frac{2}{n\pi} \quad \text{for n = odd numbers}$$

and

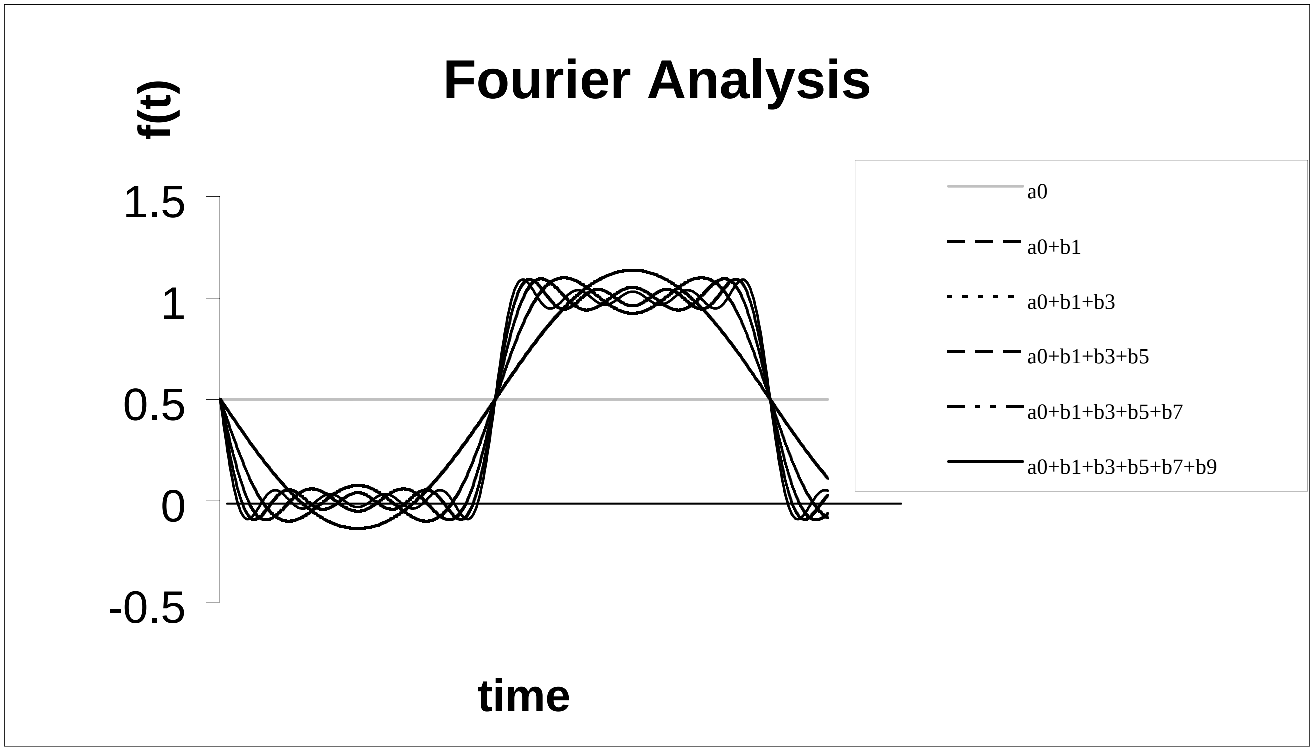
$$a_0 = \frac{2}{T}\int_0^{2\pi} f(t)\cos(0*t)\,dt = \frac{1}{\pi}\int_0^{\pi} 0*dt + \frac{1}{\pi}\int_{\pi}^{2\pi} 1*dt$$

$$a_0 = \frac{1}{\pi}\int_{\pi}^{2\pi} dt = 1$$

thus, the original function can be expanded to:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right] = \frac{1}{2} - \frac{2}{\pi}\left[\frac{\sin(1t)}{1} + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right]$$

If we added up all the terms of the Fourier Expansion, a graphical representation would look like this:



The important thing to note is that the original square wave function can be composed from adding components of multiple sine and cosine functions with frequencies that are multiples of the base frequency. The base frequency of the components is the same as the base frequency of the square wave.

Odd or Even Functions

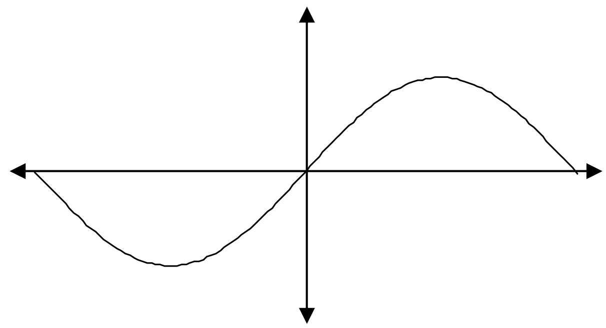
By looking at the form of the input signal, f(t), we can come up with some shortcut rules for deriving the coefficients. If we can determine if the f(t) is an odd or even function, we can determine whether the a or b coefficients are equal to zero as in the last example. A function is odd or even based on the following:

$$\text{Even Function}: \; f(-t) = f(t)$$
$$\text{Odd Function}: \; f(-t) = -f(t)$$

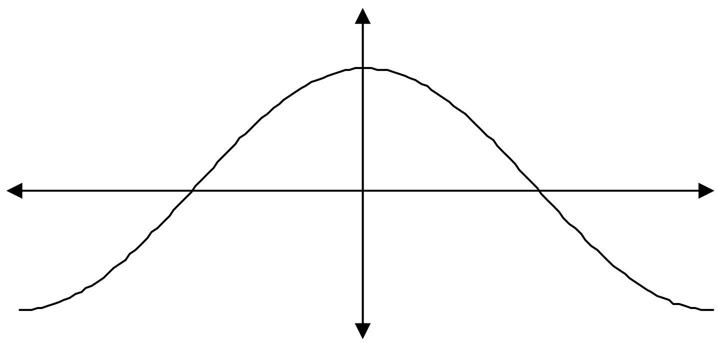
Even functions are thus functions that are symmetric about the y-axis. Odd functions are functions that are symmetric about the x-axis AND are mirror images of each other (symmetric about the origin). Many functions are neither odd nor even, but understanding this characteristic function type lets us anticipate which Fourier coefficients might be zero.

Some samples of even and odd functions.

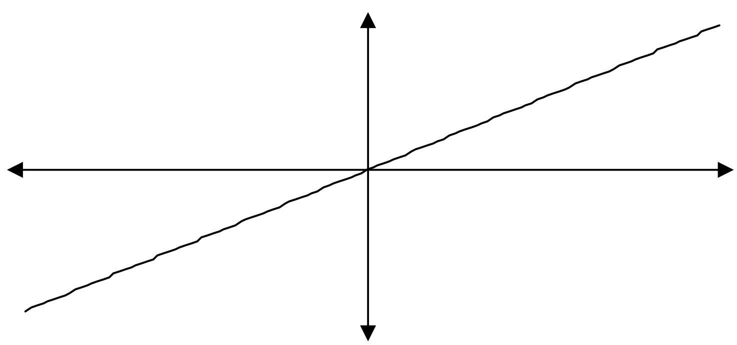
An odd function $f(t) = \sin(\omega t)$



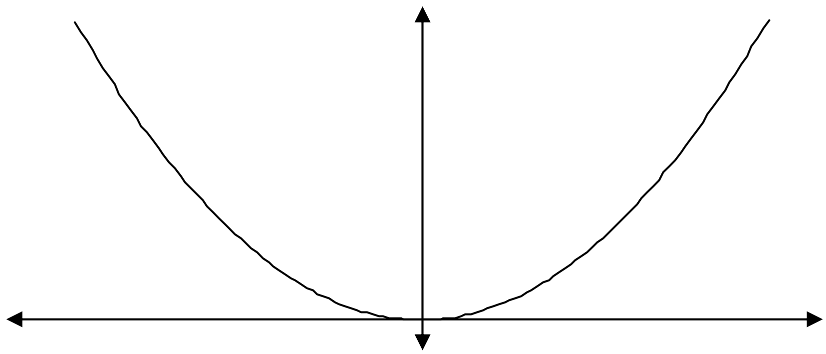
An even function $f(t) = \cos(\omega t)$



An odd function $f(t) = t$



An even function $f(t) = t^2$



Since cosines are even, other even functions are made up only of cosines. On the other hand, odd functions are made up only of sines. Thus the coefficients for the different type functions are:

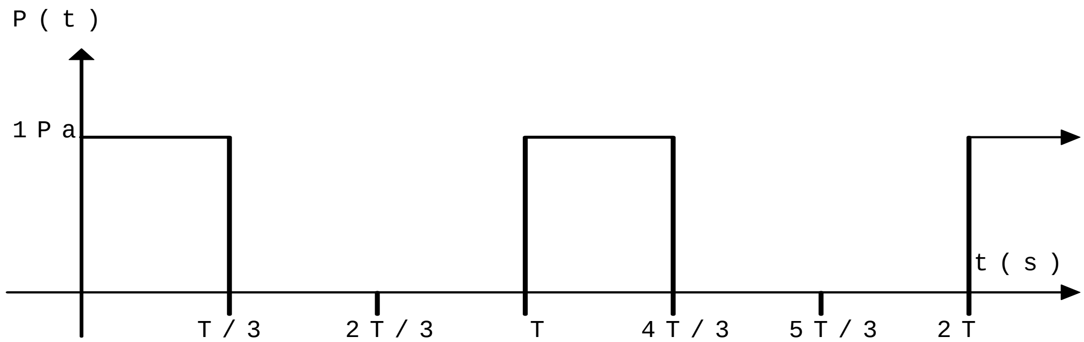
$$\text{If } f(x) \text{ is odd then } \begin{cases} a_n = 0 \\ b_n = \dfrac{2}{T}\displaystyle\int_0^T f(t)\sin(n\omega t)\,dt \quad \text{for } n = 1, 2, 3, \ldots. \end{cases}$$

$$\text{If } f(t) \text{ is even then } \begin{cases} a_n = \dfrac{2}{T}\displaystyle\int_0^T f(t)\cos(n\omega t)\,dt \quad \text{for } n = 0, 1, 2, 3, \ldots. \\ b_n = 0 \end{cases}$$

Remember, some functions are neither even or odd in which case you must simply calculate all the Fourier coefficients and see what results are obtained.

Problems

1. Given the following pressure function, ***p(t),*** which can be described as a square wave of ***1 Pa*** for ***T/3 sec***, and ***0 Pa*** for ***2T/3 sec*** shown below where ***T = 1 sec*** :



$$p(t) = \begin{cases} 1\,Pa, & 0 < t < \dfrac{T}{3} \\ 0\,Pa, & \dfrac{T}{3} < t < T \end{cases}$$

a) Is this function odd, even, both, or neither? How do you know?

b) What is the base or fundamental frequency of the square wave?

c) Perform the integrations to calculate the coefficient, "a_o".

d) Perform the integrations to calculate the coefficient, "a_n" coefficients.

e) Perform the integrations to determine the "b_n" coefficients.

f) Fill out the following table for $0 \le n \le 9$:

n	a_n (Pa)	b_n (Pa)	T_n = T / n (sec)	f_n = 1 / Tn (Hz)
0		N/A	N/A	N/A
1				
2				
3				
4				
5				
6				
7				
8				
9				

g) What is the pattern here? List the frequencies of the first nine non-zero harmonics of the fundamental that go make up the first nine terms of the Fourier Expansion.

Fourier Series – Periodic Functions

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right] \text{ or,}$$

$$f(t) = \frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t) + \cdots$$
$$+ b_1 \sin(\omega t) + b_2 \sin 2(\omega t) + \cdots$$

for a function $f(t)$ where:

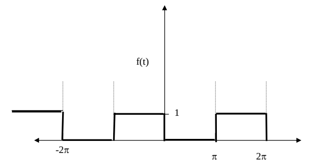
$\omega = \frac{2\pi}{T}$ the coefficients are calculated by:

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt \text{ for n = 0, 1, 2, 3, ...}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt \text{ for n = 1, 2, 3, ...}$$

Example

$$f(t) = \begin{cases} 0 & \text{when } 0 < t < \pi \\ 1 & \text{when } \pi < t < 2\pi \end{cases}$$



Note: $T = 2\pi \text{ sec} \Rightarrow \omega = \frac{2\pi}{T} = 1\frac{\text{rad}}{\text{sec}}$

Coefficients

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)dt = \frac{1}{\pi}\int_\pi^{2\pi} 1\cos(nt)dt + \frac{1}{\pi}\int_0^{\pi} 0 * \cos(nt)dt$$

$$a_n = \frac{1}{\pi}\int_\pi^{2\pi}\cos(nt)dt = \frac{1}{n\pi}\sin(nx)\Big|_\pi^{2\pi}$$

$$a_n = \frac{1}{n\pi}\left[\sin(n*2\pi) - \sin(n*\pi)\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)dt = \frac{1}{\pi}\int_\pi^{2\pi} 1\sin(nt)dt + \frac{1}{\pi}\int_0^{\pi} 0 * \sin(nt)dt$$

$$b_n = \frac{1}{\pi}\int_\pi^{2\pi}\sin(nt)dt = -\frac{1}{n\pi}\cos(nx)\Big|_\pi^{2\pi}$$

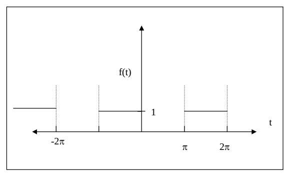
$$b_n = -\frac{1}{n\pi}\left[\cos(n*2\pi) - \cos(n*\pi)\right]$$

$$b_n = -\frac{1}{n\pi}(1 - \cos(n\pi)) \text{ for n = odd numbers otherwise } b_n = 0$$

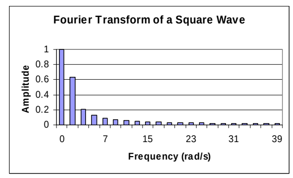
$$b_n = -\frac{2}{n\pi} \text{ for n = odd numbers}$$

Example

Time Domain



Frequency Domain

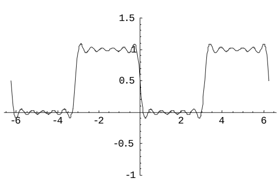


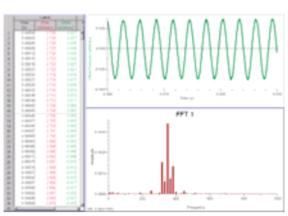
$$f(t) = \begin{cases} 0 & \text{when } 0 < t < \pi \\ 1 & \text{when } \pi < t < 2\pi \end{cases}$$

$$f(t) = \frac{1}{2} - \frac{2}{\pi}\left[\frac{\sin(1t)}{1} + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right]$$

Demos

- Mathematica
- Logger Pro

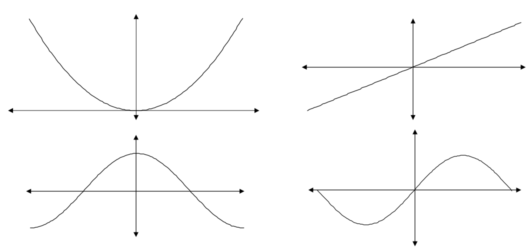




Odd and Even Functions

Even Function : $f(-t) = f(t)$

Odd Function : $f(-t) = -f(t)$



Even

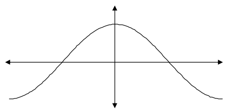
Odd

Odd and Even Functions

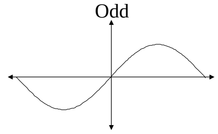
Even Function : $f(-t) = f(t)$

Odd Function : $f(-t) = -f(t)$

Even



Odd



If $f(t)$ is even then $\begin{cases} a_n = \dfrac{2}{T}\displaystyle\int_0^T f(t)\cos(n\omega t)\,dt \quad \text{for } n = 0,1,2,3,.... \\ b_n = 0 \end{cases}$

If $f(t)$ is odd then $\begin{cases} a_n = 0 \\ b_n = \dfrac{2}{T}\displaystyle\int_0^T f(t)\sin(n\omega t)\,dt \quad \text{for } n = 1,2,3,.... \end{cases}$