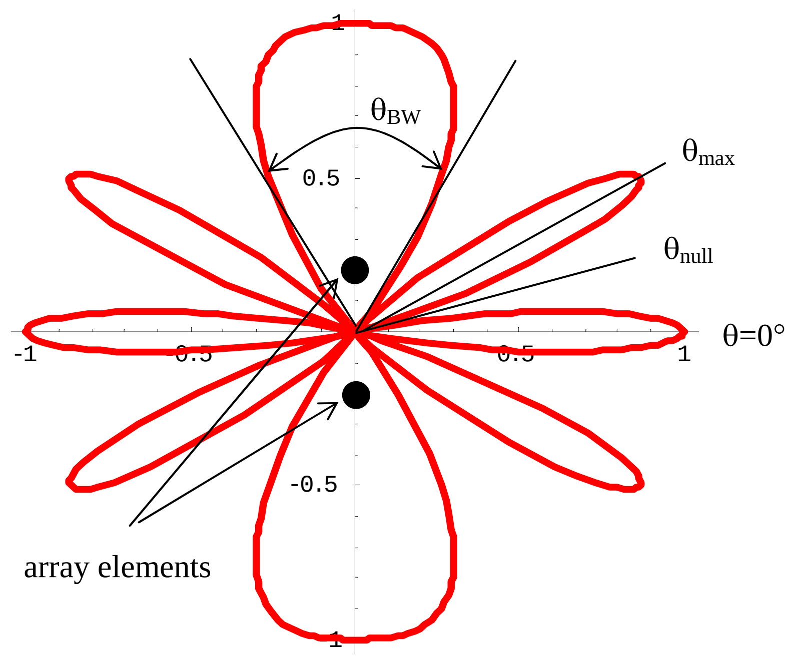
Underwater Acoustics and Sonar
SP411 Handouts and Notes
Fall 2006

Beam Pattern Function
($\lambda/d = 0.5$)



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Acknowledgement

These notes for Underwater Acoustics and Sonar (SP411) are provided to you not because of an overwhelming concern for your book buying budget, but do to the reality that the market simply does not have a major demand for an undergraduate textbook in underwater acoustics and sonar. As such, no satisfactory textbook is available and we have to write our own material.

These notes have been passed down for many years. Naval Academy legend says the first set was written by Emeritus Professor Jack Smithson. I first saw them in the late eighties while teaching Physics as a young submarine Lieutenant. Fellow Lieutenant, Mike Smith, ’79, gave me a copy and tried to persuade me to teach the course for him.

Since then, much material that was previously considered classified and therefore not included in the course has been added. I suppose in a perverse way, the infamous traitor Navy Warrant Officer John Walker takes the credit for this. For personal gain, he sold the secrets of what we could do with narrow band sonar to the Soviet Union. May he long rot in jail. I’m surprised how few Midshipmen know of his treachery. At the very least you can hate him because he has made this course more challenging than it used to be. Then learn his story and hate him for the lives he put at risk with his unconscionable acts.

I need to credit LCDR Brent Flaskerud for the first modern version of these notes, transcribed into neat MS Word files. Brent did a yeomen’s service to bring the course into the new millennium. LCDR Tom Callender added and revised the material as well.

Professor Murray Korman actually had a preliminary version of these notes published by the Naval Institute Press in 1995. He wrote most all of the homework problems. He is the master of the SP411 course and has shepherded it for several decades. He is the designer of most of the hands-on classroom demonstrations you will see this semester. I am grateful for all he does to keep this course interesting and vibrant.

List of Effective Pages

SP 411 Equations

Section 1

$$x = A\cos(\omega t + \phi); \omega = \sqrt{\frac{k}{m}}$$

$$v = \frac{dx}{dt}; a = \frac{dv}{dt}; v_{max} = \omega A; a_{max} = \omega^2 A$$

$$E = K + U = \frac{1}{2}m_{block}v^2 + \frac{1}{2}kx^2; E_{max} = \frac{1}{2}kA^2$$

$$\langle f(t) \rangle \equiv \frac{1}{T}\int_0^T f(t)\,dt$$

$$x = Ae^{-\alpha t}\cos(\omega' t + \phi)$$

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}; \alpha = \frac{b}{2m}$$

Section 2

$$s(x,t) = s_0 \sin(kx \pm \omega t)$$

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c}; \omega = \frac{2\pi}{T} = 2\pi f$$

$$c = \sqrt{\frac{B}{\rho}} = \lambda f = \frac{\omega}{k}$$

$$p_a = -B\frac{\partial s}{\partial x}; p_{a\ max} = Bks_0$$

$$u = \frac{\partial s}{\partial t}; u_{max} = \omega s_0$$

$$I(x,t) = \frac{p_a^2(x,t)}{\rho c}$$

nominal values: $\rho = 1000\ \text{kg/m}^3; c = 1500\ \text{m/s}$

$$I = p_a u; \langle I \rangle = \frac{\langle p_a^2 \rangle}{\rho c} = \frac{p_{rms}^2}{\rho c} = \frac{p_{a\ max}^2}{2\rho c}; p_{rms} = \frac{p_{max}}{\sqrt{2}}$$

$$\text{Power} = \langle I \rangle A$$

$$z_{\substack{plane\\wave}} = \rho c; p_a = zu$$

Section 3

$10^x = y$ then $\log_{10}(y) = x$

$$\log(xy) = \log(x) + \log(y)$$

$$\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$$

$$\log(x^n) = n\log(x)$$

$$L = 10\log\frac{I}{I_{ref}} = 20\log\frac{p_{rms}}{p_{ref}}$$

standard value (seawater): $p_{ref} = 1\mu\text{Pa}$

$$L_1 \oplus L_2 = 10\log\left(10^{L_1/10} + 10^{L_2/10}\right)$$

if $L_1 = L_2$ then $L_1 \oplus L_2 = L_1 + 3$ dB

if $L_1 >> L_2$ then $L_1 \oplus L_2 = L_1$

Section 4

$$c(T,D,S) = 1449.2 + 4.6T - 5.5x10^{-2}T^2$$
$$+2.9x10^{-4}T^3 + \left(1.34 - 10^{-2}T\right)(S - 35)$$
$$+1.6x10^{-2}D$$

$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta_2}{c_2}$$

$$g = \frac{\Delta c}{\Delta z}$$

$$c = c_0 + g\Delta z$$

$$R = \frac{c_n}{g\cos\theta_n}$$

$$\Delta z = R(\cos\theta_2 - \cos\theta_1)$$

$$\Delta x = -R(\sin\theta_2 - \sin\theta_1)$$

Section 6

$$I_i = I_r + I_t$$

$$I_t = TI_i \qquad I_r = RI_i$$

$$1 = T + R$$

$$n = \frac{c_1}{c_2} \qquad m = \frac{\rho_2}{\rho_1}$$

$$R(\theta_i) = \left[\frac{m\sin\theta_i - \sqrt{n^2 - \cos^2\theta_i}}{m\sin\theta_i + \sqrt{n^2 - \cos^2\theta_i}}\right]^2$$

$$R(90°) = \left(\frac{m-n}{m+n}\right)^2$$

$$\theta_c = \cos^{-1}\left(\frac{c_1}{c_2}\right)$$

$$\theta_b = \cos^{-1}\left[\sqrt{\frac{(m^2 - n^2)}{(m^2 - 1)}}\right]$$

Section 7

$$\omega = \frac{2\pi}{T}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt \quad \text{for } n = 0,1,2,3,\ldots$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin s(n\omega t)\,dt \quad \text{for } n = 1,2,3,\ldots$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right]$$

Section 8

$$ISL = 10\log\frac{I(\text{in a 1 Hz band})}{I_{ref}(\text{in a 1 Hz band})}$$

$$BL = ISL_{ave} + 10\log(\Delta f)$$

$$f_c = \sqrt{f_1 f_2}$$

$$f_2 = 2^1 f_1 \quad \text{one-octave band}$$

$$f_2 = 2^{1/2} f_1 \quad \text{half-octave band}$$

$$f_2 = 2^{1/3} f_1 \quad \text{third-octave band}$$

$$BL = \left[ISL_{ave} + 10\log(\Delta f)\right]\{\oplus SPL_{tonals}\}$$

Section 10

$$P = IA$$

$$TL \equiv 10\log\frac{I(1\text{ yd})}{I(r)}$$

$$TL = 20\log r \qquad \text{spherical only}$$

$$TL = 20\log r_0 + 10\log\frac{r}{r_0} \qquad \text{with cylindrical}$$

$$TL = 20\log r_0 + 10\log\frac{r}{r_0} + \alpha\left(r \times 10^{-3}\right)\ \text{dB}$$

with attenuation where:

$$\alpha = \left(\begin{array}{l} 0.003 + \dfrac{0.1f^2}{1+f^2} + \dfrac{40f^2}{4100+f^2} \\ +2.75\times10^{-4}f^2 \end{array}\right) \ \text{dB}/\text{kyd}$$

f is in kHz

Section 11

$$NL_{ambient} = NL_{ship} \oplus NL_{SS}$$

Section 12

$$NL_{self} = NL_{BB} \oplus NL_{tonal}\left(\oplus NL_{tonal2} \oplus NL_{tonal3} \oplus \ldots\right)$$

$$NL_{tot} = NL_{ambient} \oplus NL_{self}$$

Section 13

$$\theta_{max} \Rightarrow b(\theta) = 1$$

$$\theta_{null} \Rightarrow b(\theta) = 0$$

$$\theta_{BW} \Rightarrow b(\theta) \geq 0.5$$

Section 15

$$1 = p(D) + p(miss)$$

$$1 = p(FA) + p(null)$$

$$d = \frac{(\mu_{s+n} - \mu_n)^2}{\frac{1}{2}\left(\sigma_{s+n}^2 + \sigma_n^2\right)}$$

passive

$$DT = 5\log\left(\frac{d}{T\Delta f}\right)$$

active

$$DT = 10\log\left(\frac{d}{2T\Delta f}\right)$$

Section 9

$$L_{S/N} = (SL - TL) - (NL - DI) > ?\,DT$$

$$L_S = SL - TL$$

$$L_N = (NL - DI)$$

$$SE = L_{S/N} - DT$$

$FOM = TL_{max}$ (i.e. SE=0 so $L_{S/N}$ =DT)

Section 16

$$L_{S/N} = SL - 2TL + TS - (NL - DI) > ?\,DT$$

$$L_{S/N} = SL - 2TL + TS - RL > ?\,DT$$

$$SL = 171.5 + 10\log P_E + 10\log E + DI_T$$

Section 17

$$RL_V = SL - 40\log r + S_V + 10\log V$$

$$TS - (S_V + 10\log V) > ?\,DT$$

where:

$S_V = 10\log s_V \equiv$ Volume Scattering Strength

$V = \frac{c\tau}{2}\Psi r^2 \equiv$ reverberation volume

$\Psi = \int b(\theta,\phi)\,b'(\theta,\phi)\,d\Omega \equiv$ solid angle

based on type of arrays (see chart)

$r \equiv$ range to the target

$\tau \equiv$ pulse length

$$RL_S = SL - 40\log r + S_s + 10\log A$$

$$TS - (S_s + 10\log A) > ?\,DT$$

$S_S = 10\log s_S \equiv$ Surface Scattering Strength

$A = \frac{c\tau}{2}\Phi r \equiv$ reverberation area

$\Phi = \int b(0,\phi)\,b'(0,\phi)\,d\Phi \equiv$ equivalent beamwidth

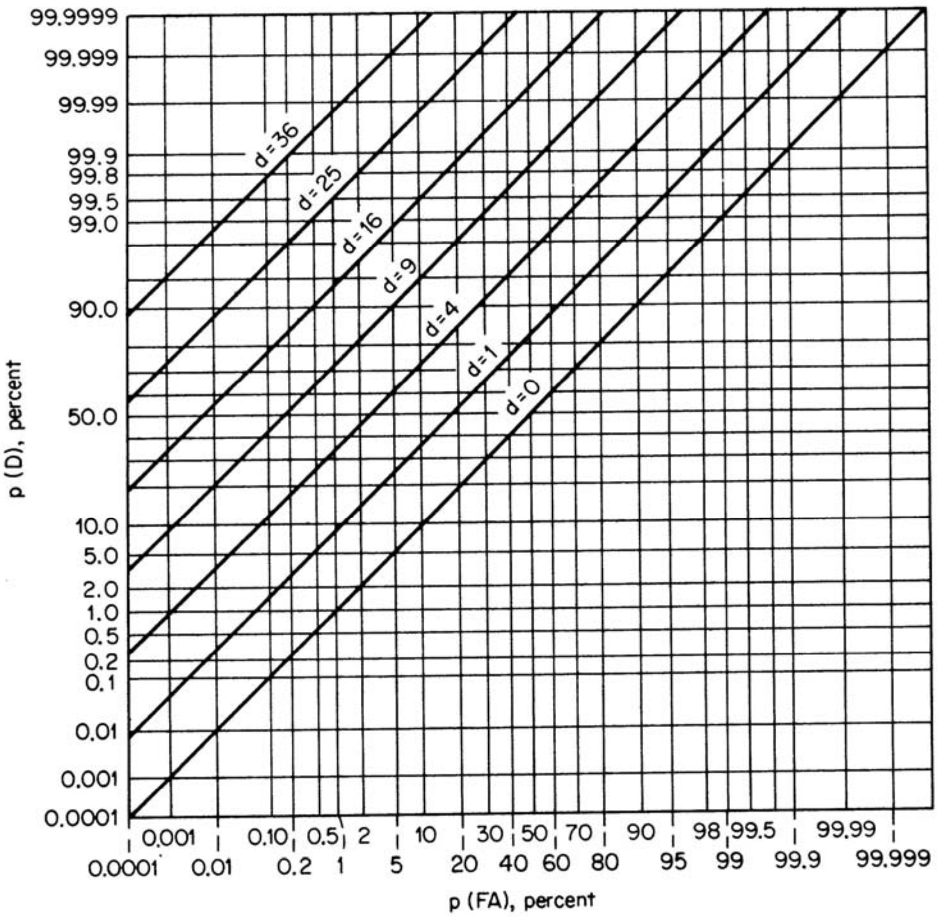
based on type of arrays (see chart)

Section 20

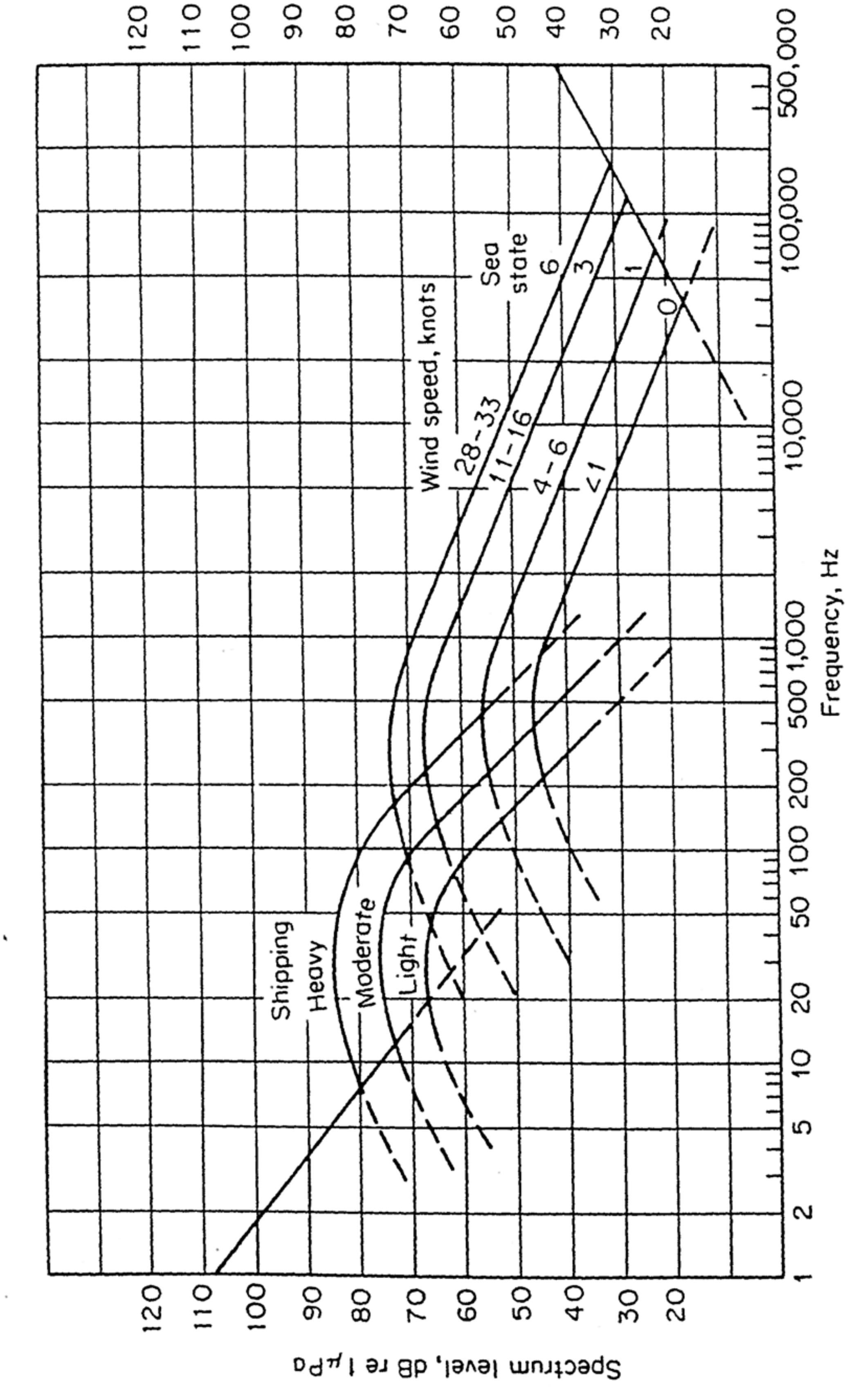
$f' = f_0\left(\frac{c \pm v_r}{c \mp v_s}\right)$ passive

$f'' = f_0\left(\frac{c \pm v_t}{c \mp v_s}\right)\left(\frac{c \pm v_s}{c \mp v_t}\right)$ active

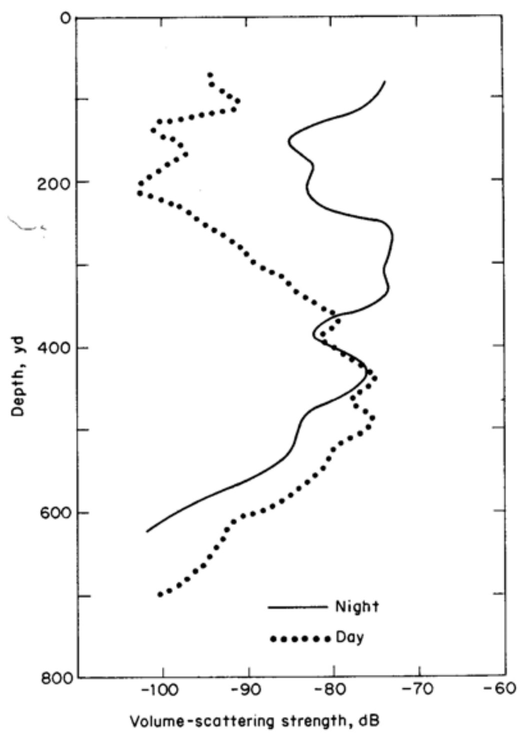
	2-element array	continuous line array	circular piston
defining parameters	element separation distance – d	array length – L	array diameter - D
beam pattern function b(θ) =	$\cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right)$	$\left(\frac{\sin\left[\frac{\pi L}{\lambda}\sin\theta\right]}{\frac{\pi L}{\lambda}\sin\theta}\right)^2$	$\left[\frac{2J_1\left(\frac{\pi D}{\lambda}\sin\theta\right)}{\frac{\pi D}{\lambda}\sin\theta}\right]^2$
directivity index DI	$10\log\left[\frac{2}{1+\left(\frac{\sin\left(2\pi d/\lambda\right)}{2\pi d/\lambda}\right)}\right]$	$10\log\frac{2L}{\lambda}$ for L >> λ	$10\log\left(\frac{\pi D}{\lambda}\right)^2$ for D >> λ
null angles b(θ) = 0 θ_{null}	$\sin\theta = (m)\frac{\lambda}{2d}$ $m = 1, 3, 5, \ldots$	$\sin\theta = (m)\frac{\lambda}{L}$ $m = 1, 2, 3, \ldots$	$\sin\theta = (z)\frac{\lambda}{D}$ $z = 1.22,\ 2.23,\ 3.24, 4.24, \ldots$ roots of $J_1\left(\frac{\pi D}{\lambda}\sin\theta\right) = 0$
side lobes b(θ)=1 θ_{max}	$\sin\theta = m\frac{\lambda}{d}$ $m = 0, 1, 2, 3\ldots.$	$\tan\left(\frac{\pi L\sin\theta}{\lambda}\right) = \left(\frac{\pi L\sin\theta}{\lambda}\right)$ $\sin\theta = y\left(\frac{\lambda}{L}\right)$ where $y = 1.43, 2.46, 3.47, 4.48,$	$\sin\theta = w\frac{\lambda}{D}$ where $w = 1.64, 2.68, 3.70, \ldots$
half power angles b(θ)=0.5 θ_{hp} $\theta_{BW}=2\theta_{hp}$ (only for beam about array axis)	$\sin\theta_{hp} = \frac{n\lambda}{4d}$ $n = 1, 3, 5, 7, \ldots$	$\sin\theta_{hp} = 0.442\frac{\lambda}{L}$	$\sin\theta_{hp} = 0.51\frac{\lambda}{D}$

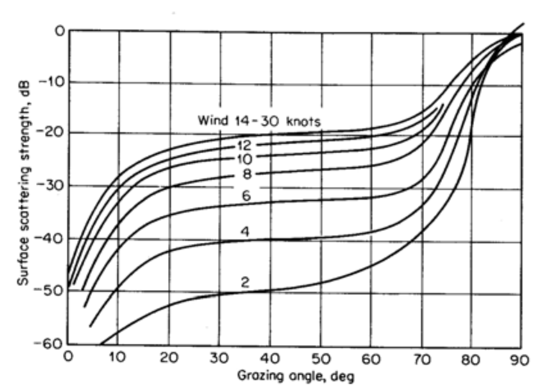


From Urick, R. J. Principles of Underwater Sound, 3rd ed. McGraw-Hill Book Company. 1983. p 383


Spectrum level, dB re 1 μPa
120
110
100
90
80
70
60
50
40
30
20
Frequency, Hz
1
2
5
10
20
50
100
200
500
1,000
10,000
100,000
500,000
Shipping
Heavy
Moderate
Light
Wind speed, knots
28–33
11–16
4–6
<1
Sea state
6
3
1
0

ARRAY TYPE	Ψ(steradians)	Φ(radians)
Circular Plane of Diameter D	$0.60\left(\frac{\lambda}{D}\right)^2$	$1.56\left(\frac{\lambda}{D}\right)$
Horizontal Line of Length L	$1.32\left(\frac{\lambda}{L}\right)$	$1.32\left(\frac{\lambda}{L}\right)$
Non-directional Point Array	4π	2π





Form	***t*** ***TS=10log(t)***	***Symbols***	***Direction of incidence***	***Conditions***
Any convex surface	$\frac{a_1 a_2}{4}$	a_1a_2 = principal radii of curvature r = range k = 2π/wavelength	Normal to surface	ka_1, ka_2 >>1 r>a
Large Sphere	$\frac{a^2}{4}$	a = radius of sphere	Any	ka>>1 r>a
Small Sphere	$61.7\frac{V^2}{\lambda^4}$	V = vol. of sphere λ = wavelength	Any	ka<<1 kr>>1
Infinitely long thick cylinder	$\frac{ar}{2}$	a = radius of cylinder	Normal to axis of cylinder	ka>>1 r > a
Infinitely long thin cylinder	$\frac{9\pi^4 a^4}{\lambda^2} r$	a = radius of cylinder	Normal to axis of cylinder	ka<<1
Finite cylinder	$\frac{aL^2}{2\lambda}$	L = length of cylinder a = radius of cylinder	Normal to axis of cylinder	ka>>1 $r > L^2/\lambda$
	$\frac{aL^2\left(\sin\beta/\beta\right)^2\cos^2\theta}{2\lambda}$	a = radius of cylinder β = kLsinθ	At angle θ with normal	

Infinite Plane surface	$\frac{r^2}{4}$		Normal to plane	
Rectangular Plate	$\left(\frac{ab}{\lambda}\right)^2\left(\frac{\sin\beta}{\beta}\right)^2\cos^2\theta$	a,b = sides of rectangle β = ka sinθ	At angle θ to normal in plane containing side a	r > a^2/λ kb >> 1 a > b
Ellipsoid	$\left(\frac{bc}{2a}\right)^2$	a, b, c = semi-major axis of ellipsoid	parallel to axis of a	ka, kb, kc >>1 r >> a, b, c
Circular Plate	$\left(\frac{\pi a^2}{\lambda}\right)\left(\frac{2J_1(\beta)}{\beta}\right)^2\cos^2\theta$	a = radius of plate β = 2kasinθ	At angle θ to normal	r > a^2/λ ka>>1
Circular Plate	$\left(\frac{4}{3\pi}\right)^2 k^4 a^6$	a = radius k = 2π//λ	Perpendicular to plate	ka<<1

Love's model for TS of single fish - $TS_{FISH} = 19.1\log L + 0.9\log f_k - 24.9$

$f_k -$ frequency in kHz

Oscillatory Motion

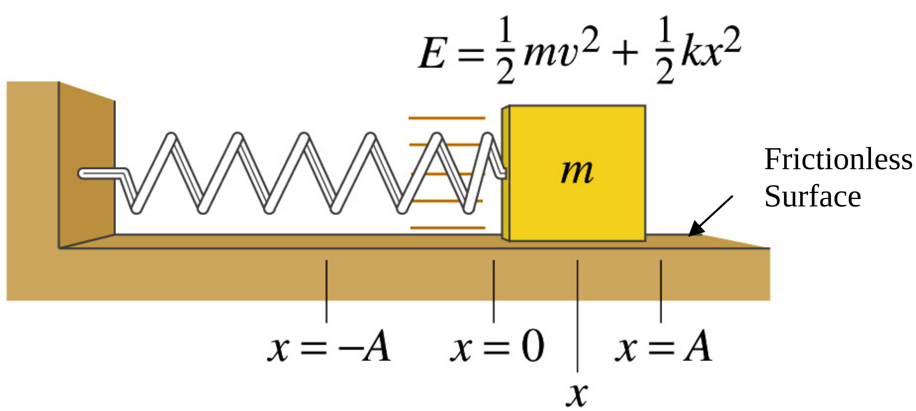
Our goal this semester is to understand how sound waves travel through the water so that we may exploit them to prosecute a target. We will start with simple models and increase complexity as we go. This course is meant to be directly applicable for the war-fighter.

To begin with lets go back to our childhood days looking out over the calm waters of the nearby pond. When you throw a rock in the water, you create a wave on the surface. If you closely watch a leaf on the surface, you will see it go up and down as the wave passes by, yet the leaf returns to its original position after the wave passes. This is a simple yet extremely important point regarding wave motion. The medium carrying the wave does not move with the wave, generally returning to its original position after the wave has gone past. The medium carrying the wave simply oscillates around an equilibrium position. To begin our study of underwater sound, we will look at the periodic nature of this motion. It is the basis of all mechanical wave motion.

Mass-Spring System

Hooke's Law and the Simple Harmonic Oscillator

An illustrative model to begin understanding acoustics is the problem of a simple mass-spring oscillating system. Begin with a mass attached to a perfect massless spring. The spring is attached to a firm wall and the mass sits on a frictionless surface. If the spring is displaced from the rest position of the system where x=0, the mass will move back and forth with a periodic motion centered about the x=0 position. This periodic motion can be described by a simple time varying equation, which should give us insight in to periodic wave motion.



From Hooke's Law, the restoring force of the spring is equal to:

$$F_{spring} = -kx$$

There is a minus sign in front of the spring constant because the force of the spring is in the opposite direction of the displacement of the mass. The displacement, x, is the distance the spring is stretched or compressed (and is equal to the displacement of the mass) from the x=0 or rest position of the spring.

We can now write an equation to relate the forces on the mass in the x-direction to the acceleration of the mass in the x-direction: (Or in other words apply Newton's second law for the motion only in the x-dimension.)

$$\sum \vec{F}_{block} = m_{block}\vec{a}$$

Since the only force on the block is due to the spring and all motion is along the x-axis, we can write the scalar equation,

$$F_{spring} = m_{block} a_x$$

$$-kx = m_{block}\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m_{block}}x = 0$$

This is a simple, second order differential equation that describes the motion of the mass. One solution for the position of the mass, x, as a function of time that satisfies the differential equation is:

$$x(t) = A\cos(\omega t + \phi)$$

where the angular frequency squared, $\omega^2 = \frac{k}{m_{block}}$ and A and ϕ, are unknown constants. Appendix A checks this solution and verifies the value of the angular frequency. We refer to quantity, $\omega t + \phi$, as the "phase" of the block's motion. The phase is generally expressed in radians and the motion repeats once the phase has changed by 2π. The amplitude of the oscillation, A and the initial phase of the oscillations ϕ, can **only** be solved for by knowing two initial conditions of the system.

Another solution to the second order differential equation is $x(t) = A\sin(\omega t + \phi)$. Another uses complex exponentials, $x(t) = Ae^{i(\omega t+\phi)}$ and is shorthand to signify only the real part of this expression is the solution to the second order differential equation. It is a worthwhile exercise for the student to show that both these solutions also satisfy the second order differential equation.

We must be able to find the velocity and acceleration of the mass as a function of time to use the initial conditions of the system. To calculate these quantities, we must just take the derivative as shown below.

$$v_x(t) = \frac{dx(t)}{dt} = A\frac{d\left[\cos(\omega t + \phi)\right]}{dt} = -\omega A \sin(\omega t + \phi)$$
$$a_x(t) = \frac{d^2x(t)}{dt^2} = -\omega^2 A\cos(\omega t + \phi) = -\omega^2 x$$

Looking at the above equations , we can obtain the maximum values of the velocity and acceleration. These maximum values are:

$$v_{max} = \omega A$$
$$a_{max} = \omega^2 A$$

An important characteristic of the system is the angular frequency. Using the above equations, and knowing a couple of the parameters of the system as a function of time, we can solve for the more easily understandable quantities, the frequency and period of the system. These can be calculated from the following equations:

$$f = \frac{\omega}{2\pi}$$
$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

Example Problem

Let's look at an example: A mass of 200 grams is connected to a light spring that has a spring constant (k) of 5.0 N/m and is free to oscillate on a horizontal, frictionless surface. If the mass is displaced 5.0 cm from the rest position and released from rest find: a) the period of its motion, b) the maximum speed and c) the maximum acceleration of the mass.

Using the relationships given above, the following can be calculated:

a) $\omega = \sqrt{k/m} == \sqrt{\dfrac{5.0\text{ N/m}}{200\text{x}10^{-3}\text{ kg}}} = 5.0\text{ rad/sec}$

$T = \dfrac{2\pi}{\omega} = 1.26\text{ seconds}$

b) Using the initial conditions that the mass was displaced 5.0 cm and let go from rest at $t = 0$ seconds

$x(0\text{ sec}) = 5.0\text{ cm} = A\,cos([(5.0\text{ rad/s})(0\text{ seconds})] + \phi)$

and $v(0\text{ sec}) = 0\text{ cm/s} = -(5.0\text{ rad/s})A\,sin([(5.0\text{ rad/s})(0\text{ sec})] + \phi)$

therefore soving for A and ϕ: $A = 5.0$ cm and $\phi = 0.0$ rad

$v_{max} = \omega A = (5.0\text{ rad/s})(5.0\text{x}10^{-2}\text{ m}) = 0.25\text{ m/s}$

c) $a_{max} = \omega^2 A = 1.25\text{ m/s}^2$

Energy in the Mass Spring System

The energy of the mass spring system can be found at any time by summing the kinetic energy of the mass with the potential energy of the spring.

$$E = K + U = \frac{1}{2} m_{block} v^2 + \frac{1}{2} k x^2$$

When the displacement of the mass from the equilibrium position is at the maximum displacement, x=A, the velocity of the spring is instantaneously zero. As there are no non-conservative forces such as friction, energy is conserved and the total energy at any time is simply

$$E_{max} = \frac{1}{2} k A^2$$

This is very powerful because it allows us to calculate the total energy of an oscillating mass very simply and then calculate the velocity when the position is known or vice versa. Conceptually, we view the continuous motion of a mass spring oscillator as the perpetual transfer of energy back and forth between kinetic and potential forms. Without any energy loss (due, for example, to friction) this transfer will continue indefinitely.

The average energy in a simple harmonic oscillator is calculated using the following definition for the average of a periodic function:

$$\left\langle f(t) \right\rangle \equiv \frac{1}{T} \int_0^T f(t)\, dt$$

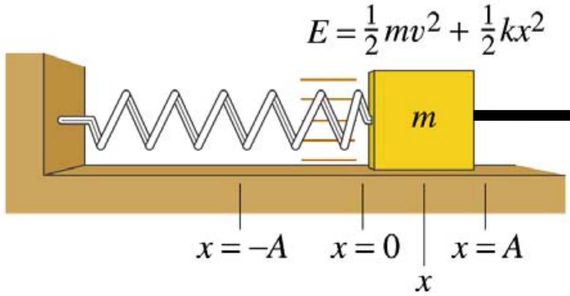
For kinetic and potential energy we find that since the time average of the square of the sine and cosine is one half, i.e. $\left\langle \sin^2 \theta(t) \right\rangle = \left\langle \cos^2 \theta(t) \right\rangle = \frac{1}{2}$, then

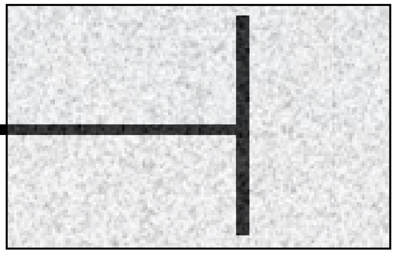
$$\left\langle K \right\rangle = \frac{1}{2} m_{block} \left\langle v^2 \right\rangle = \frac{1}{2} m_{block} \omega^2 A^2 \left\langle \sin^2(\omega t + \phi) \right\rangle = \frac{1}{4} m_{block} \omega^2 A^2 = \frac{1}{4} k A^2$$

$$\left\langle U \right\rangle = \frac{1}{2} k \left\langle x^2 \right\rangle = \frac{1}{2} k A^2 \left\langle \cos^2(\omega t + \phi) \right\rangle = \frac{1}{4} k A^2$$

This shows that on average, the kinetic energy of a simple harmonic oscillator and the potential energy of a simple harmonic oscillator are the same, each being exactly one half the total energy of the harmonic oscillator.

Damped Mass-Spring System





Hooke's Law Revisited

The approach used above for the simple harmonic oscillator will work for a damped oscillator with a small modification. Some device such as a "dashpot" provides a mechanism by which energy is removed from the system. A dashpot is like a shock absorber with a piston moving through a viscous fluid. We model the dashpot such that it provides a resistive force to the system that is proportional to the speed of the mass.

$$F_{damping} = -bv$$

The constant of proportionality, b, depends on such factors as fluid viscosity, size, shape and roughness of the piston, and the space between the piston and the fluid chamber walls. Because of this new force, our x component equation from Newton's second law gains an additional term.

$$-kx - bv_x = m_{block}a_x$$

The new equation of motion then becomes:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

A solution to the equation of motion is:

$$x = Ae^{-\alpha t}\cos(\omega' t + \phi)$$

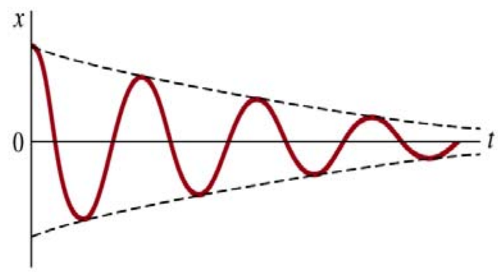
Again the initial amplitude of oscillation, A, and the initial phase, ϕ, are arbitrary constants of the second order differential equation. The angular frequency is slightly different from the undamped case:

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

The amplitude decays exponentially with time with a decay constant, α:

$$\alpha = \frac{b}{2m}$$

Appendix B shows that our solution satisfies the equation of motion and that the angular frequency and damping constants are correct. When plotted for typical values of k, m, and b, the motion of the mass looks like the graph below. As the amplitude decreases we can see that energy is leaving the system, mostly as heat generated from friction as the piston moves through the viscous fluid in the dashpot. Later in the course we will discuss losses of energy due to various mechanisms in the ocean draining energy from an acoustic wave. Although greatly simplified, the damped oscillator provides a satisfactory model of what the medium must be experiencing as the wave passes.

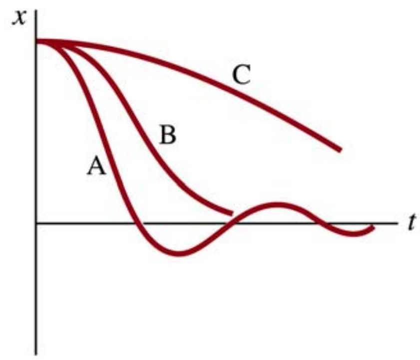


Overdamped and critically damped motion

One interesting result of the expression for the angular frequency is that if the damping constant is large enough, ω can become zero or even an imaginary number. This occurs whenever the damping constant is sufficiently large compared to the mass and the spring constant.

$$b^2 \geq 4mk$$

When this happens we say that the system is "over damped" and the motion resembles that of curve C below. Note that it can take significant time for the mass to relax to its equilibrium position in this case. When the angular frequency is exactly zero, the system is said to be "critically damped" as shown by curve B. In this case, the mass returns to the equilibrium position faster and without overshoot.



Appendix A - Checking the solution for simple harmonic motion

$$x(t) = A\cos(\omega t + \phi)$$

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$$

$$a = \frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \phi)$$

Substituting into the equation of motion:

$$\frac{d^2x}{dt^2} + \frac{k}{m_{block}}x = -A\omega^2\cos(\omega t + \phi) + \frac{k}{m_{block}}A\cos(\omega t + \phi) = 0$$

$$-\omega^2\cos(\omega t + \phi) + \frac{k}{m_{block}}\cos(\omega t + \phi) = \left(-\omega^2 + \frac{k}{m_{block}}\right)\cos(\omega t + \phi) = 0$$

So this solution works so long as $\omega^2 = \frac{k}{m_{block}}$

You should be able to repeat this process for other solutions.

Appendix B - Checking the solution for damped harmonic motion

$$x = Ae^{-\alpha t}\cos(\omega' t + \phi)$$

$$v = \frac{dx}{dt} = -Ae^{-\alpha t}\omega' \sin(\omega' t + \phi) + A(-\alpha)e^{-\alpha t}\cos(\omega' t + \phi)$$

$$= -Ae^{-\alpha t}\left[\omega' \sin(\omega' t + \phi) + \alpha\cos(\omega' t + \phi)\right]$$

$$a = \frac{d^2x}{dt^2} = -Ae^{-\alpha t}\omega'^2\cos(\omega' t + \phi) + A\alpha e^{-\alpha t}\omega' \sin(\omega' t + \phi) + A\alpha e^{-\alpha t}\omega' \sin(\omega' t + \phi) + A\alpha^2 e^{-\alpha t}\cos(\omega' t + \phi)$$

$$= Ae^{-\alpha t}\left\{2\alpha\omega' \sin(\omega' t + \phi) + \left[\alpha^2 - \omega'^2\right]\cos(\omega' t + \phi)\right\}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$Ae^{-\alpha t}\left\{2\alpha\omega' \sin(\omega' t + \phi) + \left[\alpha^2 - \omega'^2\right]\cos(\omega' t + \phi)\right\} - \frac{b}{m}Ae^{-\alpha t}\left[\omega' \sin(\omega' t + \phi) + \alpha\cos(\omega' t + \phi)\right] + \frac{k}{m}Ae^{-\alpha t}\cos(\omega' t + \phi) = 0$$

$$Ae^{-\frac{b}{2m}t}\left\{\left[2\alpha\omega' - \frac{b}{m}\omega'\right]\sin(\omega' t + \phi) + \left[\alpha^2 - \omega'^2 - \frac{b}{m}\alpha + \frac{k}{m}\right]\cos(\omega' t + \phi)\right\} = 0$$

$$\alpha = \frac{b}{2m} \qquad \frac{k}{m} - \left(\frac{b}{2m}\right)^2 - \omega'^2 = 0$$

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

Problems

1. A particle oscillates with simple harmonic motion so that its displacement varies according to the expression $x=(5.0\ \text{cm})\cos(2t+\pi/6)$, where x is in centimeters and t is in seconds. At t=0, find
    a) the displacement of the particle,
    b) its velocity, and
    c) its acceleration.
    d) Find the period and amplitude of the motion.

2. A piston in an automobile engine is in simple harmonic motion. If its amplitude of oscillation from the centerline is ±5.0 cm and its mass is 2.0 kg, find the maximum velocity and acceleration of the piston when the auto engine is running at the rate of 3600 rev/min.

3. A 20.0 g particle moves in simple harmonic motion with a frequency of 3.0 oscillations/sec and amplitude of 5.0 cm.
    a) Through what total distance does the particle move during one cycle of its motion?
    b) What is its maximum speed? Where does this occur?
    c) Find the maximum acceleration of the particle. Where in the motion does the maximum acceleration occur?

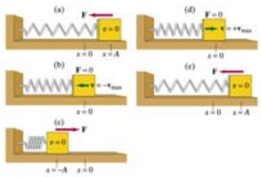
4. A 1.0 kg mass attached to a spring of force constant 25.0 N/m oscillates on a horizontal, frictionless track. At t=0, the mass is released from rest at x = -3.0 cm. (That is, the spring is compressed by 3.0 cm) Find
    a) the period of its motion,
    b) the maximum values of its speed and acceleration, and
    c) the displacement, velocity, and acceleration as functions of time.

5. A 5.0 kg mass attached to a spring of force constant 8.0 N/m vibrates in simple harmonic motion with amplitude of 10.0 cm. Calculate
    a) the maximum value of its speed and acceleration,
    b) the speed and acceleration when the mass is 6.0 cm from the equilibrium position, and
    c) the time it takes the mass to move from x = 0 to x = 8.0 cm.
    d) the total energy of the system
    e) the speed of the 5.0 kg mass when x = 5.0 cm

6. A block of unknown mass is attached to a spring of force constant 6.5 N/m and undergoes simple harmonic motion with an amplitude of 10.0 cm. When the mass is halfway between its equilibrium position and endpoint, its speed is measured to be +30 cm/s. Calculate
    a) the mass of the block,
    b) the period of the motion, and
    c) the maximum acceleration of the block.

Lesson 1 - Oscillations

- Harmonic Motion Circular Motion
- Simple Harmonic Oscillators
  - Linear - Horizontal/Vertical Mass-Spring Systems
- Energy of Simple Harmonic Motion



Math Prereqs

$$\frac{d}{d\theta}\sin\theta = \cos\theta$$

$$\frac{d}{d\theta}\cos\theta = -\sin\theta$$

$$\int_0^{2\pi}\cos\theta d\theta = \int_0^{2\pi}\sin\theta d\theta = 0$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta d\theta = \frac{1}{2\pi}\int_0^{2\pi}\sin^2\theta d\theta = \frac{1}{2}$$

Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos(\theta\pm\phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

Math Prereqs

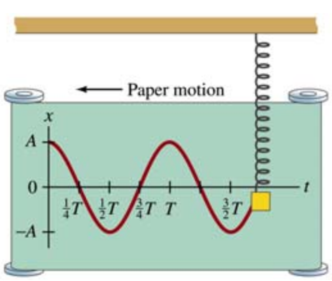
$$\langle\ \rangle = \text{"Time Average"}$$

$$\langle f(t)\rangle \equiv \frac{1}{T}\int_0^T f(t)dt$$

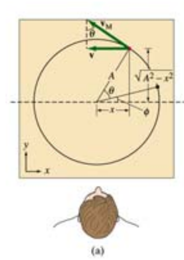
Example:

$$\left\langle \cos^2\left(\frac{2\pi}{T}t\right)\right\rangle = \frac{1}{T}\int_0^T \cos^2\left(\frac{2\pi}{T}t\right)dt = \frac{1}{T}\int_0^T\left[\frac{1}{2}+\frac{1}{2}\cos\left(2\frac{2\pi}{T}t\right)\right]dt = \frac{1}{2}$$

Harmonic

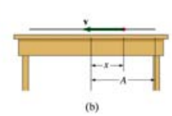


Relation to circular motion



$$x = A\cos(\theta+\phi) = A\cos(\omega t+\phi)$$

$$\omega = \frac{2\pi}{T}$$



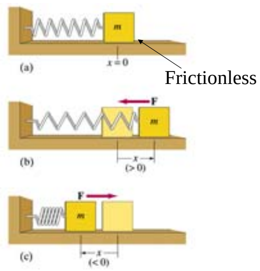
Horizontal mass-spring

$$\sum F = ma$$

Hooke's Law: $F_s = -kx$

$$-kx = m_{block}\frac{d^2x}{dt^2}$$

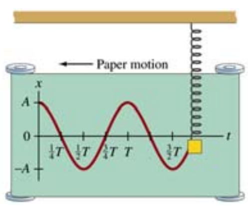
$$\frac{d^2x}{dt^2} + \frac{k}{m_{block}}x = 0$$



Solutions to differential equations

- Guess a solution
- Plug the guess into the differential equation
  - You will have to take a derivative or two
- Check to see if your solution works.
- Determine if there are any restrictions (required conditions).
- If the guess works, your guess is a solution, but it might not be the only one.
- Look at your constants and evaluate them using initial conditions or boundary conditions.

Our guess



$$x = A\cos(\omega t + \phi)$$

Definitions

$$x = A\cos(\omega t + \phi)$$

- Amplitude - (A) Maximum value of the displacement (radius of circular motion). Determined by initial displacement and velocity.
- Angular Frequency (Velocity) - (ω) Time rate of change of the phase.
- Period - (T) Time for a particle/system to complete one cycle.
- Frequency - (f) The number of cycles or oscillations completed in a period of time
- Phase - ($\omega t + \phi$) Time varying argument of the trigonometric function.
- Phase Constant - (ϕ) Initial value of the phase. Determined by initial displacement and velocity.

The restriction on the solution

$$\omega^2 = \frac{k}{m_{block}}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m_{block}}}$$

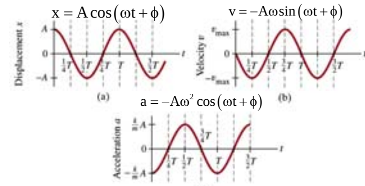
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m_{block}}{k}}$$

The constant – phase angle

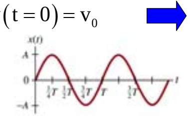
$x(t=0) = A \quad v(t=0) = 0 \quad \Rightarrow \quad \phi = 0$

$x = A\cos(\omega t + \phi)$ $\quad v = -A\omega\sin(\omega t + \phi)$

$a = -A\omega^2\cos(\omega t + \phi)$

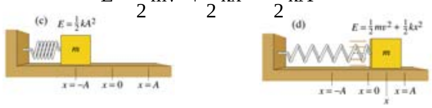


$x(t=0) = 0 \quad v(t=0) = v_0 \quad \Rightarrow \quad \phi = \frac{\pi}{2}$



Energy in the SHO

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$



$$v = \pm\sqrt{\frac{k}{m}\left(A^2 - x^2\right)}$$

Average Energy in the SHO

$$x = A\cos(\omega t + \phi)$$

$$\langle U \rangle = \frac{1}{2}k\langle x^2 \rangle = \frac{1}{2}kA^2\langle \cos^2(\omega t + \phi) \rangle = \frac{1}{4}kA^2$$

$$v = \frac{dx}{dt} = -A\omega \sin(\omega t + \phi)$$

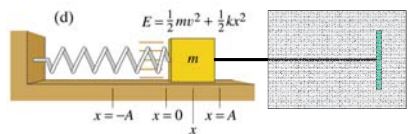
$$\langle K \rangle = \frac{1}{2}m\langle v^2 \rangle = \frac{1}{2}m\omega^2 A^2\langle \sin^2(\omega t + \phi) \rangle = \frac{1}{4}m\omega^2 A^2 = \frac{1}{4}kA^2$$

$$\langle K \rangle = \langle U \rangle$$

Example

- A mass of 200 grams is connected to a light spring that has a spring constant (k) of 5.0 N/m and is free to oscillate on a horizontal, frictionless surface. If the mass is displaced 5.0 cm from the rest position and released from rest find:
- a) the period of its motion,
- b) the maximum speed and
- c) the maximum acceleration of the mass.
- d) the total energy
- e) the average kinetic energy
- f) the average potential energy

Damped Oscillations



"Dashpot"

$$F_{damping} = -bv$$

$$-kx - b\frac{dx}{dt} = ma$$

Equation of Motion $\quad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$

Solution $\quad x = Ae^{-\alpha t}\cos(\omega' t + \phi)$

$$x = Ae^{-\alpha t}\cos(\omega' t + \phi)$$

$$v = \frac{dx}{dt} = -Ae^{-\alpha t}\omega' \sin(\omega' t + \phi) + A(-\alpha)e^{-\alpha t}\cos(\omega' t + \phi)$$

$$= -Ae^{-\alpha t}\left[\omega' \sin(\omega' t + \phi) + \alpha\cos(\omega' t + \phi)\right]$$

$$a = \frac{d^2x}{dt^2} = -Ae^{-\alpha t}\omega'^2\cos(\omega' t + \phi) + A\alpha e^{-\alpha t}\omega' \sin(\omega' t + \phi) + A\alpha e^{-\alpha t}\omega' \sin(\omega' t + \phi) + A\alpha^2 e^{-\alpha t}\cos(\omega' t + \phi)$$

$$= Ae^{-\alpha t}\left\{2\alpha\omega' \sin(\omega' t + \phi) + \left[\alpha^2 - \omega'^2\right]\cos(\omega' t + \phi)\right\}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$Ae^{-\alpha t}\left\{2\alpha\omega' \sin(\omega' t + \phi) + \left[\alpha^2 - \omega'^2\right]\cos(\omega' t + \phi)\right\} - \frac{b}{m}Ae^{-\alpha t}\left[\omega' \sin(\omega' t + \phi) + \alpha\cos(\omega' t + \phi)\right] + \frac{k}{m}Ae^{-\alpha t}\cos(\omega' t + \phi) = 0$$

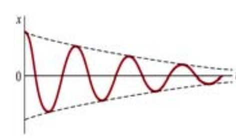
$$Ae^{-\frac{b}{2m}t}\left\{\left[2\alpha\omega' - \frac{b}{m}\omega'\right]\sin(\omega' t + \phi) + \left[\alpha^2 - \omega'^2 - \frac{b}{m}\alpha + \frac{k}{m}\right]\cos(\omega' t + \phi)\right\} = 0$$

$$\boxed{\alpha = \frac{b}{2m}} \qquad \frac{k}{m} - \left(\frac{b}{2m}\right)^2 - \omega'^2 = 0 \Rightarrow \boxed{\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}}$$

Damped frequency oscillation

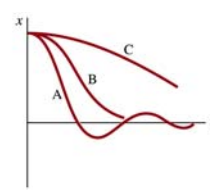
$$\alpha = \frac{b}{2m}$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$



$$b^2 \geq 4mk$$

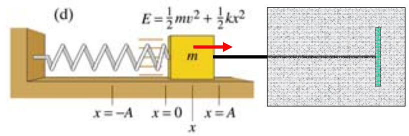
B - Critical damping (=)
C - Over damped (>)



Giancoli 14-55

- A 750 g block oscillates on the end of a spring whose force constant is k = 56.0 N/m. The mass moves in a fluid which offers a resistive force F = -bv where b = 0.162 N-s/m.
  - What is the period of the motion? What if there had been no damping?
  - What is the fractional decrease in amplitude per cycle?
  - Write the displacement as a function of time if at t = 0, x = 0; and at t = 1.00 s, x = 0.120 m.

Forced vibrations



$$F_{ext} = F_0 \cos\omega t \qquad -kx - b\frac{dx}{dt} + F_0\cos\omega t = ma$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos\omega t$$

$$x = A_0 \sin(\omega t + \phi_0)$$

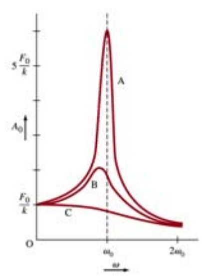
Resonance

$$x = A_0 \sin(\omega t + \phi_0)$$

Natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$

$$A_0 = \frac{F_0}{m\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \frac{b^2\omega^2}{m^2}}}$$

$$\phi_0 = \tan^{-1}\left(\frac{m\left(\omega^2 - \omega_0^2\right)}{b\omega}\right)$$

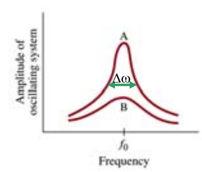


Quality (Q) value

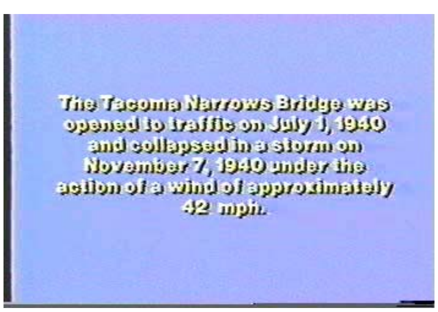
- Q describes the sharpness of the resonance peak
- Low damping give a large Q
- High damping gives a small Q
- Q is inversely related to the fraction width of the resonance peak at the half max amplitude point.

$$Q = \frac{m\omega_0}{b}$$

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q}$$



Tacoma Narrows Bridge



Tacoma Narrows Bridge (short clip)

Waves and the One-Dimensional Wave Equation

Earlier we talked about the waves on a pond. Before we start looking specifically at sound waves, let's review some general information about waves.

Types

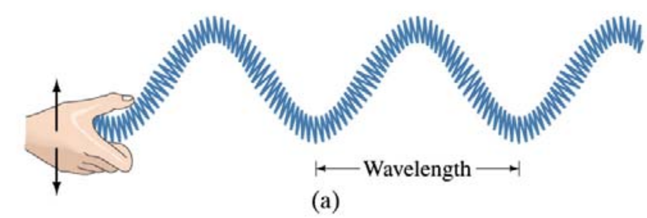
There are two general classifications of waves, longitudinal and transverse:

Transverse Wave – A traveling wave in which the particles of the disturbed medium move perpendicularly to the wave velocity. An example is the wave pulse on a stretched rope that occurs when the rope is moved quickly up and down.

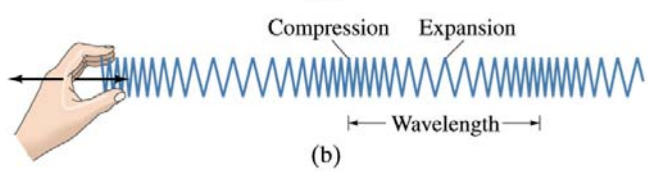
Longitudinal Wave – A traveling wave in which the particles of the medium undergo displacement parallel to the direction of the wave motion. Sound waves are longitudinal waves.

One thing to note is that some waves exhibit characteristics of both types of waves. The waves on our pond are a combination of both types.

Transverse:

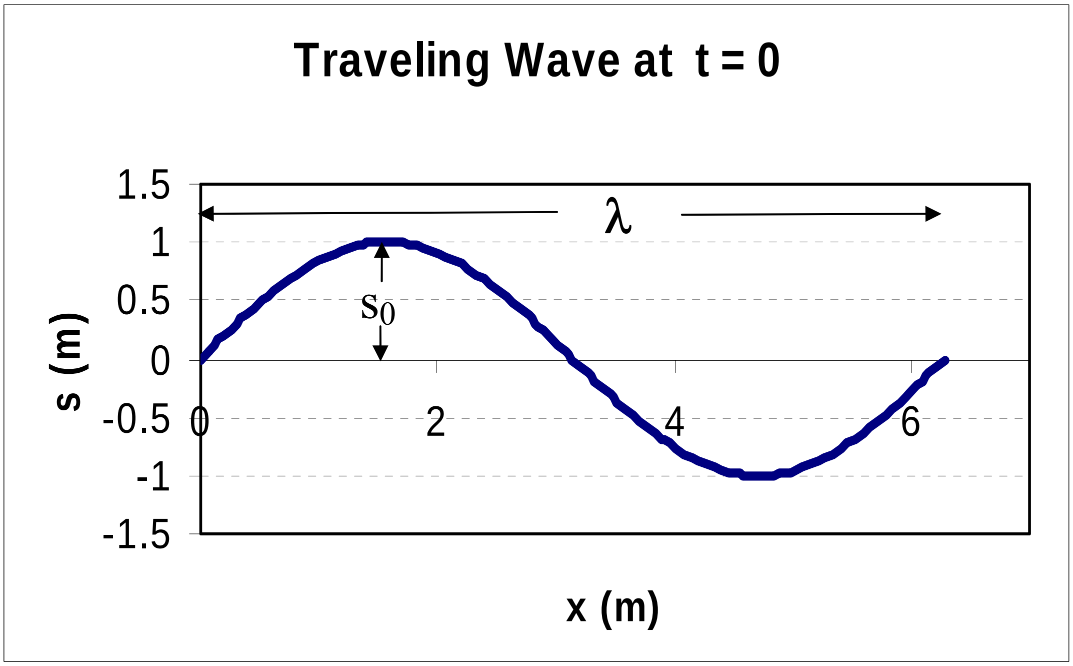


Longitudinal:



Characteristics

Just like the periodic motion of the simple harmonic oscillator, waves have certain characteristics. The ones we will concentrate on are the frequency, period, wave speed and the wavelength. Recall from SP211, a picture of a transverse wave in a medium at some time, maybe t=0 sec.



We wrote an equation to describe this picture:

$$s(x) = s_0 \sin\left(\frac{2\pi}{\lambda} x\right)$$

where:

s = particle displacement – Distance that the fluid particle is moved from its equilibrium position at any time, t.

s_o = maximum particle displacement or amplitude

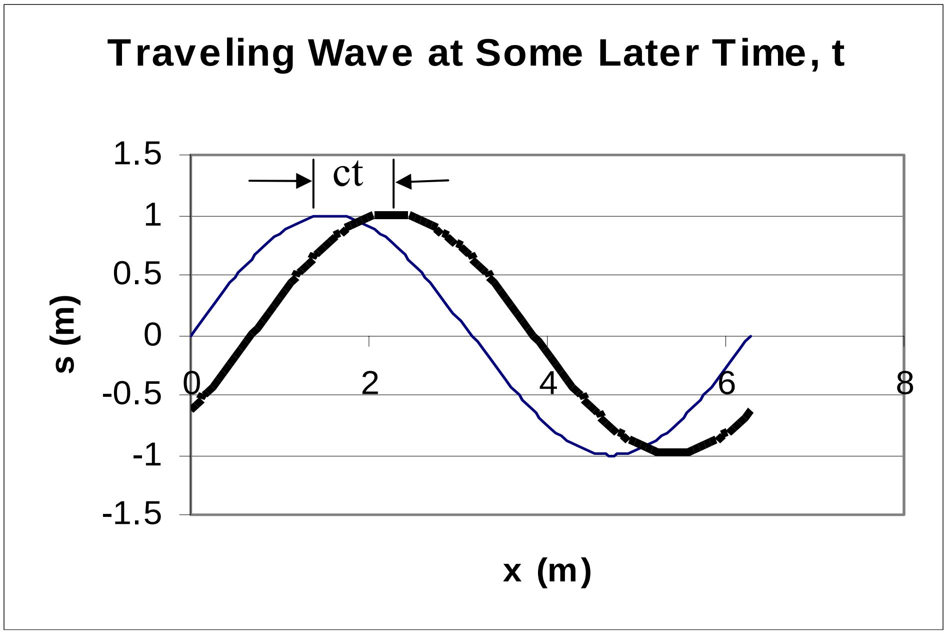
λ = distance over which the wave begins to repeat

k = $\frac{2\pi}{\lambda}$ = a conversion factor that relates the change in phase (angle) to a spatial displacement. We call k the wavenumber.

When we let this wave begin to move to the right with a speed, c, the position is shifted in the governing equation from x to x-ct.

$$s(x,t) = s_o \sin\left[\frac{2\pi}{\lambda}(x - ct)\right]$$

Below is a picture of the same traveling wave shown at some later time, t.



Now, if instead of taking a snap shot of the wave in the medium at two different times, what if we had set a sensor somewhere in space – maybe at x = 0 m, and recorded the wave's displacement over time. The equation governing the wave would become:

$$s(0,t) = s_o \sin\left[\frac{2\pi}{\lambda}(0-ct)\right] = -s_o \sin\left[\frac{2\pi c}{\lambda}t\right] = -s_o \sin\left[\frac{2\pi}{T}t\right] = -s_o \sin[\omega t]$$

where

T = period – Time to complete one cycle.

c = $\frac{\lambda}{T}$ = wave velocity – Distance that wave energy travels per unit time.

ω = $\frac{2\pi}{T}$ = a conversion factor that relates the change in phase (angle) to a temporal displacement. We call ω the angular frequency.

f = $\frac{1}{T}$ = frequency, is the inverse of the period. It is the number of cycles per unit time that pass the origin.

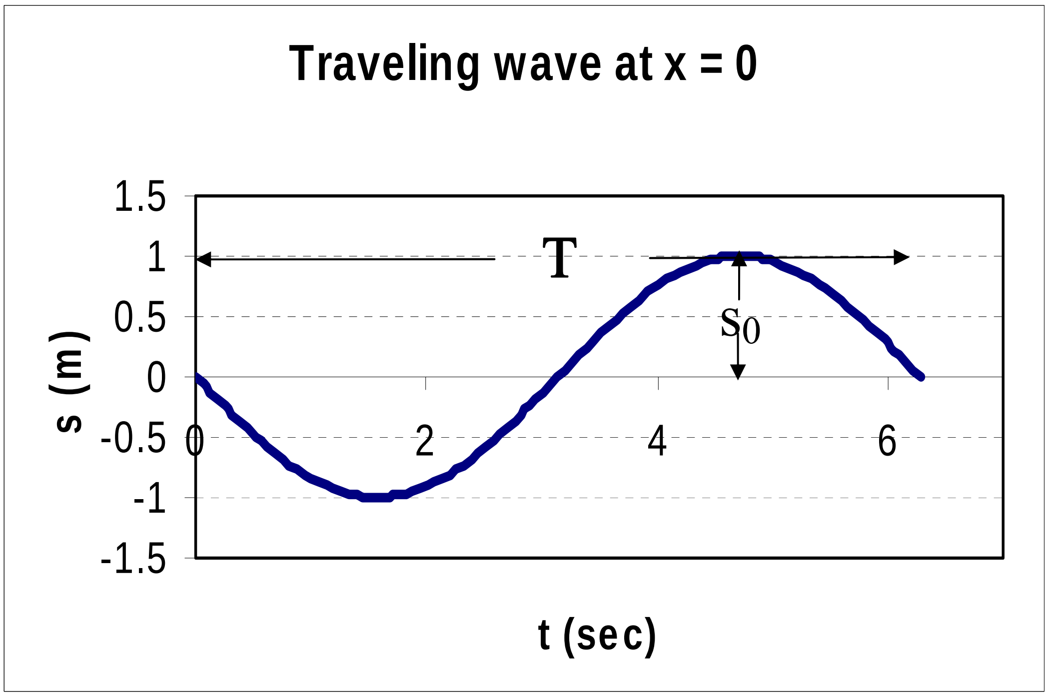
Note that we have employed a similar strategy regarding the group of constants in front of the time variable that we used when discussing the wavenumber, k. Since the wave repeats every 2π change in phase and that corresponds to a time period, T, angular frequency, ω=2π /T, is nothing more than a conversion factor from time to phase angle. The symmetry with wavenumber is striking causing many people to identify the wave number as the "special frequency" and to specifically refer to angular frequency, ω, as the "temporal frequency"

To be clear, the speed of the wave c, is not the speed of the medium. It is the speed of the wave disturbance envelope and is often called the "phase speed." It is the speed you would need to run next to the medium in order to stay in phase with a point on the disturbance.

The speed of the medium is also called the particle speed and is found by taking the derivative of the displacement with respect to time.

$u = \frac{\partial s}{\partial t}$ = particle speed – Distance that the medium travels per unit time.

Note that the average value of the particle velocity over any cycle is zero.



Putting these three pictures together, we have an expression for a traveling wave in a medium

$$s(x,t) = s_o \sin\left[\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right]$$

or more compactly,

$$s(x,t) = s_o \sin[kx - \omega t]$$

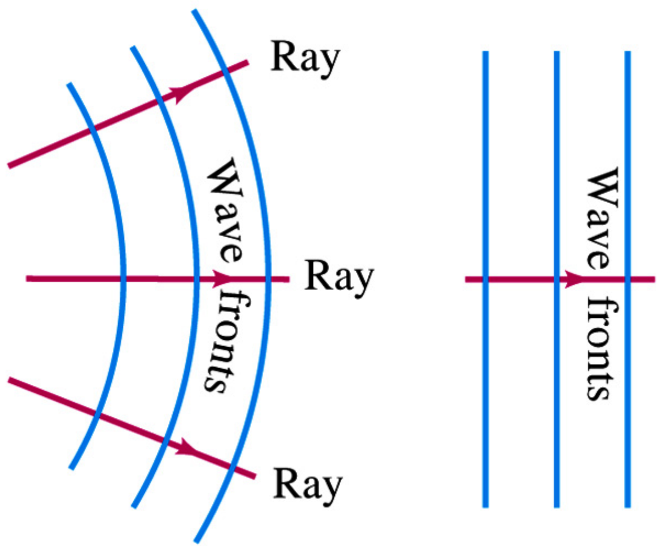
We also have a new way of defining the speed of the wave. It makes good sense that the wave speed is the distance the wave travels in one cycle (the wavelength) divided by the time it takes the wave to complete one cycle. It is a simple matter to substitute the frequency, f, for the period:

$$c = \frac{\lambda}{T} = f\lambda$$

The wave speed can also be calculated from the angular frequency and the wavenumber:

$$c = \frac{\lambda}{T}\left(\frac{2\pi}{2\pi}\right) = \frac{\omega}{k}$$

We call waves modeled using this result "plane waves" because in three dimensions the locus of points all having the same phase are planes. We call these planes "wavefronts" and often draw them as lines on a page separated by one wavelength. In fact, the wavefronts are actually parallel planes. We also find it convenient to show the direction the wave is traveling using a "ray" which is constructed perpendicular to the wavefronts.



Spherical Wave Plane Wave

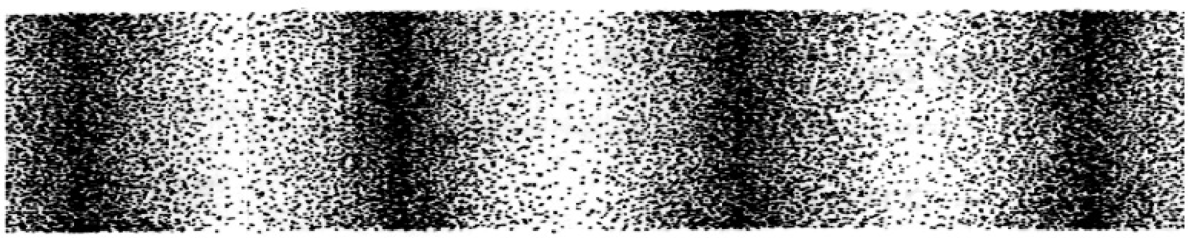
Sound Waves

When sound travels in a fluid, i.e a gas or a liquid, the displacement must be in the longitudinal direction because fluids are poor at transmitting the shear forces necessary to sustain a transverse wave. Below is a cartoon of the longitudinal displacement of a sound wave.

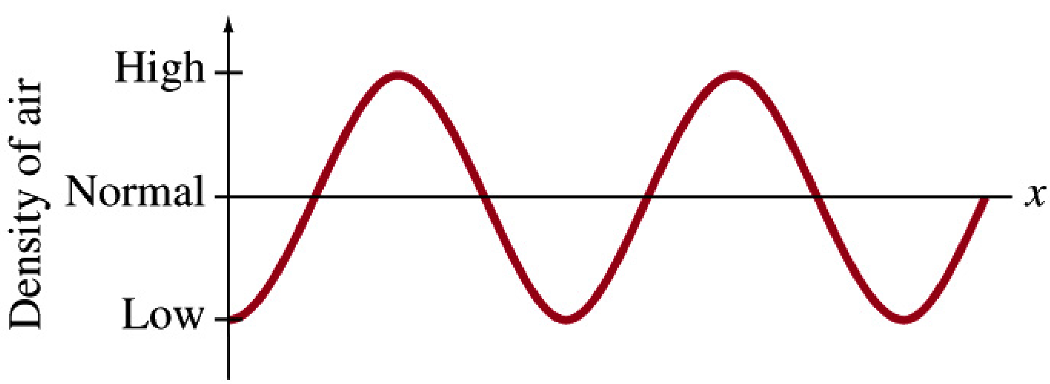
We call the locations where the fluid is displaced into clumps of closely spaced molecules condensations (high density) and the locations where the fluid molecules are sparsely spaced, rarefactions (low density).

The intermolecular forces tend to push out on each other at the condensations just as a compressed spring pushes back on a mass. The gas laws suggest the high density regions of a gaseous fluid are at a higher pressure (force per unit area) and the low density regions are at a lower pressure. The same is true for liquid fluids.

In addition to describing sound waves in fluids by the displacement of the molecules, we can also describe the wave by the velocity of the molecules or the variations in density and pressure.



(a)



(b)

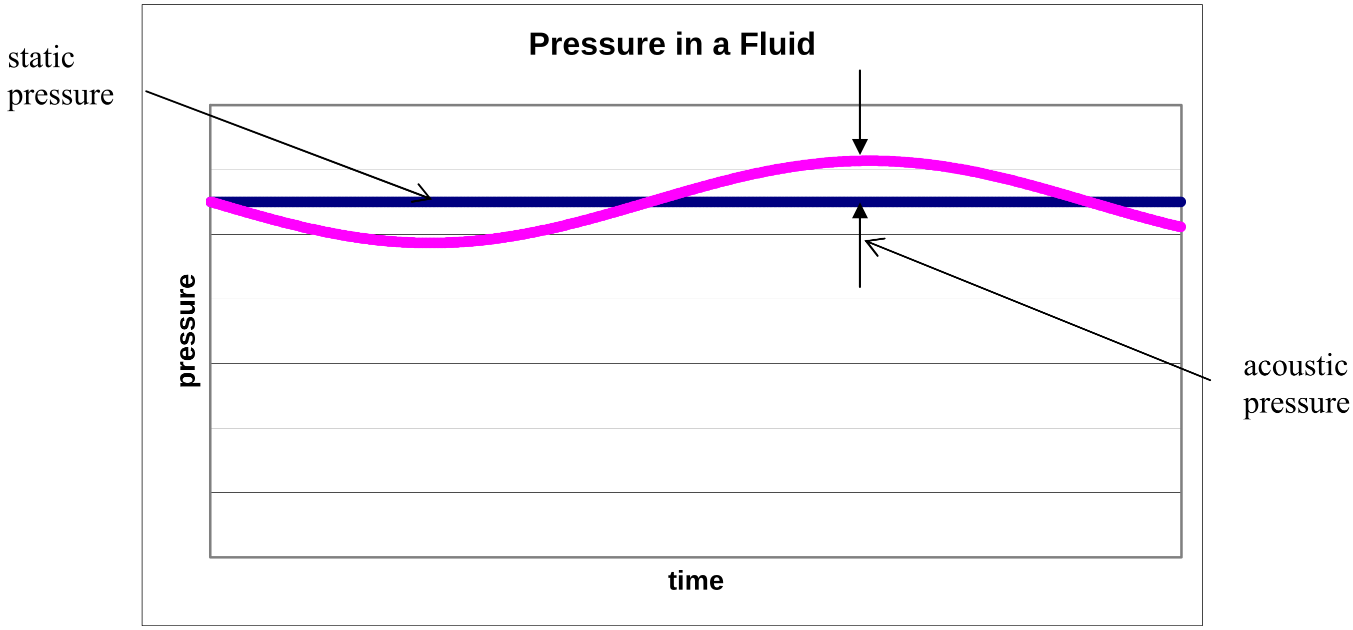
Acoustic Presssure

In the case of pressure, static pressure from the height of the column of fluid above the wave are always present. This force is constant with time. In SP211 we learned how to calculate this pressure, p, using the following equation:

$$p = p_o + \rho g h$$

where ρ is the density of the fluid and h is the height of the fluid column.

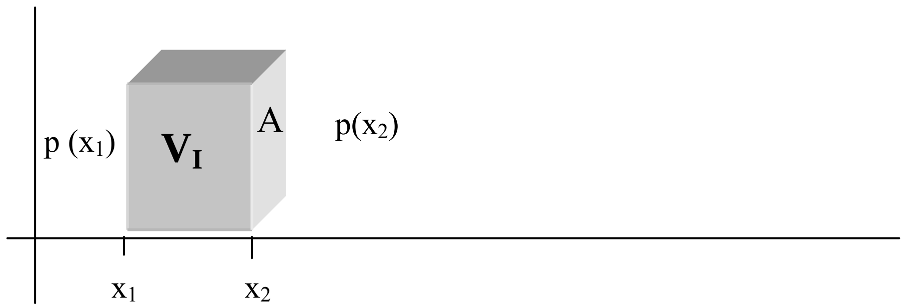
The acoustic pressure due to the condensations and rarefactions sits on this static pressure and oscillates around it due to the presence of the acoustic wave motion. While we could consider the entire pressure variation in describing an acoustic wave, we will, by convention, instead consider only the pressure variation from the static pressure.



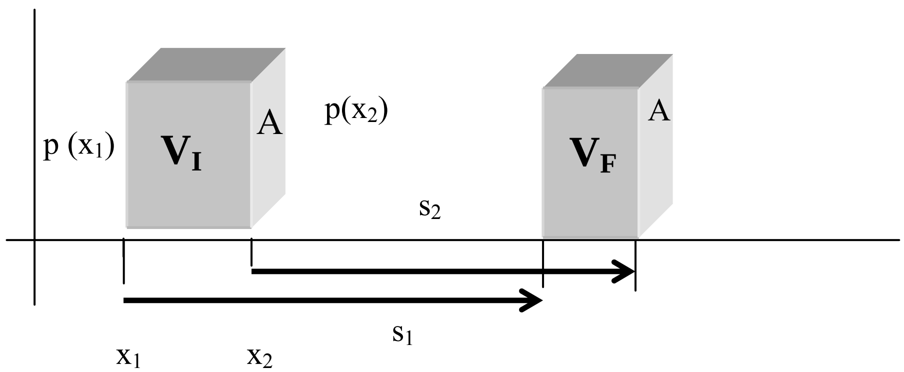
We saw that simple harmonic motion has a governing differential equation called the "equation of motion" whose solution gives the position of a mass as a function of time. In the case of a traveling wave, there is an analogous equation whose solution describes the medium's particle displacement as a function of position and time. This partial differential equation is known as "the wave equation." In the next section we will show how the wave equation follows directly from some fundamental Physics principles.

Sound Waves in a medium – the wave equation

Initial Position – V_I



Later Position – V_F



To derive the one-dimensional wave equation, let's look at the motion of a small volume of fluid. We can relate its motion to the spring-mass system from the previous section. If we apply a pressure gradient to the fluid volume, V_I, (such as an acoustic pressure from an acoustic wave) it will move and compress the volume of fluid. The pressure on the left face of the fluid block is $p_1(x_1)$, while that exerted on the right face is $p_2(x_2)$. If there is a differential pressure, Δp, then the fluid block might move to the right, and, as the block accelerates, it will change to volume, V_F. We will make some assumptions regarding the movement of the block:

1. The process is adiabatic – no heat is lost or gained by the presence of the acoustic wave. This is a reasonable assumption because for acoustic wave frequencies in the ocean, the wavelength is too long and thermal conductivity of seawater too small for significant heat flow to take place.
2. Changes in particle displacement of the fluid from equilibrium are small.
3. The fluid column is not deformed (shear deformation) by differential pressure.

To fully describe the motion of sound in the fluid from first principles, we will examine three well known Physics laws – Newton's Second Law, an equation of state, and conservation of mass. These laws, coupled with the assumptions above provide a robust and powerful model for underwater sound.

Newton's Second Law

Newton's Second Law is customarily used by examining the forces in a particular direction and then summing them as vectors. In the case of our fluid volume, V_I, the forces in the x direction are:

$$\sum F_x = p(x_1)A - p(x_2)A = -\Delta pA$$

This net force across the volume is equal to the mass times the acceleration of the volume. The mass is found by multiplying the initial density by the initial volume ($\Delta x = x_2 - x_1$)

$$m = \rho_I A\Delta x$$

The acceleration in the x direction is the second time derivative of average displacement

$$a_x = \frac{\partial^2 \overline{s}}{\partial t^2}$$

and

$$\overline{s} = \frac{s_1 + s_2}{2}$$

Substituting into Newton's Second Law,

$$\sum F_x = ma_x \quad \text{becomes}$$

$$-\Delta pA = \rho_I A\Delta x\left(\frac{\partial^2 \overline{s}}{\partial t^2}\right) \quad \text{and rearranging gives}$$

$$-\frac{\Delta p}{\Delta x} = \rho_I\left(\frac{\partial^2 \overline{s}}{\partial t^2}\right) \text{ or more appropriately}$$

$$\frac{\partial p_a}{\partial x} = -\rho\left(\frac{\partial^2 s}{\partial t^2}\right)$$

In the final result, acoustic pressure was used since the derivative of the static pressure is zero. Additionally, the instantaneous density and displacement for an infinitesimally small volume are substituted.

Equation of State and Conservation of Mass

Even though we think of liquids mostly as incompressible fluids, in reality, they are not. The Bulk Modulus of Elasticity describes how much the volume of the liquid changes for a given change in pressure. In equation form this is:

$$B \equiv -\frac{p(x_2)-p(x_1)}{(V_F - V_I)/V_I} \approx -\frac{p_a}{\Delta V/V_I}$$

The significance of the negative sign in above equation is that when p_a is positive, then $V_F<V_I$ and ΔV is negative.

Using solid geometry we can develop an expression to relate the acoustic pressure to the displacement of the small volume in the above figure. Implied in this argument is the law of conservation of mass. We are not allowing any of the medium to escape the volume, nor are we allowing any additional mass to seep in.

$$V_I = A\Delta x$$

$$V_F = A(\Delta x + \Delta s)$$

(Note: $\Delta s = s_2 - s_1$ is a negative number)

$$\frac{(V_F - V_I)}{V_I} = \frac{\Delta V}{V_I} = \frac{(A(\Delta x + \Delta s) - A\Delta x)}{A\Delta x} = \frac{\Delta s}{\Delta x}$$

Thus substituting in the last two equations and rearranging the definition of the Bulk Modulus of Elasticity:

$$p_a = -B\frac{\Delta V}{V_I}$$

$$p_a = -B\frac{\Delta s}{\Delta x} \quad \text{or more correctly}$$

$$p_a = -B\frac{\partial s}{\partial x}$$

Substituting this last result into our previous relationship between pressure and displacement:

$$\frac{\partial p_a}{\partial x} = -\rho\frac{\partial^2 s}{\partial t^2} = \frac{\partial}{\partial x}\left(-B\frac{\partial s}{\partial x}\right) = -B\frac{\partial^2 s}{\partial x^2}$$

The One Dimensional Wave Equation

Substituting the conclusion from conservation of mass and equation of state into Newton''s Second Law results in the one-dimensional wave equation that we can use to describe the displacement, s, from their rest position of particles in a medium, with respect to time and position. This equation is a partial differential equation with a solution that varies with time and position. As with the mass-spring system equations, if we can find an equation that satisfies this second order differential equation, the equation could be used to describe the motion of the particles in the medium.

$$\frac{\partial^2 s}{\partial x^2} = \left(\frac{\rho}{B}\right)\frac{\partial^2 s}{\partial t^2}$$

One solution that we will use was described above as a plane wave and has the form:

$$s(x,t) = s_o \sin(kx \pm \omega t)$$

Recall that:

s_0 = amplitude of the oscillation or maximum displacement
k = $2\pi/\lambda$ is the wave number
ω = $2\pi f = 2\pi/T$ is the angular frequency
± determines the direction that the wave travels
(+ is for a wave traveling to the left, - is for a wave traveling to the right)

To check the validity of this solution we must take the appropriate second derivatives:

$$\frac{\partial^2}{\partial x^2}\left[s_o \sin(kx - \omega t)\right] = -s_o k^2 \sin(kx - \omega t)$$
$$\frac{\partial^2}{\partial t^2}\left[s_o \sin(kx - \omega t)\right] = -s_o \omega^2 \sin(kx - \omega t)$$

Substitution into the wave equation

$$-s_o k^2 \sin(kx - \omega t) = -\left(\frac{\rho}{B}\right)s_o \omega^2 \sin(kx - \omega t)$$

or

$$k^2 = \left(\frac{\rho}{B}\right)\omega^2$$

Rearranging and recalling that the speed of the wave, c = ω/k,

$$\frac{\omega^2}{k^2} = c^2 = \frac{B}{\rho}$$

This is a fairly profound result. It tells us that the plane wave solution for particle displacement is a "good" solution provided the speed of the wave is not arbitrary, but exactly equal to the square root of the bulk modulus divided by the density. When the bulk modulus and density of water are used, a nominal value for the speed of sound in water is 1500 m/s. This agrees with measured results.

Had we used an equation of state for a gas instead of a liquid, we would have arrived at a similar result following a similar procedure. The plane wave solution would still solve the wave equation, but the wave speed would become:

$$c^2 = \frac{\gamma nRT}{m}$$

When typical room temperature numbers are used, this results in a nominal speed of sound in air of 340 m/s.

The rules of differential equations make no statement about the uniqueness of a solution to the wave equation. Many other solutions exist as well. Had the solution been expressed as a cosine vice a sine, the wave equation would still have been satisfied. Additionally, complex exponentials could have been used as a solution due to Euler's identity.

$$s(x,t) = s_o e^{i(kx-\omega t)}$$

This expression is really shorthand for the real (or imaginary) part of the complex exponential. A Gaussian pulse of the following form also satisfies the wave equation.

$$s(x,t) = s_o e^{-\left(\frac{kx-\omega t}{\omega\tau}\right)^2}$$

Additionally, if a certain frequency wave satisfies the differential equation, all multiples or harmonics of that frequency must also work.

$$s(x,t) = s_o \sin(nkx \pm n\omega t)$$

Rules for differential equations also specify that linear combinations of solutions are also solutions. This is called the principle of superposition. A method using the theory developed by a French mathematician named Fourier will allow disturbances of almost any shape to be constructed using series of harmonic plane waves. These disturbances will still themselves be solutions to the wave equation.

Alternate Views for Describing an Acoustic Wave – The Pressure Field

So far, we have viewed sound moving in a fluid as a harmonic traveling wave, considering only particle displacements. This is not a unique view. Just as an electromagnetic wave can be seen as an oscillating electric field or and oscillating magnetic field, so too can a sound wave be seen as an oscillating pressure field, an oscillating velocity field or an oscillating density field. Of course, the fundamental difference remains that the electromagnetic wave is always a vector field, while the sound wave in a fluid is generally a scalar field.

Using the solution for the wave equation, $s(x,t) = s_0 \sin(kx - \omega t)$, we can find the equations for two of these fields. First the we will find the acoustic pressure. Previously we found the relationship of the acoustic pressure p_a, and the displacement of the small volume from the equation of state. Using this we get:

$$p_a(x,t) = -B\frac{\partial s}{\partial x}$$

$$p_a(x,t) = -B\frac{\partial\left[s_0 \sin(kx-wt)\right]}{\partial x} = -Bs_0 k\cos(kx-wt)$$

The first important observation about the pressure field relative to the displacement field is that they are 90 degrees out of phase with each other. This means that when the particle displacement of the medium is at a maximum, the acoustic pressure is at a minimum. Additionally, when the displacement is zero, the maximum acoustic pressure is:

$$p_{a\,max} = Bs_o k = \rho c^2 s_o k$$

By convention, acousticians prefer not to use an engineering modulus, B, instead substituting $B=\rho c^2$.

Alternate Views – The Velocity Field and Specific Acoustic Impedance

The particle velocity is not the wave velocity. The speed that the wave travels is a function of the medium and is a constant. The speed of sound, c, is given by the equations:

$$c = \sqrt{\frac{B}{\rho}} = \frac{\lambda}{T} = f\lambda = \frac{\omega}{k}$$

The particle velocity of the medium, on the other hand tells us how fast the molecules in the fluid are moving. It is found by simply taking the time derivative of the equation describing the position of the medium, the plane wave solution.

$$u(x,t) = \frac{\partial s}{\partial t}$$

$$u(x,t) = \frac{\partial\left[s_0 \sin(kx-\omega t)\right]}{\partial t} = -s_0\omega\cos(kx-\omega t)$$

where

$$u_{max} = s_0\omega = s_0 ck$$

It is noteworthy that for a plane wave, the particle velocity and the particle displacement are 90 degrees out of phase, but that the velocity and acoustic pressure are in phase. We can also find the maximum particle velocity from the characteristics of the wave.

In your electrical engineering classes, you were introduced to a quantity called "impedance." It was the ratio of the driving "force" in a circuit, the voltage, to the rate at which charge passes by a point in the circuit, the current.

$$Z_{\text{electric}} \equiv \frac{\tilde{V}}{\tilde{I}}$$

By analogy, the driving force in an acoustic wave is the pressure and the rate at which particles in the medium pass a particular point is the velocity. It is no accident that we define the specific acoustic impedance as the ratio of the pressure to the particle velocity.

$$z \equiv \frac{p(x,t)}{u(x,t)}$$

For the case of a plane wave we have found expressions for both the pressure and velocity fields.

$$z \equiv \frac{p}{u} = \frac{-\rho c^2 s_0 k \cos(kx - \omega t)}{-s_0 ck \cos(kx - \omega t)} = \rho c$$

The specific acoustic impedance relates the characteristics of a sound wave to the properties of the medium in which it is propagating. Nominal values for the density, ρ, and the wave speed, c, for water are $\rho = 1000\ \text{kg/m}^3$ and $c = 1500$ m/s. Do not be confused into thinking that specific acoustic impedance is always the product of density and the speed of sound. This is only true for a plane wave. For other geometries, for instance a spherically spreading wave, the specific acoustic impedance is a different expression – even in the same fluid.

More on Continuity of Mass – The Density Field

When motivating the wave equation, it was mentioned that the mass in our test fluid volume was not changing. Specifically, the initial mass in position I is the same as that in position F.

$$\rho_I V_I = \rho_F V_F$$

Recalling our expression for the equation of state and substituting,

$$p_a = -B\frac{V_F - V_I}{V_I} = -B\frac{\frac{\rho_I}{\rho_F}V_I - V_I}{V_I} = -B\left(\frac{\rho_I}{\rho_I} - 1\right) = -B\left(\frac{\rho_I - \rho_F}{\rho_F}\right) \approx B\left(\frac{\rho_F - \rho_I}{\rho_I}\right)$$

We find that the fractional change in density, $\left(\frac{\rho_F - \rho_I}{\rho_I}\right)$ is directly proportional to the pressure.

This fractional change in density is called a condensation variable. It is often written,

$$\frac{\rho(x,t) - \rho_o}{\rho_0} = \frac{p_a}{B} = s_o k \cos(kx - \omega t)$$

We have developed four different descriptions for a traveling acoustic plane wave; particle displacement, particle velocity, acoustic pressure and fractional change in density. Particle displacement is 90 degrees out of phase with the other three, but all four discriptions travel with the same wave speed and have the same period and wavelength. All four can be used to properly model acoustic effects.

Energy in a Sound Wave

Missing in the discussion of wave equations and their solution is any mention of energy. We started the semester with a review of simple harmonic (sinusoidal) motion. The reason we did this should be apparent to you by now. As a plane wave traverses any medium, all specific particle locations undergo simple harmonic motion as the wave passes by. Because of this, we can use the basic SP211 equations for kinetic and potential energy of the medium. The only modification is to replace mass with density so as to calculate energy density or energy per unit volume. This is a logical modification since the medium carrying the wave is continuous. It would make no sense to identify a particular piece of mass, nor the total mass. The equations for kinetic and potential energy density in a simple harmonic oscillator are respectively as follows

$$\varepsilon_K = \frac{1}{2}\rho u^2$$
$$\varepsilon_P = \frac{\frac{1}{2}k_{Hooke}s^2}{V} = \frac{\frac{1}{2}m\omega^2 s^2}{V} = \frac{1}{2}\rho\omega^2 s^2$$

Since we have equations for particle displacement and particle velocity, we can simply substitute these into the above.

$$\varepsilon_K = \frac{1}{2}\rho u^2 = \frac{1}{2}\rho\left[s_o\omega\cos(kx-\omega t)\right]^2 = \frac{1}{2}\rho\omega^2 s_o^2\cos^2(kx-\omega t)$$
$$\varepsilon_P = \frac{1}{2}\rho\omega^2 s^2 = \frac{1}{2}\rho\omega^2\left[s_o\sin(kx-\omega t)\right]^2 = \frac{1}{2}\rho\omega^2 s_o^2\sin^2(kx-\omega t)$$

It should be clear that the total energy is the sum of the potential and kinetic energy and that when the kinetic energy is maximum the potential energy is zero and vice versa. The question of how the energy is partitioned depends on when you ask the question.

The average energy in a simple harmonic oscillator is calculated using the following definition for a periodic function:

$$\langle f(t)\rangle \equiv \frac{1}{T}\int_0^T f(t)\,dt$$

For kinetic and potential energy we find that since the time average of $\langle\sin^2\theta(t)\rangle = \langle\cos^2\theta(t)\rangle = \frac{1}{2}$,

$$\langle\varepsilon_K\rangle = \frac{1}{2}\rho\omega^2 s_o^2\langle\cos^2(kx-\omega t)\rangle = \frac{1}{4}\rho\omega^2 s_o^2 = \frac{1}{4}\rho u_{max}^2$$
$$\langle\varepsilon_P\rangle = \frac{1}{2}\rho\omega^2 s_o^2\langle\sin^2(kx-\omega t)\rangle = \frac{1}{4}\rho\omega^2 s_o^2 = \frac{1}{4}\rho u_{max}^2$$

This shows that on average, the kinetic energy of a plane wave and the potential energy of a plane wave are the same, each being exactly one half the total energy of the harmonic oscillator. The total average energy density of the wave is then,

$$\langle \varepsilon \rangle = \langle \varepsilon_K \rangle + \langle \varepsilon_P \rangle = \frac{1}{2}\rho u_{max}^2$$

Using the acoustic impedance, $u = \frac{p}{z} = \frac{p}{\rho c}$ allows us to write the total energy in terms of maximum pressure.

$$\langle \varepsilon \rangle = \frac{1}{2}\frac{p_{a\,max}^2}{\rho c^2}$$

Acoustic Intensity

Acoustic intensity, I, is defined as the amount of energy passing through a unit area per unit time as the wave propagates through the medium. As we described in SP211, energy moved per unit time is power which has units of Watts. Intensity then must have units of Watts/m^2.

$$I = \left[\frac{Power}{Area}\right] = \left[\frac{Work}{time} \times \frac{1}{Area}\right] = \left[\frac{Force \times displacement}{Area \times time}\right]$$

$$I = \left[Pressure \times velocity\right]$$

This unit analysis suggests acoustic intensity can be calculated from the product of acoustic pressure and particle velocity.

$$I = p_a u \quad \text{where}$$

$$p_a = p_{a\,max}\sin\left(kx - \omega t\right) \text{ and } u = u_{max}\sin\left(kx - \omega t\right)$$

$$\text{but } u = \frac{p}{\rho c}$$

$$I\left(x,t\right) = \frac{p_a^2\left(x,t\right)}{\rho c}$$

One important thing to note is that since the <u>acoustic pressure</u> is a time-varying quantity, so is the <u>intensity</u>.

We will use a more meaningful quantity, the time average acoustic intensity. The average intensity of an acoustic wave is the time average of the pressure over a single period of the wave and is given by the equation:

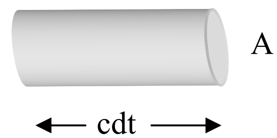
$$\langle I \rangle = \frac{\langle p_a^2 \rangle}{\rho c}$$

Since the time averaged acoustic pressure is $\langle p_a^2 \rangle = \frac{1}{2}p_{a\,max}^2$, the average acoustic intensity can be written:

$$\langle I \rangle = \frac{1}{2}\frac{p_{a\,max}^2}{\rho c}$$

This result looks remarkably similar to the average energy density of a traveling plane wave. In fact this is not accidental. If you consider a plane wave as a cylinder of length cdt and cross section A, the total energy in this cylinder, dE, would be the product of the energy density and the volume.

$$\langle dE \rangle = \langle \varepsilon \rangle A c dt$$



Rearranging we see an alternative expression for average acoustic intensity,

$$\langle I \rangle = \left\langle \frac{1}{A}\frac{dE}{dt} \right\rangle = \langle \varepsilon \rangle c$$

Since $\langle \varepsilon \rangle = \frac{1}{2}\frac{p_{a\max}^2}{\rho c^2}$, the average acoustic intensity is again $\langle I \rangle = \frac{1}{2}\frac{p_{a\max}^2}{\rho c}$.

This result is pleasing in that it agrees with an analogy suggested earlier between voltage and pressure. In your electrical engineering class, you learned that electric power was voltage squared divided by impedance. Average power was found using

$$\langle P \rangle = \frac{1}{2}\frac{V_{\max}^2}{Z}$$

Now we have found that average acoustic power per unit area is simply acoustic pressure squared divided by specific acoustic impedance.

$$\langle I \rangle = \frac{\langle P \rangle}{A} = \frac{1}{2}\frac{p_{a\max}^2}{z}$$

This sheds light on why the modifier "specific" precedes acoustic impedance. By analogy, specific acoustic impedance, z, must be acoustic impedance Z, divided by area.

To further make use of electrical engineering backround, time averaged pressure may also be determined by:

$$\langle p_a^2 \rangle = p_{rms}^2$$

$$\sqrt{\langle p_a^2 \rangle} = p_{rms} = \frac{p_{\max}}{\sqrt{2}}$$

therefore:

$$\langle I \rangle = \frac{p_{\max}^2}{2\rho c} = \frac{p_{rms}^2}{\rho c}$$

Lastly, from this point further, unless otherwise noted, when we refer to the intensity of the wave, we actually mean the time-averaged intensity.

Review and definition:

1. displacement (s) ≡ distance fluid particle moves from equilibrium (meters)
2. period (T) ≡ time required to complete one complete oscillation (seconds)
3. particle velocity (u) ≡ displacement/time (meters/second) = $\frac{\partial s}{\partial t}$
4. wave speed (c) ≡ speed the wave front is moving (meters/second) where $c = \sqrt{\frac{\beta}{\rho}} = \frac{\omega}{k} = \lambda f$
5. frequency (f) ≡ 1/T (Hz or 1/second)
6. angular frequency (ω) ≡ 2πf (radians/second)
7. wave lengths (λ) ≡ distance between same amplitude points of two successive wave fronts (meters)
8. wave number (k) ≡ 2π/λ (1/m)
9. wave fronts ≡ surface over which all particles vibrate in phase
10. acoustic ray ≡ a vector perpendicular to the wave front pointing in the direction of propagation at one specific
11. static pressure (p_s) ≡ pressure of environment minus any changes due to sound wave (Pa or N/m^2)
12. acoustic pressure (p_a) ≡ pressure fluctuations due to presence of wave motion of particle displacement (Pa)
13. instantaneous pressure (p_{tot}) ≡ static plus acoustic pressure at any one instant
14. plane waves ≡ small segment of a spherical wavefront at a long distance from the source
15. rms pressure (p_{rms}) = $\sqrt{\langle p_a^2 \rangle}$ ≡ root mean square value of the acoustic pressure (Pa)
16. Intensity (I) = $p^2_{a\ max}/2\rho c = p^2_{rms}/\rho c$
17. acoustic impedance (z) ≡ p/u =ρc = ρω/k
18. Bulk Modulus of Elasticity (B) ≡ provides relationship between change in pressure to change in volume of unit of fluid
19. density (ρ) ≡ mass contained in a unit volume of fluid (kg/m^3)

Problems

1. A sound wave propagates a point about 50 meters below the surface of a calm sea. The instantaneous pressure at the point is given by: $p = 6x10^5 + 1000\sin(400\pi t)$, where t is in seconds and p in Pascals.
   a) What is the value of static pressure at the point?
   b) What is the value of maximum (or peak) acoustic pressure at the point?
   c) What is the root-mean-square acoustic pressure?
   d) What is the acoustic pressure when t=0, 1.25, 2.5, 3.75, 5.00 milliseconds?
   e) What is the average acoustic intensity of the sound wave? (The density of the water is 1000 kg/m^3 and the sound speed is 1500 m/sec.)
   f) What is the intensity level, L, in dB re 1 μPa?

2. A plane acoustic wave is propagating in a medium of density ρ=1000 kg/m^3. The equation for a particle displacement in the medium due to the wave is given by: $s = (1x10^{-6})\cos(8\pi x - 12000\pi t)$, where distances are in meters and time is in seconds.
   a) What is the rms particle displacement?
   b) What is the wavelength of the sound wave?
   c) What is the frequency?
   d) What is the speed of sound in the medium?
   e) What is the value of maximum (or peak) particle velocity?
   f) What is the value of maximum acoustic pressure?
   g) What is the specific acoustic impedance of the medium?
   h) What is the bulk modulus of the medium?
   i) What is the acoustic intensity of the sound wave?
   j) What is the acoustic power radiated over a 3 m^2 area?

3. A plane acoustic wave is propagating in a medium of density ρ and sound speed c. The equation for pressure amplitude in the medium due to the wave is given by: $p = p_0 \cos(kx - \omega t)$, where p_0 is the maximum pressure amplitude of the sound.
   a) Show that the equation above can be written in the form, $p = p_0 \cos\frac{2\pi}{\lambda}(x - ct)$.
   b) Show that maximum pressure amplitudes (compressions) can be found at the following locations in space: x=nλ+ct where n= 0, 1, 2, 3, …
   c) Show that maximum pressure amplitudes (rarefactions) can be found at the following location in space: x=(n+1/2)λ+ct, where n = 0, 1, 2, 3, …

4. A plane acoustic wave travels to the left with amplitude 100 Pa, wavelength 1.0 m and frequency 1500 Hz; $p_1 = 100Pa\cos\left(\frac{2\pi x}{m} + \frac{3000\pi}{\sec}t\right)$, while another plane wave travels to the right with amplitude 200 Pa, wavelength ½ m and frequency 750 Hz: $p_2 = 200Pa\cos\left(\frac{\pi x}{m} + \frac{1500\pi}{\sec}t\right)$.

a) Find the rms average total pressure. Your answer will not depend on distance x. (Hint: rms average pressure $\equiv \sqrt{\langle P_{total}^2 \rangle}$, where the <> symbol denotes a time average.

b) If $p_1 = p_0 \cos(kx - \omega t)$ and $p_1 = p_0 \cos(kx - \omega t + \phi)$, find the rms average total pressure.

5. Given the following equation for an acoustic wave, originating from a source in the ocean

$$p(x,t) = 8x10^5 \text{ Pa} \sin\left(\frac{2\pi}{13\text{ m}}x - \frac{2\pi[160]}{\text{sec}}t\right)$$

Determine the following:

a) The wavelength
b) The rms pressure of the wave
c) What is the frequency of the wave?
d) The time averaged intensity of the acoustic wave

6. If the particle displacement can be found to be:

$$s(x,t) = 6x10^{-6} \text{ m} \cos\left(\frac{2\pi}{13\text{ m}}x - \frac{2\pi[160]}{\text{sec}}t\right)$$

a) What is the value of the peak particle velocity?
b) What would be the maximum acoustic pressure if the Bulk Modulus of Elasticity of the medium were $2.0x10^9$ N/m^2?

7. If a pressure pulse from a small explosion in water is known to be equal to

$$\text{p} = (1000\text{Pa})\,\text{e}^{-\left(\frac{\text{t}}{0.1\text{sec}}\right)^2} \quad \text{at x} = 0$$

a) Construct a solution to the wave equation for the pulse propagating to the right. This expression must be in the form of a function of x and t.
b) Sketch p(x,t) from part a) for time t = 0, t = 0.1 s, and t = 0.2 s.

8. If an acoustic pressure pulse in water at x = 0 is known to be

$$\text{p}(\text{t}) = \frac{\text{p}_\text{o}}{1 + \left(\frac{\text{t}}{\tau}\right)^2} \quad \text{where } \tau = 1 \text{ millesec, } \text{p}_\text{o} = 1 \text{ Pa}$$

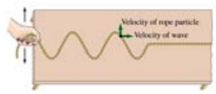
a) Find a wave expression for the pressure pulse traveling in the x-direction to the left.
b) Find an expression for the intensity of the waveform found in part a).

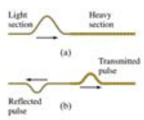
9. What is the speed of sound in yards per second in:

a) air?
b) water?

Waves

- Traveling Waves
  - Types
  - Classification
  - Harmonic Waves
  - Definitions
  - Direction of Travel
- Speed of Waves
- Energy of a Wave





Types of Waves

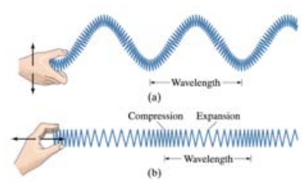
- Mechanical Waves - Those waves resulting from the physical displacement of part of the medium from equilibrium.
- Electromagnetic Waves - Those wave resulting from the exchange of energy between an electric and magnetic field.
- Matter Waves - Those associated with the wave-like properties of elementary particles.

Requirements for Mechanical Waves

- Some sort of disturbance
- A medium that can be disturbed
- Physical connection or mechanism through which adjacent portions of the medium can influence each other.

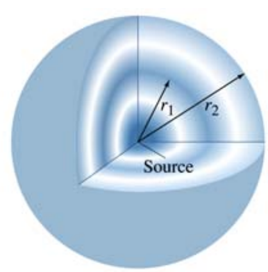
Classification of Waves

- Transverse Waves - The particles of the medium undergo displacements in a direction perpendicular to the wave velocity
  - Polarization - The orientation of the displacement of a transverse wave.
- Longitudinal (Compression) Waves - The particles of the medium undergo displacements in a direction parallel to the direction of wave motion.
  - Condensation/Rarefraction

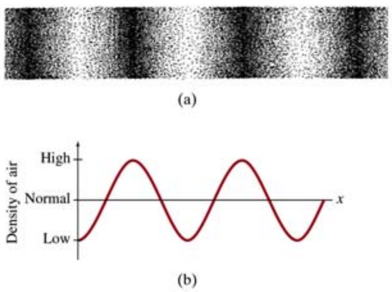


Waves on the surface of a liquid

3D Waves

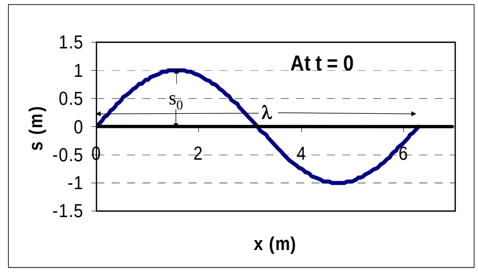


Sound Waves



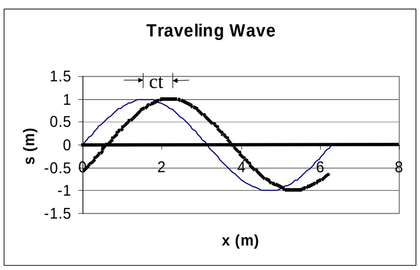
Harmonic Waves

- Transverse displacement looks like:



$$s(x) = s_0 \sin\left(\frac{2\pi}{\lambda} x\right)$$

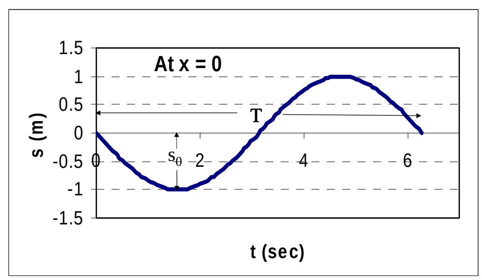
Let the wave move



$$s(x,t) = s_0 \sin\left[\frac{2\pi}{\lambda}(x - ct)\right]$$

Standing at the origin

- Transverse displacement looks like:

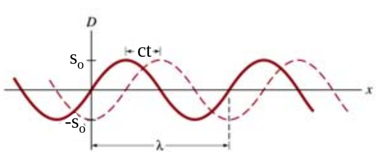


$$s(0,t) = s_o \sin\left[\frac{2\pi}{\lambda}(0 - ct)\right] = s_o \sin\left[-\frac{2\pi c}{\lambda} t\right] = -s_o \sin\left(\frac{2\pi}{T} t\right)$$

Phase Velocity

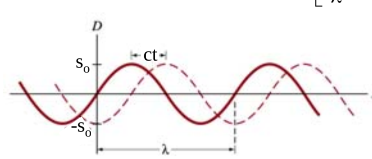
$$c = \frac{\text{distance moved in one cycle}}{\text{time required for one cycle}} = \frac{\lambda}{T} = f\lambda$$

- Wave velocity is a function of the properties of the medium transporting the wave



That negative sign

- Wave moving right $\quad s(x,t) = s_o \sin\left[\frac{2\pi}{\lambda} x - \frac{2\pi}{T} t\right]$
- Wave moving left $\quad s(x,t) = s_o \sin\left[\frac{2\pi}{\lambda} x + \frac{2\pi}{T} t\right]$



Alternate notation

$$s(x,t) = s_0 \sin\left[\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right]$$

$$\boxed{s(x,t) = s_o \sin[kx - \omega t]}$$

Wave number $k = \dfrac{2\pi}{\lambda}$

Angular frequency $\omega = \dfrac{2\pi}{T}$

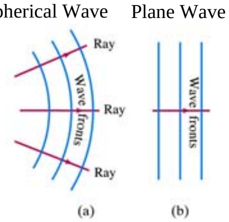
$$c = \frac{\lambda}{T} = \frac{\lambda}{2\pi}\frac{2\pi}{T} = \frac{\omega}{k}$$

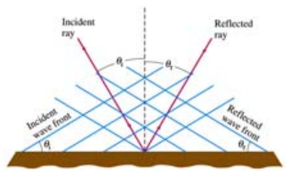
Definitions

- Amplitude - (s_o) Maximum value of the displacement of a particle in a medium (radius of circular motion).
- Wavelength - (λ) The spatial distance between any two points that behave identically, i.e. have the same amplitude, move in the same direction (spatial period)
- Wave Number - (k) Amount the phase changes per unit length of wave travel. (spatial frequency, angular wavenumber)
- Period - (T) Time for a particle/system to complete one cycle.
- Frequency - (f) The number of cycles or oscillations completed in a period of time
- Angular Frequency - (ω) Time rate of change of the phase.
- Phase - ($kx - \omega t$) Time varying argument of the trigonometric function.
- Phase Velocity - (v) The velocity at which the disturbance is moving through the medium

Two dimensional wave motion

Spherical Wave Plane Wave

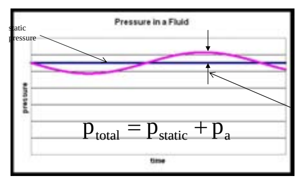


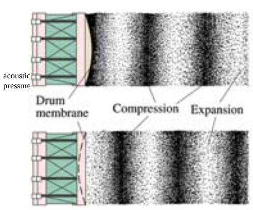


$$\theta_i = \theta_r$$

Acoustic Variables

$$s(x,t) = s_o \sin[kx - \omega t]$$

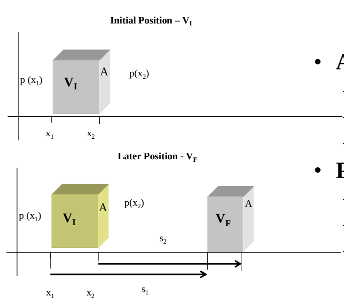




- Displacement
- ParticleVelocity $u = \dfrac{\partial s}{\partial t}$
- Pressure
- Density $\dfrac{\rho(x,t) - \rho_o}{\rho_o}$

Condensation = Compression
Rarefaction = Expansion

A microscopic picture of a fluid



- **Assumptions:**
  - Adiabatic
  - Small displacements
  - No shear deformation
- **Physics Laws:**
  - Newton's Second Law
  - Equation of State
  - Conservation of mass

The Wave Equation

Newton's Second Law/
Conservation of Mass

$$\frac{\partial p_a}{\partial x} = -\rho\left(\frac{\partial^2 s}{\partial t^2}\right)$$

Equation of State/
Conservation of Mass

$$p_a = -B\frac{\partial s}{\partial x}$$

PDE – Wave Equation

$$\frac{\partial^2 s}{\partial x^2} = \left(\frac{\rho}{B}\right)\frac{\partial^2 s}{\partial t^2}$$

Solutions to differential equations

- Guess a solution
- Plug the guess into the differential equation
  - You will have to take a derivative or two
- Check to see if your solution works.
- Determine if there are any restrictions (required conditions).
- If the guess works, your guess is a solution, but it might not be the only one.
- Look at your constants and evaluate them using initial conditions or boundary conditions.

The Plane Wave Solution

$$s(x,t) = s_o \sin(kx \mp \omega t) \Rightarrow \frac{\partial^2 s}{\partial x^2} = \left(\frac{\rho}{B}\right)\frac{\partial^2 s}{\partial t^2}$$

$$-s_o k^2 \sin(kx - \omega t) = -\left(\frac{\rho}{B}\right) s_o \omega^2 \sin(kx - \omega t)$$

$$k^2 = \left(\frac{\rho}{B}\right)\omega^2$$

$$\frac{\omega}{k} = c = \sqrt{\frac{B}{\rho}}$$

General rule for wave speeds

$$c = \sqrt{\frac{\text{Elastic Property}}{\text{Inertial Property}}}$$

Longitudinal wave in a long bar
$$c = \sqrt{\frac{\text{Young's modulus}}{\text{density}}} = \sqrt{\frac{Y}{\rho}}$$

Longitudinal wave in a fluid
$$c = \sqrt{\frac{\text{Bulk modulus}}{\text{density}}} = \sqrt{\frac{B}{\rho}}$$

Sound Speed

$$c = \sqrt{\frac{\text{Bulk modulus}}{\text{density}}} = \sqrt{\frac{B}{\rho}}$$

	Air	Sea Water
Bulk Modulus	1.4(1.01 x $10^{5)}$ Pa	2.28 x 10^9 Pa
Density	1.21 kg/m^3	1026 kg/m^3
Speed	343 m/s	1500 m/s

Variation with Temperature:

Air $$v \approx (331 + 0.60T)\frac{m}{s}$$

Seawater $$v \approx (1449.05 + 4.57T - .0521T^2 + .00023T^3)\frac{m}{s}$$

Example

- A plane acoustic wave is propagating in a medium of density ρ=1000 kg/m3. The equation for a particle displacement in the medium due to the wave is given by:
  $$s = (1x10^{-6})\cos(8\pi x - 12000\pi t)$$
  where distances are in meters and time is in seconds.
- What is the rms particle displacement?
- What is the wavelength of the sound wave?
- What is the frequency?
- What is the speed of sound in the medium?

Alternate Solutions

$$s(x,t) = s_o \cos(kx \pm \omega t)$$

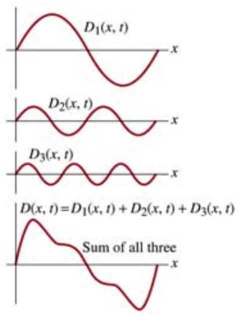
$$s(x,t) = s_o e^{i(kx - \omega t)}$$

$$s(x,t) = s_o e^{-\left(\frac{kx - \omega t}{\omega\tau}\right)^2}$$

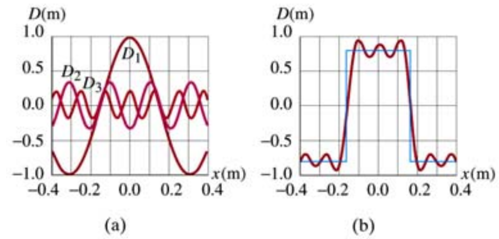
$$s(x,t) = s_o \sin(nkx \pm m\omega t)$$

Superposition

- Waves in the same medium will add displacement when at the same position in the medium at the same time.
- Overlapping waves do not in any way alter the travel of each other (only the medium is effected)



Superposition



- Fourier's Theorem – any complex wave can be constructed from a sum of pure sinusoidal waves of different amplitudes and frequencies

Alternate Views

Particle Displacement $\boxed{s(x,t) = s_o \sin(kx \pm \omega t)}$

Particle Velocity $u = \dfrac{\partial s}{\partial t} = -s_0 \omega \cos(kx - \omega t)$

Pressure $p_a(x,t) = -B\dfrac{\partial s}{\partial x} = -\rho c^2 s_0 k\cos(kx - wt)$

Density $\dfrac{\rho(x,t) - \rho_o}{\rho_0} = \dfrac{p_a}{B} = -s_o k \cos(kx - \omega t)$

Pitch is frequency

Audible	20 Hz – 20000 Hz
Infrasonic	< 20 Hz
Ultrasonic	>20000 Hz

Middle C on the piano has a frequency of 262 Hz. What is the wavelength (in air)?

1.3 m

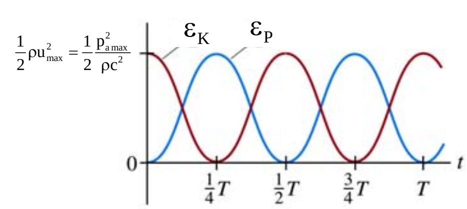
Specific Acoustic Impedance

- Like electrical impedance $Z_{electric} \equiv \dfrac{\tilde{V}}{\tilde{I}}$
- Acoustic analogy
  - Pressure is like voltage
  - Particle velocity is like current
- Specific acoustic Impedance: $z \equiv \dfrac{p(x,t)}{u(x,t)}$
- For a plane wave: $z \equiv \dfrac{p}{u} = \dfrac{-\rho c^2 s_0 k \cos(kx - \omega t)}{-s_0 ck \cos(kx - \omega t)} = \rho c$

Energy Density in a Plane Wave

$$\varepsilon_K = \frac{1}{2}\rho u^2 = \frac{1}{2}\rho\left[s_o \omega \cos(kx - \omega t)\right]^2 = \frac{1}{2}\rho\omega^2 s_o^2 \cos^2(kx - \omega t)$$

$$\varepsilon_P = \frac{1}{2}\rho\omega^2 s^2 = \frac{1}{2}\rho\omega^2\left[s_o \sin(kx - \omega t)\right]^2 = \frac{1}{2}\rho\omega^2 s_o^2 \sin^2(kx - \omega t)$$

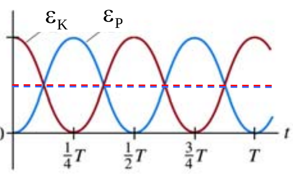


Average Energy Density

$$\langle \varepsilon_K \rangle = \frac{1}{2}\rho\omega^2 s_o^2 \langle \cos^2(kx-\omega t) \rangle = \frac{1}{4}\rho\omega^2 s_o^2 = \frac{1}{4}\rho u_{max}^2$$

$$\langle \varepsilon_P \rangle = \frac{1}{2}\rho\omega^2 s_o^2 \langle \sin^2(kx-\omega t) \rangle = \frac{1}{4}\rho\omega^2 s_o^2 = \frac{1}{4}\rho u_{max}^2$$

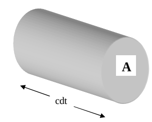
$$\frac{1}{2}\rho u_{max}^2 = \frac{1}{2}\frac{p_{a\,max}^2}{\rho c^2}$$



$$\langle \varepsilon \rangle = \langle \varepsilon_K \rangle + \langle \varepsilon_P \rangle = \frac{1}{2}\rho u_{max}^2$$

Or $\langle \varepsilon \rangle = \dfrac{1}{2}\dfrac{p_{a\,max}^2}{\rho c^2}$

Average Power and Intensity



$$\langle dE \rangle = \langle \varepsilon \rangle A c dt$$

$$\langle P \rangle = \left\langle \frac{dE}{dt} \right\rangle = \langle \varepsilon \rangle A c$$

$$\langle I \rangle = \frac{\langle P \rangle}{A} = \langle \varepsilon \rangle c = \frac{1}{2}\rho c u_{max}^2 = \frac{1}{2}\frac{p_{a\,max}^2}{\rho c} = \frac{1}{2} p_{a\,max} u_{max}$$

Instantaneous Intensity

$$I(x,t) = p_a(x,t)\,u(x,t) = \frac{[p_a(x,t)]^2}{z} = z[u(x,t)]^2$$

$$I = \left[\frac{Power}{Area}\right] = \left[\frac{Work}{time} \times \frac{1}{Area}\right] = \left[\frac{Force \times displacement}{Area \times time}\right]$$

$$I = [Pressure \times velocity]$$

$$P = \frac{V^2}{Z} = ZI^2 = VI$$

Root Mean Square (rms) Quantities

$$\sqrt{\langle p_a^2 \rangle} = p_{rms} = \frac{p_{max}}{\sqrt{2}}$$

therefore:

$$\langle I \rangle = \frac{p_{max}^2}{2\rho c} = \frac{p_{rms}^2}{\rho c}$$

Logarithms and Levels

Logarithms are used to compare two quantities to one another quickly with an easy frame of reference. It is particularly useful if there is a large difference in orders of magnitude between quantities as in acoustic pressure or acoustic energy calculations. We will see how useful logarithms can be in our next lesson. For now, let's concentrate on review of some of the basic principles leading up to our use of logarithms.

Unless otherwise stated, we will be working solely with logarithms that are in base 10 (Briggsian) . Some useful relationships to remember when working with logarithms are:

1. $y = 10^x$ then $\log_{10}(y) = x$

2. $\log(xy) = \log(x) + \log(y)$

3. $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$

4. $10\log(x^n) = n10\log(x)$

Intensity Level

In the last lesson, we defined the time average intensity in relation to the time average or rms pressure as well as the maximum acoustic pressure.

$$\langle I \rangle = \frac{\langle p^2 \rangle}{\rho c} = \frac{p_{max}^2}{2\rho c}$$

The intensity is a useful quantity because it quantifies the power in an acoustic wave, but because of the large variation in magnitudes of Intensity, it is more useful to use logarithms to compare intensities. The below table demonstrates the wide variation in Intensity for typical sounds in air.

We will start by defining a new quantity, **L**, the **intensity level**, which has units of dB.

$$L \equiv 10\log\frac{\langle I \rangle}{I_0}$$

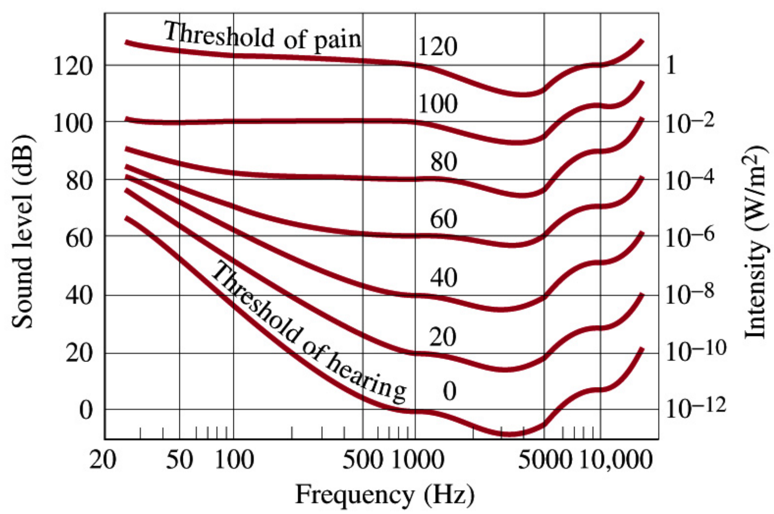
where:

<I> is the time average intensity of the sound wave.

I_0 is the reference level used for comparison purposes.

Source	**Intensity (W/m^2)**	**Intensity Level (dB)**
Jet Plane	100	140
Pain Threshold	1	120
Siren	$1x10^{-2}$	100
Busy Traffic	$1x10^{-5}$	70
Conversation	$3x10^{-6}$	65
Whisper	$1x10^{-10}$	20
Rustle of leaves	$1x10^{-11}$	10
Hearing Threshold	$1x10^{-12}$	1

The reference intensity in air is typically 1×10^{-12} W/m^2. Using this simple definition you see that intensities spanning 14 orders of magnitude become **intensity levels** between 1 and 140. This is an appealing scale because our ears seem to judge loudness on a logarithmic vice linear scale. Additionally, if you tried to graph various intensities, say as a function of frequency, your scale would likely only display the loudest noise with all others jammed along the abscissa. When intensity levels are plotted, the graph becomes much more useful.



The units of decibels were constructed for intensity level definition. A “bel” was named after Alexander Graham Bell and defined:

$$\text{"bel"} \equiv \log \frac{I}{I_0}$$

A “decibel” adopts the standard metric prefix and is 1/10th of a bel.

Reference Intensity

We have already noted above that the reference intensity when calculating intensity level for sounds in air is conventionally 1 x 10^{-12} W/m^2, the hearing threshold. This is not always the case. In fact for water, it is conventional to use a standard reference pressure, p_o. The most common reference pressure for water is 1μPa. This should not alarm you since the two can be converted using the specific acoustic impedance and assuming a plane wave.

$$I_0 = \frac{p_0^2}{\rho c}$$

Thus for water with a nominal density, ρ=1000 kg/m^3 and the nominal speed of sound, c = 1500 m/s, the reference intensity would be:

$$I_0 = \frac{(1\,\mu\text{Pa})^2}{(1000\,\text{kg/m}^3)(1500\,\text{m/s})} = 6.67\text{x}10^{-19}\ \text{W}/\text{m}^2$$

Similarly, one can work backwards from the reference intensity in air and determine that the reference pressure is about 20 μPa (ρ = 1.21 kg/m^3, c = 343 m/s).

Unfortunately, you must be very observant when using decibels to understand the reference level used in the calculation of an intensity level. While the numbers stated here for water and air are the most common today, up until the early 1970's, the standard reference pressure level for sound in water was the microbar (μbar). To remove any ambiguity, intensity levels are generally stated with the reference included as follows:

$$L = 40\ \text{dB}_{\text{re 1 μPa}}$$

Of course, this puts additional burden on you when submitting answers on homework, tests and quizzes.

Sound Pressure Level (SPL)

If the reference is provided as a pressure, and we know the about the pressure of the sound wave, we do not actually need to convert both to intensities because we can relate the pressure of a sound wave directly to the reference pressure using our basic rules for logarithms.

$$L = 10\log\frac{\langle I\rangle}{I_0} = 10\log\frac{\frac{\langle p^2\rangle}{\rho c}}{\frac{p_0^2}{\rho c}} = 10\log\frac{\langle p^2\rangle}{p_0^2} = 10\log\left(\frac{\sqrt{\langle p^2\rangle}}{p_0}\right)^2$$

A better equation for the intensity level is then:

$$L = 20\log\left(\frac{\sqrt{\langle p^2\rangle}}{p_0}\right) = 20\log\frac{p_{rms}}{p_o} \qquad \text{where,}\quad p_{rms} = \sqrt{\langle p^2\rangle} = \frac{p_{max}^2}{2}$$

In this form, the intensity level is often called the "sound pressure level." The sound pressure level and the intensity level must be equal provided the reference values correspond ($I_0 = \frac{p_0^2}{\rho c}$).

Note that the form $10\log\left(\frac{X_1}{X_2}\right)$ is used for energy quantities (power, intensity). These are sometimes called "mean squared" quantities. The form $20\log\left(\frac{x_1}{x_2}\right)$ is used for acoustic pressure and other "root mean squared" quantities such as voltage.

As a quick example, a sound wave in water with an rms pressure of 100 μPa would have an intensity level or sound pressure level (in dB):

$$L = 20\log\frac{100\ \mu\text{Pa}}{1\ \mu\text{Pa}}$$

$$L = 40\ \text{dB}_{\text{re}\,1\,\mu\text{Pa}}$$

As stated above, the reference pressure is given in this answer so that we know the intensity level is a comparison of the intensity to the reference pressure. In the future, all intensity levels for sound in water can be **assumed** to be referenced to 1 μPa unless otherwise stated. For sound in air, the standard reference pressure is 20 μPa.

About the Decibel (dB)

A couple of things to note about this new unit, dB:

1) Remember that decibels are often used to deal with values that differ over many orders of magnitude thus allowing for much smaller differences in dB. For instance, a hydrophone with a source level of 120 dB emits a sound wave with a rms pressure of 1,000,000 Pa. A hydrophone with a source level of 100 dB emits a sound wave with an rms pressure of 100,000 Pa. Thus in this instance, a difference of 20 dB equals a **difference of 900,000 Pa**.

2) Every time you see the units of dB, you should think of a **ratio**. By definition, a level in dB is related to the **ratio** of rms pressure to a reference pressure (in water p_{ref} = 1μPa). When expressing a sound pressure level referenced to 1 μPa, the units are noted as **dB/1 μPa** or **dB re 1 μPa**. Sound levels in air use 20 μPa as the reference level, the average human hearing threshold at for a 1 kHz signal. Acoustic signals in water were originally referenced to 1 μbar. You can show that sound levels referenced to the new 1 μPa reference level are therefore 100 dB higher than those referenced to 1 μbar.

$$20\log\left(\frac{10^5\ \mu\text{Pa}}{1\ \mu\text{Pa}}\right) = 100\ \text{dB}$$

Later we will see how a difference of levels of two sources, in dB, is related to the **ratio** of the pressure (or intensity) of the two sources.

3) Since $L = 10\log\frac{\langle I\rangle}{I_0}$, a 10 dB intensity level means that $\langle I\rangle$ is 10 times greater than I_0 and a 3 dB intensity level increase corresponds to a doubling of the energy level.

4) Intensity levels and sound pressure levels both use the symbol, L. As we move through the course, we will discuss source levels, noise levels, and reverberation levels. A common procedure in the Navy is to assign a subscript such as L_S for a source level. Several standard textbook have adopted the convention of putting the subscript before the "L." In this case, SL would mean source level.

Working with intensity levels

For this course, we will need to work with intensity levels in many ways. Some examples of using intensity levels are given below:

Subtracting Intensity Levels

Finding the difference between two intensity levels is a little bit different. **The difference in the two intensity levels represents the ratio of the intensities or pressure**:

$$L_2 - L_1 = 10\,log\frac{\langle I_2\rangle}{I_0} - 10\,log\frac{\langle I_1\rangle}{I_0}$$
$$L_2 - L_1 = 10\,log\langle I_2\rangle - 10\,log\,I_0 - \left[10\,log\langle I_1\rangle - 10\,log\,I_0\right]$$
$$L_2 - L_1 = 10\,log\langle I_2\rangle - 10\,log\langle I_1\rangle$$
$$L_2 - L_1 = 10\,log\frac{\langle I_2\rangle}{\langle I_1\rangle}$$

or substituting in the definition for the intensity:

$$L_2 - L_1 = 10\,log\frac{\frac{\langle p_2^2\rangle}{\rho c}}{\frac{\langle p_1^2\rangle}{\rho c}}$$
$$L_2 - L_1 = 10\,log\frac{\langle p_2^2\rangle}{\langle p_1^2\rangle}$$
$$L_2 - L_1 = 20\,log\frac{\langle p_2\rangle}{\langle p_1\rangle}$$

So if a noisy sub was emitting a source level of 140 dB and a quiet sub was emitting a source level of 80 dB, the difference between the two intensity levels would be:

$$L_2 - L_1 = 140\ \mathrm{dB} - 80\ \mathrm{dB} = 60\ \mathrm{dB}$$

For perspective, this represents a **ratio** of the intensity of both submarines To find the actual ratio (not in dB):

$$L_2 - L_1 = 10\,log\frac{\langle I_2\rangle}{\langle I_1\rangle} \quad \text{or}$$

$$\frac{\langle I_2\rangle}{\langle I_1\rangle} = 10^{\frac{L_2 - L_1}{10}} \quad \text{or}$$

$$I_2 = 10^6 * I_1$$

Or ratio of the acoustic pressures emitted:

$$L_2 - L_1 = 20\log\frac{\langle p_2\rangle}{\langle p_1\rangle} \quad \text{or}$$

$$\frac{\langle p_2\rangle}{\langle p_1\rangle} = 10^{\frac{L_2 - L_1}{20}} \quad \text{or}$$

$$p_2 = 10^3 * p_1$$

In other words, the acoustic pressure of the sound wave from the louder sub is 3 orders of magnitude or a thousand times greater than that of the quiet sub.

This example illustrates why it is so much more efficient to reference all intensities or pressures to intensity levels to provide an easier comparison between numbers that can be so many orders of magnitude different.

Adding Incoherent Intensity Levels

Noise in the ocean is the combination of noise from many different sources. How can we add two intensity levels together? We want to add two intensity levels, L_1 and L_2, where:

$$L_1 = 10\log\frac{\langle I_1\rangle}{I_0} \quad \text{and} \quad L_2 = 10\log\frac{\langle I_2\rangle}{I_0}$$

$$"L_1 + L_2" = 10\log\frac{I_{tot}}{I_0}$$

$$\text{but } I_{tot} = \langle I_1\rangle + \langle I_2\rangle$$

First, let's rewrite equation, $L_1 = 10\log\frac{\langle I_1\rangle}{I_0}$ to solve for $\langle I_1\rangle$

$$\frac{L_1}{10} = \log\frac{\langle I_1\rangle}{I_0}$$

$$10^{\frac{L_1}{10}} = \frac{\langle I_1\rangle}{I_0}$$

$$\text{similarly,}\quad 10^{\frac{L_2}{10}} = \frac{\langle I_2\rangle}{I_0}$$

$$\text{therefore}\langle I_1\rangle = I_0 10^{\frac{L_1}{10}} \text{ and } \langle I_2\rangle = I_0 10^{\frac{L_2}{10}}$$

$$\text{so}\quad I_{tot} = I_0\left(10^{\frac{L_1}{10}} + 10^{\frac{L_2}{10}}\right)$$

$$L_{tot} = L_1 \oplus L_2 = 10\log\left(\frac{I_{tot}}{I_0}\right)$$

$$L_{tot} = 10\log\left(10^{\frac{L_1}{10}} + 10^{\frac{L_2}{10}}\right)$$

We used the notation with a circle around the plus sign to represent the power sum of two decibel quantities.

$$L_{tot} = L_1 \oplus L_2$$

To add intensity levels, there are two shortcuts that can be used for some problems to make it easier than using the above equation:

1. if $L_1 = L_2$ then $L_{tot} = 10\log\left(10^{\frac{L_1}{10}} + 10^{\frac{L_2}{10}}\right) = L_1 + 3\text{ dB}$ this is because:

$$L_{tot} = 10\log\left(10^{\frac{L_1}{10}} + 10^{\frac{L_1}{10}}\right) = 10\log\left[2\left(10^{\frac{L_1}{10}}\right)\right] = 10\log\left[10^{\frac{L_1}{10}}\right] + 10\log[2] = L_1 + 3\text{ dB}$$

The rules of logarithms, specifically the second rule above, tells us the only time it is appropriate to actually add dB would be when intensity was multiplied by some quantity as in the case of the gain provided by an amplifier. In this example, if an amplifier had doubled the intensity as if there were two source intensities, we say the amplifier provided a 3 dB increase and we simply add the 3 dB to the initial intensity level.

2. if $L_1 >> L_2$ (or vice versa), then $L_{tot} \approx L_1$ (or vice versa). Here, "much more than" is defined as 10 dB or $I_1 > 10 * I_{2,}$

Problems

1. Using the rules for logs:
a) Simplify the following relationship for this empirical sound level of a signal in water ($dB_{re\ 1\mu Pa}$): $L = 20\log_{10}\left(\frac{x^a y^b}{z^c}\right)\ dB_{re\,1\mu Pa}$.
b) If x=5, y=8, z=10, a=10, b=5, and c=11, what is the resulting level (in $dB_{re\ 1\mu Pa}$).

2. Cavitation may take place at the face of a sonar transducer when the sound peak pressure amplitude being produced exceeds the hydrostatic pressure in water.
a) For a hydrostatic pressure of 600,000 Pa, what is the highest intensity that may be radiated without producing cavitation?
b) What is the intensity level in dB re 1 μPa?
c) How much acoustic power is radiated if the transducer face has an area of 1/3 m^2?

3. If $P_{2\ rms}$ = 100 μPa and $P_{1\ rms}$ = 25 μPa, what is:
a) $\frac{\langle I_2 \rangle}{\langle I_1 \rangle} = ?$
b) $L_2 - L_1$ =?

4. If $L_1 = L_2$ = 60 $dB_{re\ 1\ \mu Pa}$, L_3 = 57 $dB_{re\ 1\ \mu Pa}$, L_4 = 50 $dB_{re\ 1\ \mu Pa}$, and L_5 = 65 $dB_{re\ 1\ \mu Pa}$, what is L_{tot}, the some of all the levels.

5. What is the intensity of a 0 $dB_{(re\ 1\ \mu Pa)}$ acoustic wave in water?

6. If $L_1 = L_2$, then prove that $L_{tot} = L_1$+3 dB (the 3 dB rule).

7. If $P_{1,rms}$ = 200 μPa and $P_{2,rms}$ = 10μPa, determine (assume P_0 = 1μPa):
a) L_1,
b) L_2,
c) $L_1 \oplus L_2$,
d) L_1-L_2,
e) What does the previous result tell us?

8. If L_1=30 dB re 1μPa and L_2=65 dB re 1μPa, what is P_2/P_1?

9. Show that a plane wave having an effective acoustic pressure of 1 μbar in air has an intensity level of 74 dB re 0.0002 μbar.

10. Find the intensity (W/m^2) produced by an acoustic plane wave in water of 120 dB sound pressure level relative to 1 μbar.

11. What is the ratio of the sound pressure in water for a plane wave to that of a similar wave in air of equal intensity? c_{air} = 343 m/s, ρ_{air} = 1.21 kg/m^3, c_{water} = 1500 m/s, ρ_{water} = 1000 kg/m^3.

12. If the intensity level in seawater is 160 dB re 1 μPa, what is the rms acoustic pressure in μPa?

a) What is the rms acoustic pressure in μPa if the intensity level is 160 dB re 1 μbar?
b) What is the rms acoustic pressure in μPa if the intensity level is 160 dB re 0.0002 μbar?

13. Over a certain band of frequencies in the deep ocean basis, the noise level due to surface water turbulence (due to wind) is 62 dB and the noise level due to distant shipping is 65 dB. What is the total noise level?

14. The rms pressure from a low frequency sound source is 200 μPa. What is the combined rms pressure for both sources? What is the combined source level in dB re 1 μPa?

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THE ELUSIVE DECIBEL: THOUGHTS ON SONARS AND MARINE MAMMALS

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INTRODUCTION

A few years ago, there was considerable controversy over the effects of a proposed global acoustic experiment designed to measure the temperature of the world's oceans[1]. The focus of concern was the possible effect of the acoustic signals on whales and other marine life. There is continued interest in the effects of underwater sound on marine animals, according to a recent news item in *The Economist*[2] based on related scientific correspondence in *Nature*[3]. The thesis is that loud signals from experimental sonars harm marine mammals, or at least harass them enough to unacceptably alter their behaviour patterns. In the various discussions of this important issue that can be found in the press and on the internet, one often sees questionable comparisons being made, such as the acoustic output of a naval sonar being compared with the noise from a jet aircraft. Some misunderstandings between professionals in different fields can be traced to the multiple uses of the term "decibel". Acoustical terms can be confusing, even for experts. It is not at all surprising that well-intentioned articles sometimes fail to present situations clearly. By definition, the decibel is a relative unit, not an absolute unit with a physical dimension; unless the standard of comparison is cited, the term "decibel" is to all intents and purposes useless. The confusion is not helped by the use of the decibel to specify distinctly different physical quantities, or the same physical quantity with different reference levels. Some reporters—and even some scientists—are getting their "apple" decibels mixed up with their "orange" decibels, as it were.

The decibel (abbreviated dB) is simply a numerical scale used to compare the values of like quantities, usually power or intensity. Acousticians introduced the decibel to devise a compressed scale to represent the large dynamic range of sounds experienced by people from day to day, and also to acknowledge that humans—and presumably other animals—perceive loudness increases in a logarithmic, not linear, fashion. An intensity ratio of 10 translates into a level difference of 10 decibels[4]; a ratio of 100 translates into a level difference of 20 dB; 1000 into 30 dB; and so on. (The term "level" usually implies a decibel scale.) In a uniform acoustic medium, the magnitude of the acoustic intensity is proportional to the square of the pressure for a freely-propagating sound wave. Accordingly, the level difference in

decibels associated with two sound pressure values (measured in the same medium) is determined by calculating the ratio of the pressures, squaring this number, taking the logarithm (base 10), and multiplying by 10.[5] If one chooses a standard reference pressure value, then sound pressure levels can be specified in decibels relative to that reference, but this should be stated along with the number, for clarity[6].

The following is a typical erroneous statement found in the press, on radio, on television, and on internet discussion groups. Referring to an experimental sonar source that produces very loud low-frequency sound, *The Economist* wrote: "It has a maximum output of 230 decibels, compared with 100 decibels for a jumbo jet." Regardless of the author's intention, the implication is that a whale would experience an auditory effect from the sonar that would be substantially greater than that of a person exposed to the jet aircraft. However, this type of comparison is misleading for at least three reasons: (1) the reference sound pressures used in underwater acoustics and in-air acoustics are not the same; (2) it compares a source level with a received level; and (3) there is no obvious connection between an annoying or harmful sound level for a human in air and an annoying or harmful sound level for a marine animal in water. In the remainder of this note, we will expand on these topics somewhat, attempt to correct the mistaken impression, and try to direct attention to the real issue at the heart of the controversy.

1. STANDARD REFERENCE SOUND PRESSURES IN AIR AND IN WATER

The standard reference pressures used in underwater acoustics and in-air acoustics are not the same. In water, acousticians use a standard reference sound pressure of 1 micropascal (i.e. 10^{-6} newtons per square metre), abbreviated μPa. In air, acousticians use a higher standard reference sound pressure of 20 μPa. The in-air standard was chosen so that the threshold of hearing for a person with normal hearing would correspond to 0 dB at a frequency of 1000 Hz. Adopting different standards for air and water inevitably leads to a confusing consequence: the same sound pressure that acousticians label 0 decibels in air would be labelled 26 decibels in water. Presumably, both factions of acousticians had equally good reasons for proposing their respective standards, and this dichotomy is now entrenched in an ANSI standard[6], which is unlikely to change. Accordingly, the following dictum should always be observed, especially when dealing with cross-disciplinary issues: *It is essential that sound levels stated in decibels include the reference pressure.*

2. SOURCE LEVEL AND RECEIVED LEVEL

The erroneous statement compares a *source* level with a *received* level. In underwater acoustics, a source level usually represents the sound level at a distance of one metre from the source, while a received level is the sound level at the listener's actual position, which could be considerably more distant with a correspondingly reduced sound level. In an unbounded uniform medium, loudness decreases rapidly with increasing source-receiver distance, 6 dB less per doubling of distance. For example, *The Economist* (and even *Nature*), in referring to the 230 dB sonar source level, neglected to mention the reference distance of 1 metre. In contrast, the 100 dB number that *The Economist* associated with a jumbo jet is not a source level at all, but is typical of a received noise level measured during jet airplane take-off, averaged over several microphones situated several hundred to some thousands of metres from the runway[7]. *It is incorrect to compare a source level at 1 metre with a received noise level at an unspecified (and probably much larger) distance.*

Combining these two remarks, the output of the sonar source should have been written as 230 dB re 1 μPa at 1 m, while the jumbo jet noise level should have been written as 100 dB re 20 μPa. The inclusion of the reference values shows that these are not like quantities, and that the numbers are not directly comparable. *The Encyclopedia of Acoustics*[8] offers 120 dB re 20 μPa as a typical noise level associated with jet aircraft take-off measured at 500 m distance (although there is sure to be a wide variation about this number, depending on the type of aircraft, etc.). With the assumption of spherical spreading, referencing this level back to 1 metre distance adds 54 dB. Switching to the 1 μPa standard reference adds another 26 dB. Accordingly, the source level of a large jet looks more like 120 + 54 + 26 = 200 dB re 1 μPa at 1 m, compared with 230 dB re 1 μPa at 1 m for the sonar. Both of these are loud sources, but now at least the comparison is sensible. The ratio of sound pressures is around 32, rather than over 3 million, as some commenters would have you believe!

There are other minor issues that could be discussed. The signal from the sonar source is narrowband, and the concentration of all the signal at one frequency may be particularly troublesome for an animal who has a cavity that resonates at that frequency. On the other hand, the jet noise is broadband, and the acoustic signal was probably passed through a filter that approximately matches the sensitivity of the human ear before the measurement was made, so this measurement would be meaningless for an animal with a different hearing sensitivity curve. Much more could be said about these issues, but the principal reason for raising them is to underscore the message that the sonar / jet plane comparison has little validity.

3. WHAT HURTS?

There is no clear connection between a harmful sound level for a human in air and that for an animal in water. All creatures have evolved and adapted to their respective environments and there is no reason why human hearing characteristics should apply to any other animal, including whales. If a given sound pressure hurts a human, would the same sound pressure level in water hurt a whale (or a fish, or a shrimp)? Is the threshold of pain higher? Is it lower? Particularly when comparing acoustic effects in media of widely different impedance, is acoustic pressure the relevant acoustic quantity, or is it acoustic intensity?[9] In the end, it is the answers to these and related questions that really matter, not juggling decibels. To properly answer these questions and to determine the “community” noise standards for marine animals, scientific research is necessary—just as it was for humans. Some of this work has already been done, and an excellent review[10] of the state of knowledge up to 1995 is a good starting point for acousticians and biologists interested in deepening their understanding. A single example cannot represent the whole range of species under consideration, but is typical: The response threshold (determined through behavioural studies) of a Beluga at 1000 Hz is just over 100 dB re 1 μPa. Of course, this says nothing about the Beluga’s threshold of pain, and says nothing about what sound level would unacceptably alter its behaviour. *It is unwise to assume that the auditory experience of any animal would be the same as that of a human exposed to the same sound level.*

CONCLUSION

As sonar engineers, marine biologists, and environmentally conscious citizens continue to discuss these important issues, we should at least agree to use the same acoustical units to convey our points of view, to avoid confusion and misrepresentation. Some sensible acousticians have advocated abandoning the use of the decibel—which is partly to blame for our woes—in favour of good old SI (i.e., metric) units for sound pressure, acoustic intensity, power, etc. Until that happy day dawns, let us include reference values with our decibels, so we don't end up with fruit salad dBs. Ultimately, what is important is to determine what underwater sound levels are harmful to marine life. We must develop mitigation measures to allow underwater acoustic systems to be operated while ensuring the protection of the marine environment with due diligence.

ACKNOWLEDGEMENT

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[2] "Quiet, please. Whales navigating", *The Economist*, 1998 March 7, page 85.

[3] R. Frantzis, "Does acoustic testing strand whales?", *Nature* **392**, 1998 March 5, page 29.

[4] In fact, this defines 1 bel, named after Alexander Graham Bell. The bel turned out to be too large for practical purposes and the decibel—which is 1/10 of a bel—is the preferred unit. Also, one decibel is about the smallest incremental change of sound pressure level a person can sense.

[5] Mathematically, this is equivalent to taking the logarithm of the pressure ratio and multiplying by 20, but knowing when to multiply by 10 or 20 in such calculations is an endless source of confusion to the neophyte, so we advocate the definition in the main text.

[6] *American National Standard Preferred Reference Quantities for Acoustical Levels*, ANSI S1.8-1969, page 8.

[7] Malcolm J. Crocker, editor, *The Encyclopedia of Acoustics* (John Wiley and Sons, Inc., New York, 1997), page 1095.

[8] Malcolm J. Crocker, editor, *The Encyclopedia of Acoustics* (John Wiley and Sons, Inc., New York, 1997), page 11.

[9] The suggestion that acoustic intensity has more bearing than sound pressure in this context has been seriously proposed by some acousticians; however, the available evidence gives the nod to sound pressure, not intensity.

[10] W. John Richardson *et al*., *Marine Mammals and Noise* (Academic Press, New York, 1995).

Lesson 3 - Logs and Levels

Math Prereqs

$$y = 10^x \quad \text{then} \quad \log_{10}(y) = x$$

$$\log(xy) = \log(x) + \log(y)$$

$$\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$$

$$10\log(x^n) = n10\log(x)$$

Examples

- Without using your calculator, find the following: ($\log_{10}(2) = 0.30$)
- $\log_{10}(10^{-3}) =$
- $\log_{10}(1 \times 10^{12}) =$
- $\log_{10}(2 \times 10^{12}) =$
- $\log_{10}(200) =$
- $\log_{10}(200) - \log_{10}(10) =$
- $\log_{10}(2^{10}) =$

-3, 12, 12.3, 2.3, 1.3, 3

Pitch is frequency

Audible	20 Hz – 20000 Hz
Infrasonic	< 20 Hz
Ultrasonic	>20000 Hz

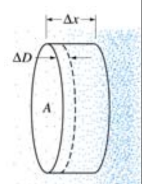
Middle C on the piano has a frequency of 262 Hz. What is the wavelength (in air)?

1.3 m

Intensity of sound

$$\langle I \rangle = \frac{\langle p^2 \rangle}{\rho c} = \frac{p_{max}^2}{2\rho c}$$

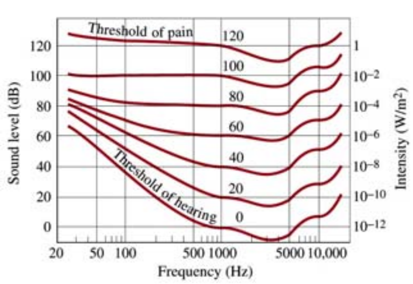
- Loudness – intensity of the wave. Energy transported by a wave per unit time across a unit area perpendicular to the energy flow.



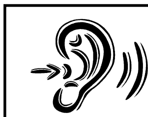
Source	Intensity (W/m^2)	Sound Level
Jet Plane	100	140
Pain Threshold	1	120
Siren	1×10^{-2}	100
Busy Traffic	1×10^{-5}	70
Conversation	3×10^{-6}	65
Whisper	1×10^{-10}	20
Rustle of leaves	1×10^{-11}	10
Hearing Threshold	1×10^{-12}	1

Sound Level - Decibel

$$L = 10\log\left(\frac{\langle I \rangle}{I_0}\right) \qquad I_0 = 1\times10^{-12}\ \frac{W}{m^2}$$



Why the decibel?



- Ears judge loudness on a logarithmic vice linear scale
- Alexander Graham Bell $\quad \text{"bel"} \equiv \log\frac{I}{I_0}$
- deci = $\frac{1}{10}$
- 1 bel = 10 decibel

$$L(\text{in dB}) = 10\log\left(\frac{\langle I \rangle}{I_0}\right)$$

Reference Level Conventions

$$I_0 = \frac{p_0^2}{\rho c}$$

Location	Reference Intensity	Reference Pressure
Air	1 x 10^{-12} W/m^2	20 µPa
Water	6.67 x 10^{-19} W/m^2	1 uPa

Historical Reference

- 1 microbar
- 1 bar = 1 x 10^5 Pa
- 1 µbar = 1 x 10^5 µPa

$$20\log\left(\frac{10^5\ \mu\text{Pa}}{1\ \mu\text{Pa}}\right) = 100\ \text{dB}$$

- So to convert from intensity levels referenced to 1 µbar to intensity levels referenced to 1 µPa, simply add 100 dB

Sound Pressure Level

Mean Squared Quantities:
Power, Energy, **Intensity**

$$L = 10\log\left(\frac{\langle I\rangle}{I_0}\right)$$

"Intensity Level"

Root Mean Squared Quantities:
Voltage, Current, **Pressure**

$$L = 20\log\left(\frac{\sqrt{\langle p^2\rangle}}{p_0}\right) = 20\log\frac{p_{rms}}{p_o}$$

"Sound Pressure Level"

Subtracting Intensity Levels

$$L_2 - L_1 = 10\log\frac{\langle I_2\rangle}{I_0} - 10\log\frac{\langle I_1\rangle}{I_0}$$

$$L_2 - L_1 = 10\log\frac{\langle I_2\rangle}{\langle I_1\rangle}$$

$$L_2 - L_1 = 20\log\frac{\langle p_2\rangle}{\langle p_1\rangle}$$

Two Submarines

- If a noisy sub was emitting a source level of 140 dB and a quiet sub was emitting a source level of 80 dB,
- What is the difference in noise levels?
- what does this mean in terms of relative intensity and acoustic pressure?

Adding Levels

$$L_1 = 10\log\frac{\langle I_1\rangle}{I_0} \quad \text{and} \quad L_2 = 10\log\frac{\langle I_2\rangle}{I_0}$$

$$"L_1 + L_2" = 10\log\frac{I_{tot}}{I_0}$$

$$\text{but } I_{tot} = \langle I_1\rangle + \langle I_2\rangle$$

$$L_{tot} = L_1 \oplus L_2 = 10\log\left(10^{\frac{L_1}{10}} + 10^{\frac{L_2}{10}}\right)$$

Total Noise

- On a particular day, noise from shipping is 53 dB and noise from rain and biologics is 50 dB. What is the total noise level from the two sources?

Adding Equal Noise Levels

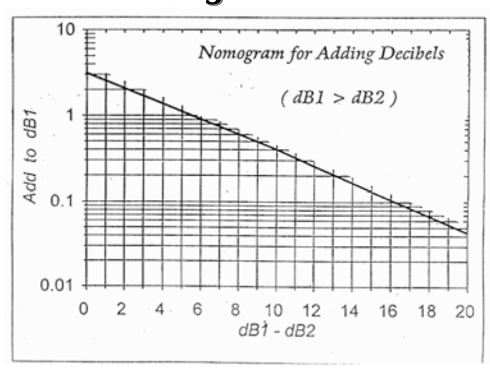
- Two transducers are both transmitting a source intensity level of 90 dB. What is the total source intensity level?

$$\text{If } \; L_1 = L_2 \;\; \text{then } L_{Total} = L_1 \oplus L_2 = L_1 + 3\text{dB}$$

Two Submarines

- If a noisy sub was emitting a source level of 140 dB and a quiet sub was emitting a source level of 80 dB, what is the total noise from the two submarines?

Adding Decibels



Backups

Reference Values

$$I_0 = \frac{p_0^2}{\rho c}$$

Air: $$I_0 = \frac{(20\ \mu\text{Pa})^2}{(1.2\ \text{kg/m}^3)(343\ \text{m/s})} = 1\text{x}10^{-12}\ \text{W/m}^2$$

Water: $$I_0 = \frac{(1\ \mu\text{Pa})^2}{(1000\ \text{kg/m}^3)(1500\ \text{m/s})} = 6.67\text{x}10^{-19}\ \text{W/m}^2$$

Sound Pressure Level

$$L = 10\log\frac{\langle I\rangle}{I_0} = 10\log\frac{\frac{\langle p^2\rangle}{\rho c}}{\frac{p_0^2}{\rho c}} = 10\log\frac{\langle p^2\rangle}{p_0^2} = 10\log\left(\frac{\sqrt{\langle p^2\rangle}}{p_0}\right)^2$$

$$L = 20\log\left(\frac{\sqrt{\langle p^2\rangle}}{p_0}\right) = 20\log\frac{p_{rms}}{p_o}$$

Subtracting Intensity Levels

$$L_2 - L_1 = 10\log\frac{\langle I_2\rangle}{I_0} - 10\log\frac{\langle I_1\rangle}{I_0}$$

$$L_2 - L_1 = 10\log\langle I_2\rangle - 10\log I_0 - \left[10\log\langle I_1\rangle - 10\log I_0\right]$$

$$L_2 - L_1 = 10\log\langle I_2\rangle - 10\log\langle I_1\rangle$$

$$L_2 - L_1 = 10\log\frac{\langle I_2\rangle}{\langle I_1\rangle}$$

$$L_2 - L_1 = 10\log\frac{\frac{\langle p_2^2\rangle}{\rho c}}{\frac{\langle p_1^2\rangle}{\rho c}}$$

$$L_2 - L_1 = 10\log\frac{\langle p_2^2\rangle}{\langle p_1^2\rangle}$$

$$L_2 - L_1 = 20\log\frac{\langle p_2\rangle}{\langle p_1\rangle}$$

Addition

$$"L_1 + L_2" = 10\log\frac{I_{tot}}{I_0}$$

but $I_{tot} = \langle I_1\rangle + \langle I_2\rangle$

$$\frac{L_1}{10} = \log\frac{\langle I_1\rangle}{I_0}$$

$$10^{\frac{L_1}{10}} = \frac{\langle I_1\rangle}{I_0}$$

similarly, $10^{\frac{L_2}{10}} = \frac{\langle I_2\rangle}{I_0}$

therefore $\langle I_1\rangle = I_0 10^{\frac{L_1}{10}}$ and $\langle I_2\rangle = I_0 10^{\frac{L_2}{10}}$

so $I_{tot} = I_0\left(10^{\frac{L_1}{10}} + 10^{\frac{L_2}{10}}\right)$

$$L_{tot} = L_1 \oplus L_2 = 10\log\left(\frac{I_{tot}}{I_0}\right)$$

$$L_{tot} = 10\log\left(10^{\frac{L_1}{10}} + 10^{\frac{L_2}{10}}\right)$$

Speed of Sound in the Sea

The speed of a wave propagating through a medium is not a constant. This is especially true for the non-homogeneous medium, the ocean. The speed of sound through water has been found to be mainly a function of three factors. They are **temperature, pressure or depth and salinity**. Because the speed is not constant, sound does not travel along straight paths.

Temperature

In general, for most areas of the ocean, the water temperature decreases from the surface to the bottom, but there are many local variations. Shallow layers see the most variation with time and depth (ie. Surface mixing, solar heating, currents, seasonal variations, etc). In vary deep water, the temperature eventual becomes constant with depth at about 4 C.

Depth

Hydrostatic pressure makes sound velocity increase with depth because of variations in the bulk modulus, *B*. This effect is linear in the first approximation with an increase of 0.017 m/s per meter increase in depth.

Recall in Physics I we showed that pressure varies with depth according to the simple formula,

$$P = P_o + \rho g h$$

Leroy formula (1968) gives a precise hydrostatic pressure:

$$P = \left[1.0052405\left(1 + 5.28 \times 10^{-3} \sin\phi\right) z + 2.36 \times 10^{-6} z^2 + 10.196\right] \times 10^4 \text{ Pa}$$

ϕ - latitude in degrees

z - depth in meters

(From: Lurton, X. An Introduction to Underwater Acoustics, 1st ed. London, Praxis Publishing LTD, 2002, p37)

Salinity

The change in the mix of pure water and dissolved salts effects sound velocity. Salinity is expressed in practical salinity units (p.s.u.). These unit have the same magnitude as the traditional parts per thousand (‰). Most oceans have a salinity of 35 p.s.u., although salinity can vary locally based on hydrological conditions. Closed seas have a greater difference in their salinity (38 p.s.u. for Mediterranean Sea due to evaporation, 14 p.s.u. for Baltic Sea due to large freshwater input). Salinity varies very little with depth, but there can be stronger variations near river estruaries, melting ice, etc.

Velocity Models

In the 1940's, sound velocity variations and their affect on acoustic propagation were first noticed and studied. It is very difficult to locally measure sound velocity, but easy to measure the parameters that affect it (temperature, salinity, and depth). Several models have been created to predict sound velocity. A good first approximation is that developed by Medwin (1975). It is simple but limited to 1000 meters in depth:

$$c(t,z,S) = 1449.2 + 4.6t - 5.5x10^{-2}t^2 + 2.9x10^{-4}t^3 + (1.34 - 10^{-2}t)(S-35) + 1.6x10^{-2}z$$

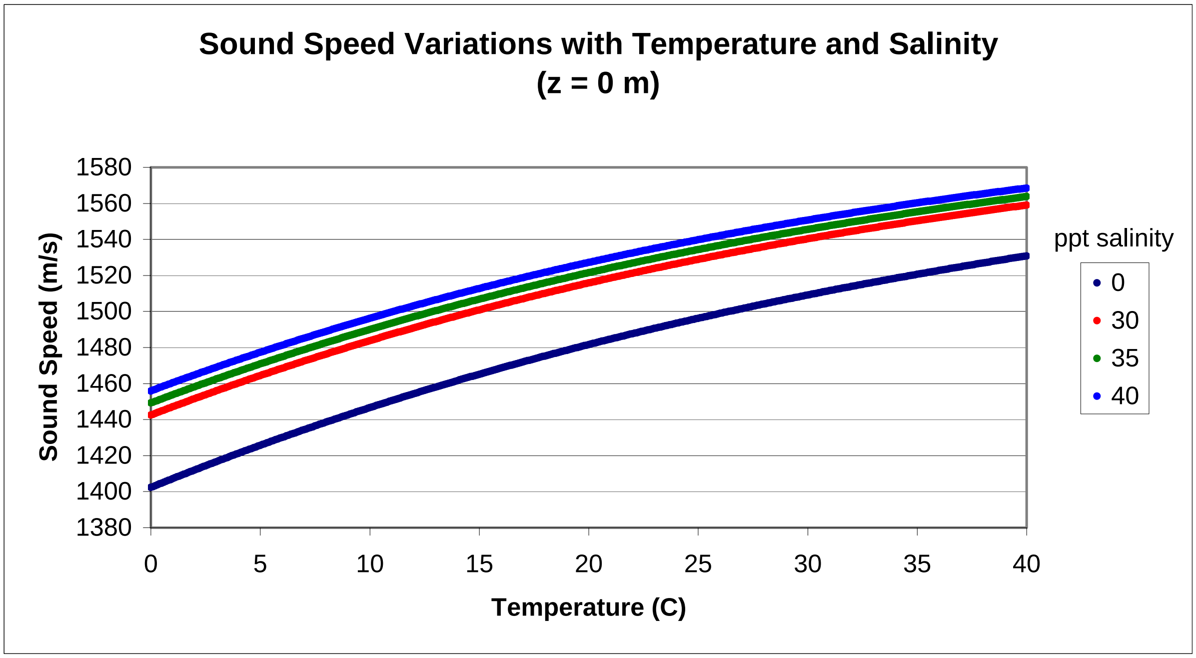
with the following limits:

$0 \le t \le 35°$ C

$0 \le S \le 45$ p.s.u.

$0 \le z \le 1000$ meters

Where c is the speed of sound as a function of temperature, t, depth, z, and salinity, S.



(From: Lurton, X. An Introduction to Underwater Acoustics, 1st ed. London, Praxis Publishing LTD, 2002, p37)

More recent and accurate models have been developed and include Chen and Millero (1977). Their model is endorsed by UNESCO and used as the standardized reference model:

$$c = c_0 + c_1P + c_2P^2 + c_3P^3 + AS + BS^{\frac{3}{2}} + CS^2$$

P = Pressure from Leroy Formula

$$c_o = 1402.388 + 5.03711t - 5.80852x10^{-2}t^2 + 3.3420x10^{-4}t^3 - 1.478x10^{-6}t^4 + 3.1464x10^{-9}t^5$$

$$c_1 = 0.153563 + 6.8982x10^{-4}t - 8.1788x10^{-6}t^2 + 1.3621x10^{-7}t^3 - 6.1185 + 1.3621x10^{-10}t^4$$

$$c_2 = 3.126x10^{-5} - 1.7107x10^{-6}t + 2.5974x10^{-8}t^2 - 2.5335x10^{-10}t^3 + 1.0405x10^{-12}t^4$$

$$c_3 = -9.7729x10^{-9} + 3.8504x10^{-10}t - 2.3643x10^{-12}t^2$$

$$A = A_o + A_1P + A_2P^2 + A_3P^3$$

$$A_1 = 9.4742x10^{-5} - 1.258x10^{-5}t - 6.4885x10^{-8}t^2 + 1.0507x10^{-8}t^3 - 2.0122x10^{-10}t^4$$

$$A_2 = -3.9064x10^{-7} + 9.1041x10^{-9}t - 1.6002x10^{-10}t^2 + 7.988x10^{-12}t^3$$

$$A_3 = 1.1x10^{-10} + 6.649x10^{-12}t - 3.389x10^{-13}t^2$$

$$B = -1.922x10^{-2} - 4.42x10^{-5}t + \left(7.3637x10^{-5} + 1.7945x10^{-7}t\right)P$$

$$C = -7.9836x10^{-6}P + 1.727x10^{-3}$$

Where,

t - temperature (° C)
z - depth (m)
S - salinity (p.s.u.)

As you can see, the speed of propagation has a very complicated dependence on these three factors. Some thumbrules that you can use to relate the dependence of the speed of sound in seawater to each of the factors are:

$$1°\text{ C increase in temperature} \Rightarrow 3\text{ m/s increase in speed}$$
$$100\text{ meters of depth} \Rightarrow 1.7\text{ m/s increase in speed}$$
$$1\text{ ppt increase in salinity} \Rightarrow 1.3\text{ m/s increase in speed}$$

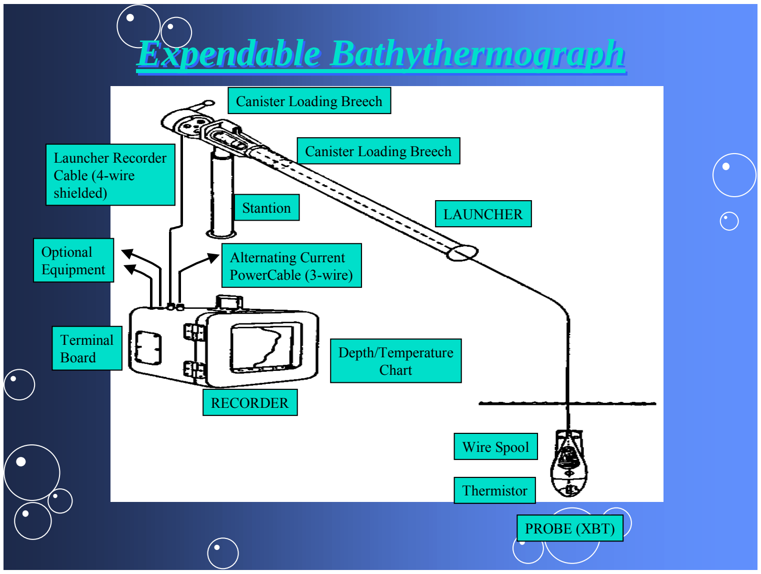
(From: Principles of Naval Weapons Systems, Edited by Joseph B. Hall, CDR, USN, Dubuque, IA: Kendall/Hunt Publishing Co, 2000, p.179)

Seawater contains many inhomogenieties, including bubble layers close to the surface, mineral particles in suspension, and living organisms. These are all potential scatterers of acoustic waves, especially at higher frequencies.

Measuring the Speed of Sound in the Ocean

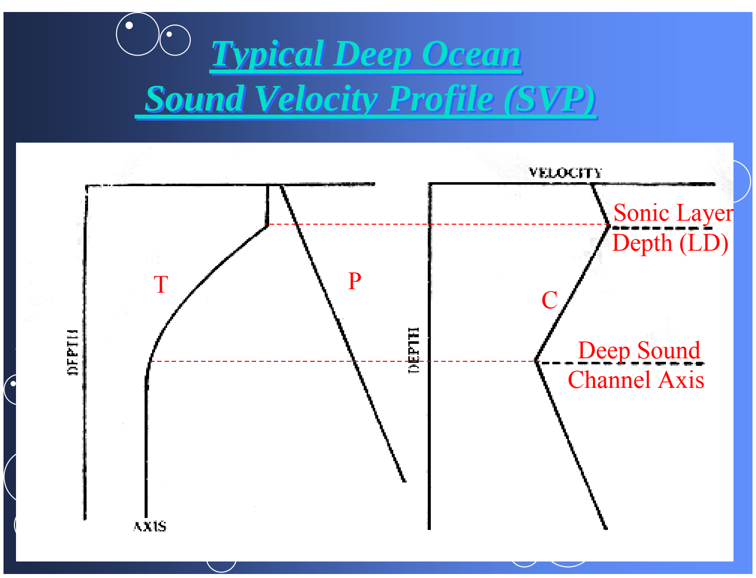
To predict the direction of propagation of a sound wave in the ocean, we must know the speed of sound as a function of position (or depth) in the ocean water. To measure the speed of sound in water, the Navy has developed several tools to measure the temperature of the seawater as a function of depth or the velocity of sound directly.

The most widely used tool is an Expendable BathyThermograph or XBT (picture compliments of ES419). XBTs are launched from submarines, surface ships and even aircraft. These measure the temperature of the water as the device sinks at a known rate and transmits this back to the launching platform. This provides a detailed plot of temperature as a function of depth. Neglecting salinity, the Sound Velocity Profile or SVP can be calculated as a function of depth and temperature (since these cause the greatest variation in the speed of sound in seawater.)



Many modern submarines are often equipped with velocimeters that calculate the speed of sound *in situ*. Other submarines have systems that calculate and record sound speed using temperature and depth measurements from onboard ships instruments.

Expendable Bathythermographs produce graphs of water temperature and sound speed as a function of water depth as seen below. In the next lesson we will examine typical plots in more detail for tactical significance. For now you should familiarize yourself with the basic shape of these typical plots.

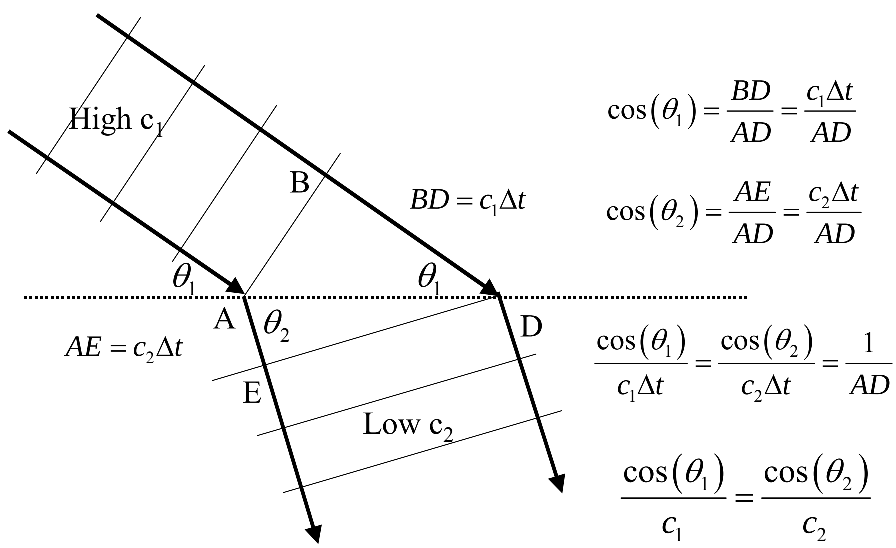


Using a Sound Velocity Profile and Snell's Law

We will now shift from thinking of sound as a wave and using the wave equations to sound as a ray and using Snell's Law. We can look at either the grazing angles, referenced to the horizontal and used when looking at refraction, or incidence angles, referenced to the vertical and used for refraction and backscattering.

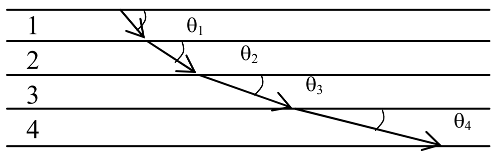
In the below sketch, a plane wave is moving towards a boundary beyond which the speed of sound is much slower. As the wavefronts hit the boundary they slow down and bend more normal to the boundary. Specific examination of the wave after the right edge hits the boundary at point A shows that the left side of the wavefront must travel a distance from B to D expressed as the product of the sound speed c_1 and some time interval Δt. In that same time interval the right edge of the wave front moves from A to E expressed as the product of sound speed c_2 and

Refraction



some time interval Δt. Using trigonometry we see that the ratio of the cosine of the grazing angle to the speed of sound remains constant across the boundary. This observation is called Snell's Law.

Snell's law and ray theory are well suited for each other. Imagine that a sound ray is transmitted through a series of mediums label 1 through 4 with sequentially increasing sound speed. In each medium, the angle the ray makes with the horizontal, θ, will depend on the angle it has in the previous medium and the speed of sound for each medium. The figure below depicts the relation.



where $c_1 < c_2 < c_3 < c_4$ and $\theta_1 > \theta_2 > \theta_3 > \theta_4$

According to Snell's Law

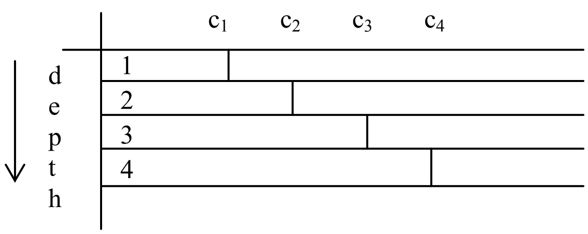
$$\frac{cos(\theta_1)}{c_1} = \frac{cos(\theta_2)}{c_2} = \frac{cos(\theta_3)}{c_3} = ... = \frac{cos(\theta_n)}{c_n} = \text{constant}$$

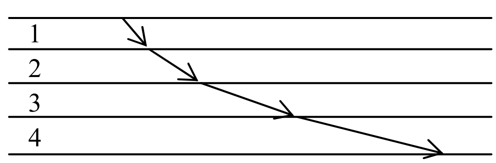
Notice that when a ray is in a layer and horizontal, $\theta = 0°$ and the $\cos(\theta) = 1$. We call the speed of sound when the ray is horizontal, c_o.

Sound Rays Travel in Arcs

Using Snell's Law from above, we can approximate the behavior of a sound ray as it travels through a medium where the speed of sound is changing at a constant rate. Let's take the example where the speed of sound increases as a function of depth as shown.

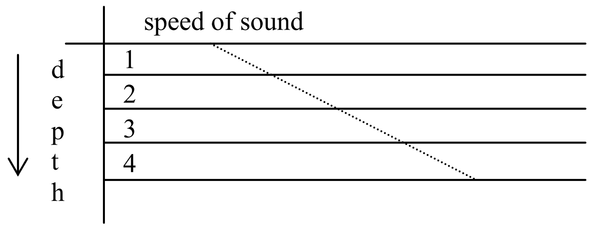
If the speed of sound increased in each layer as shown, a sound ray would travel in a path the same as the one already shown.





(Notice that the sound ray is bending back towards the layers where the sound speed is lower. This can be used later to qualitatively determine the ray path for sound in water.)

More realistically though, the speed of sound changes as a continuous function. If we use a continuous function instead of the step function for the speed of sound vs. depth, the speed of sound as a function of depth can be described by a simple linear equation. This result can be used to find functions for the radius of the path of the sound ray as well as other quantities.



(Korman, M.S. Principles of Underwater Sound and Sonar, the preliminary edition. Dubuque, IA: Kendall/Hunt Publishing Company, 1995, pgs 145-147)

The speed of sound, shown as the dotted line, can be expressed as (c_1 is the surface temperature):

$$c = c_1 + gz$$

where g is the gradient, $g = \frac{\Delta c}{\Delta z}$. From Snell's Law and inserting our relation ship for c, yields:

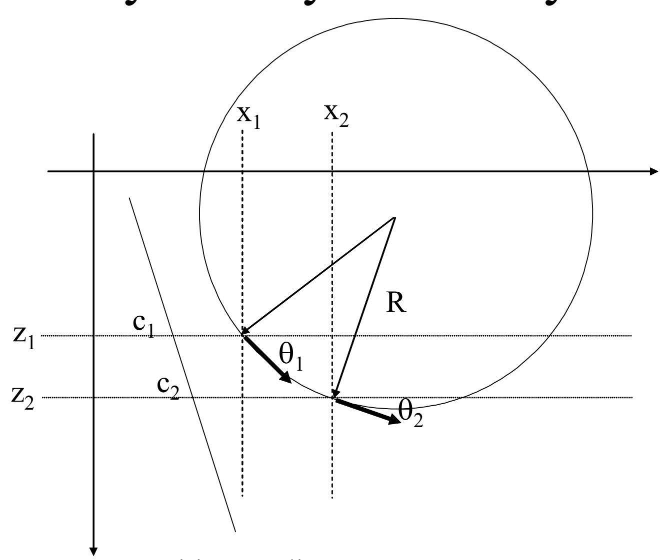
$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c}$$

$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta_2}{c_1 + gz}$$

$$z = R\left(\cos\theta - \cos\theta_1\right)$$

where R is defined as: $R \equiv \frac{c_1}{g\cos\theta_1}$. Soon we will show R is the radius of curvature of the sound ray. θ is always measured clockwise from the horizontal axis.

Ray Theory Geometry



Positive gradient, g

In polar coordinates we know that the slope of a line is

$$\frac{dz}{dx} = \tan\theta$$

From above we see that $dz = -R\sin\theta d\theta$. To find how the ray angle varies with distance x,

$$dx = -R\frac{\sin\theta}{\tan\theta}d\theta = -R\cos\theta d\theta$$

Integrating both sides gives the result that:

$$x - x_1 = -R\left[\sin\theta - \sin\theta_1\right]$$

Integrating both sides of dz gives:

$$z - z_1 = R\left[\cos\theta - \cos\theta_1\right]$$

Rearranging these two equations:

$$x - x_1 - R\sin\theta_1 = -R\sin\theta$$

$$z - z_1 + R\cos\theta_1 = R\cos\theta$$

Or

$$x - x_p = -R\sin\theta$$

$$z - z_p = R\cos\theta$$

With

$$x_p = x_1 - R\sin\theta_1$$

$$z_p = z_1 + R\cos\theta_1$$

Squaring the top two equations and adding the results gives the equation of a circle,

$$\left(x - x_p\right)^2 + \left(z - z_p\right)^2 = R^2$$

Specifically, the circle has radius, $R \equiv \frac{c_1}{g\cos\theta_1}$, and is centered at the point (x_p,z_p). Thus we have shown that a sound ray in a layer of constant sound speed will travel along the arc of a circle.

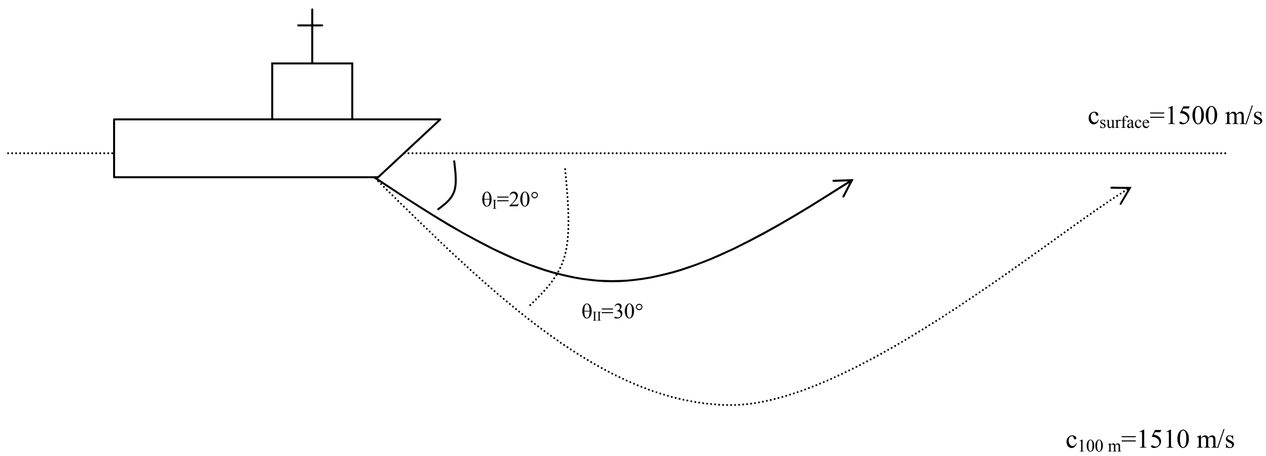
To sum up the results then:

$R = \dfrac{c_n}{g\,cos\,\theta_n}$	Radius of arc of the circle
$g = \dfrac{\Delta c}{\Delta z}$	Gradient (How fast the speed of sound changes per meter change in depth.)
$\dfrac{c_n}{cos\,\theta_n} = \text{constant}$	Snell's Law
$\Delta z = z_2 - z_1 = R(cos\,\theta_2 - cos\,\theta_1)$	Vertical Displacement
$\Delta x = x_2 - x_1 = -R(sin\,\theta_2 - sin\,\theta_1)$	Horizontal Displacement
$t_n = \left\| \dfrac{1}{g_n} \ln\left(\dfrac{\tan\left(\frac{\theta_n}{2}\right)}{\tan\left(\frac{\theta_{n-1}}{2}\right)} \right) \right\|$	Time to travel in layer n
$s_n = \dfrac{c_{n-1}}{g\cos\theta_{n-1}}(\theta - \theta_{n-1})$	Curvilinear Path Length

These equations will only work for one specific sound ray emanating from a source in an environment with a constant gradient. The last two equations in the table are presented without proof, but are useful results from many standard sources.

Example 1

Let’s look at the following example.



Sound leaves the ship at two different angles, θ_1 and θ_2. Note the path travelled by each ray is different and if we calculate the parameters R, Δx and Δz, each of these will be different for each ray.

For both rays, the gradient, g, is a constant. This is calculated as such:

$$g = \frac{\Delta c}{\Delta z} = \frac{(1500 - 1510)\text{m/s}}{(0 - 100)\text{m}}$$

$$g = 0.1\,\text{sec}^{-1}$$

We must now calculate the radius of curvature, R of each ray separately:

$$R_I = \frac{c}{g\cos\theta_1} = \frac{1500\text{ m/s}}{(0.1\,\text{sec}^{-1})(\cos 20°)}$$

$$R_I = 16{,}000\text{ meters}$$

and

$$\text{R}_{\text{II}} = \frac{\text{c}}{\text{g}\cos\theta_{\text{II}}} = \frac{1500\text{ m/s}}{(0.1\text{ sec}^{-1})(\cos 30°)}$$

$$\text{R}_{\text{II}} = 17{,}300\text{ meters}$$

The skip distance, X, is the distance between successive places where the sound ray stikes the surface. The easiest way to calculate this is to calculate the displacement, Δx, from where the sound strikes the surface first to where the sound has leveled off or gone horizontal (θ_2 = 0°). Thus:

$$X = 2\Delta x = -2R(\sin 0° - \sin\theta)$$

$$X = 2R\sin\theta$$

where θ is the angle of reflection from the surface. So for each ray:

$$X_I = 2(16{,}000\ \text{m})\sin 20°$$
$$X_I = 11{,}000\ \text{m}$$

and

$$X_{II} = 2(17{,}300\ \text{m})\sin 30°$$
$$X_{II} = 17{,}300\ \text{m}$$

The results of the calculations for each ray are significantly different from each other and show how the ray paths depend on the initial angle of the ray.

We can do the same for the depth the rays to. The maximum depth excursion of the ray below its starting depth occurs when the ray goes horizontal again ($\theta_2 = 0°$) or:

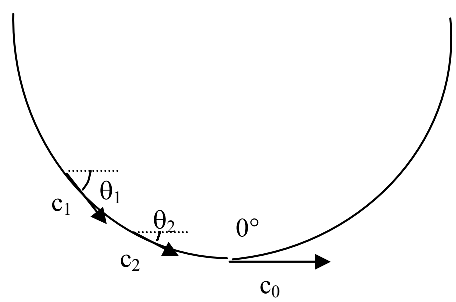
$$\Delta z_{\max} = R(\cos 0° - \cos\theta)$$
$$\Delta z_{\max} = R(1 - \cos\theta)$$

so for each ray:

$$\Delta z_{\max,I} = 965\ \text{m}\ (3170\ \text{feet})$$
$$\Delta z_{\max,II} = 2320\ \text{m}\ (7600\ \text{feet})$$

Example 2

Also try the following example problem.[1]



Use the figure above and the following information to answer the questions.

a. If $\theta_2 = 30°$, $c_2 = 1299$ m/s, $c_1 = 964$ m/s, what is θ_1?

Using Snell's Law we have:

$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta_2}{c_2}$$
$$\cos\theta_1 = c_1\frac{\cos\theta_2}{c_2}$$
$$\theta_1 = \cos^{-1}\left[\frac{964}{1299}\cos 30°\right] = 50°$$

[1] From: Korman, M. S. Principles of Underwater Sound and Sonar, the preliminary edition, p. 144.

b. Determine c_0.

Again using Snell's law and the $\theta_0 = 0°$

$$\frac{c_2}{\cos\theta_2} = c_0$$

$$c_0 = \frac{1299\, m/s}{\cos 30°} = 1500\text{ m/s}$$

c. What is the gradient if Δz = 3000 m between points "1" and "0"?

$$g = \frac{\Delta c}{\Delta z} = \frac{c_0 - c_1}{\Delta z} = \frac{1500 m/s - 964 m/s}{3000 m}$$

$$= .18\text{ s}^{-1}$$

d. What is the radius of the sound ray path?

$$R = \frac{c_1}{g\cos\theta_1}$$

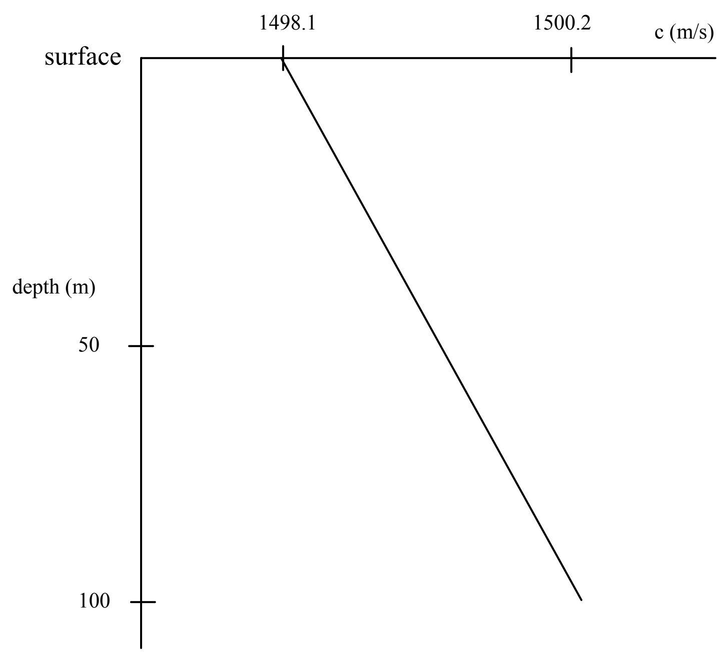
$$R = \frac{964\text{ m/s}}{\left(0.18\text{ s}^{-1}\right)\left(\cos(50°)\right)}$$

$$R = 8330\text{ meters}$$

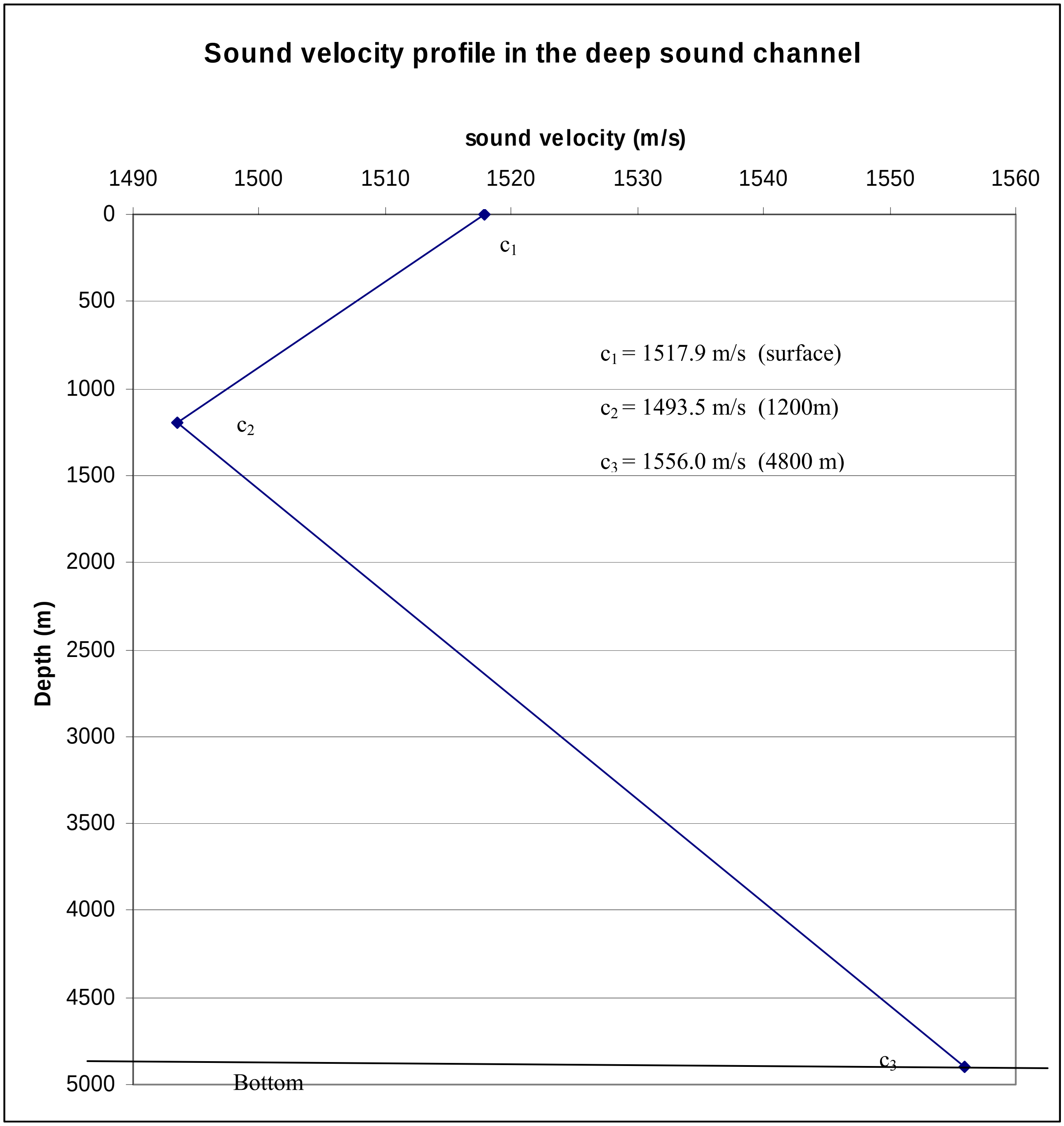
Problems

1. A submerged submarine is at 10 meters.
   a) Use Medwin's Equation to determine the speed of sound in the water if the salinity is 35.0 ppt and the seawater injection temperature is 20.0°C.
   b) If the submarine in the problem above submerges to 200. meters and the seawater injection temperature goes down to 5.00°C, what is the new sound speed?
   c) Determine the average gradient between the two depths in the problems above.
   d) Is this a positive or negative sound gradient?
   e) Sketch the SVP and sketch the approximate of several sound rays emanating from the sub when it is at a depth of 10.0 meters.
   f) If sound radiates from the sub at a depth of 10.0 m at an initial angle of 15° with respect to the horizontal, determine the angle of depression of the sound when it has reached a depth of 500. meters (assume the gradient is constant.)
   g) Determine the Radius of Curvature of the sound ray.
   h) Determine the horizontal displacement of the sound ray as it goes from 10.0 meters to 500. meters.

2. Use the following SVP to complete the next problems:
   a) Calculate the gradient of the SVP.
   b) If a sound ray exits horizontally from a sub that is at 50.0 m, what will be its grazing angle when it hits the surface of the ocean?
   c) If a ray reflects off the surface of the ocean at an angle of 2.15° (assume the surface is perfectly flat), what will be the skip distance of the sound ray?
   d) This is an example of:
      i) a positive gradient
      ii) a negative gradient
   e) If a sub is at 50.0 m, what is the largest angle below the horizontal where the ray will **not** reach 100. m?
   f) What is the skip distance of the limiting ray?



Use the following SVP for the remaining problems:



3. Compute the sound speed gradients for $0 < z < 1200$ m and $1200 < z < 4800$.

4. A ray starts at 1525 m with a grazing angle if 15 degrees (recall a positive grazing angle is pointed below the horizontal).
   a) What are the sound speeds at depths of 1525 m and 2440 m?
   b) Does the ray curve upward or downward?
   c) What is the grazing angle at 2440 m?
   d) At what sound speed will the ray become horizontal (a horizontal ray has a grazing angle of 0 degrees)?

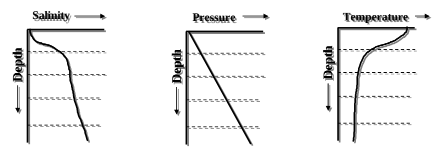
5. A sound source is at a depth of 1200 m.
   a) At what angle with respect to the horizontal does the ray have to make at 1200 m so that that when it reaches the surface the grazing angle is 0 degrees? This is called the surface limiting ray.
   b) What angle with respect to the horizontal does a ray have to make at 1200 m so that when it reaches the bottom at 4900 m, the angle is 0 degrees? This is called the bottom limiting ray.
   c) At what depth below 1200 m is the sound speed equal to that at the surface?
   d) At what angle with respect to the horizontal does a ray have to make at 1200 m so that when it reaches the the depth found in c), the grazing angle is 0 degrees? This is called the lower limiting ray
   e) Compute the radius of the surface limiting ray.
   f) Compute the radius of the bottom limiting ray.
   g) Compute the radius of the lower limiting ray.
   h) Compute the horizontal distance that the bottom limiting ray travels from the source until it grazes the bottom.
   i) Compute the horizontal distance that the surface limiting ray travels from the source until it just grazes the surface.

6. A ray leaving a sound source at 1200 m points downward with an angle of 30 degrees with respect to the surface.
   a) How far will it travel horizontally until its angle with the horizontal is 25 degrees?
   b) At what depth does the ray in a) make an angle of 25 degrees with respect to the horizontal.

Speed of Sound in Water

Medium Effects: Elasticity and Density

Variable Effects of:



Speed of Sound Factors

- Temperature
- Pressure or Depth
- Salinity

1° C increase in temperature $\Rightarrow$ 3 m/s increase in speed

100 meters of depth $\Rightarrow$ 1.7 m/s increase in speed

1 ppt increase in salinity $\Rightarrow$ 1.3 m/s increase in speed

Temperature, Pressure, and Salinity

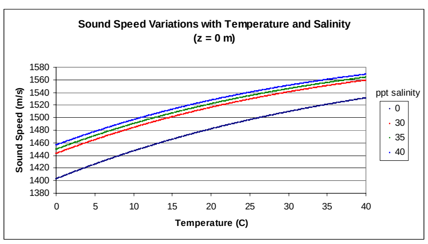
$$c(t,z,S) = 1449.2 + 4.6t - 5.5\text{x}10^{-2}t^2 + 2.9\text{x}10^{-4}t^3 + \left(1.34 - 10^{-2}t\right)(S-35) + 1.6\text{x}10^{-2}z$$

with the following limits:

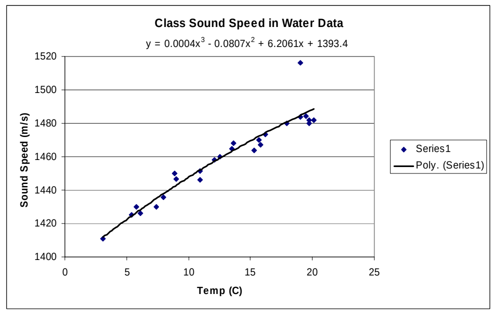
$0 \le t \le 35°$ C

$0 \le S \le 45$ p.s.u.

$0 \le z \le 1000$ meters



Class Sound Speed Data



More Curve Fitting

Chen and Millero

$$c = c_0 + c_1P + c_2P^2 + c_3P^3 + AS + BS^{\frac{3}{2}} + CS^2$$

P = Pressure from Leroy Formula

$$c_o = 1402.388 + 5.03711t - 5.80852\text{x}10^{-2}t^2 + 3.3420\text{x}10^{-4}t^3 - 1.478\text{x}10^{-6}t^4 + 3.1464\text{x}10^{-9}t^5$$

$$c_1 = 0.153563 + 6.8982\text{x}10^{-4}t - 8.1788\text{x}10^{-6}t^2 + 1.3621\text{x}10^{-7}t^3 - 6.1185 + 1.3621\text{x}10^{-10}t^4$$

$$c_2 = 3.126\text{x}10^{-5} - 1.7107\text{x}10^{-6}t + 2.5974\text{x}10^{-8}t^2 - 2.5335\text{x}10^{-10}t^3 + 1.0405\text{x}10^{-12}t^4$$

$$c_3 = -9.7729\text{x}10^{-9} + 3.8504\text{x}10^{-10}t - 2.3643\text{x}10^{-12}t^2$$

$$A = A_o + A_1P + A_2P^2 + A_3P^3$$

$$A_1 = 9.4742\text{x}10^{-5} - 1.258\text{x}10^{-5}t - 6.4885\text{x}10^{-8}t^2 + 1.0507\text{x}10^{-8}t^3 - 2.0122\text{x}10^{-10}t^4$$

$$A_2 = -3.9064\text{x}10^{-7} + 9.1041\text{x}10^{-9}t - 1.6002\text{x}10^{-10}t^2 + 7.988\text{x}10^{-12}t^3$$

$$A_3 = 1.1\text{x}10^{-10} + 6.649\text{x}10^{-12}t - 3.389\text{x}10^{-13}t^2$$

$$B = -1.922\text{x}10^{-2} - 4.42\text{x}10^{-5}t + \left(7.3637\text{x}10^{-5} + 1.7945\text{x}10^{-7}t\right)P$$

$$C = -7.9836\text{x}10^{-6}P + 1.727\text{x}10^{-3}$$

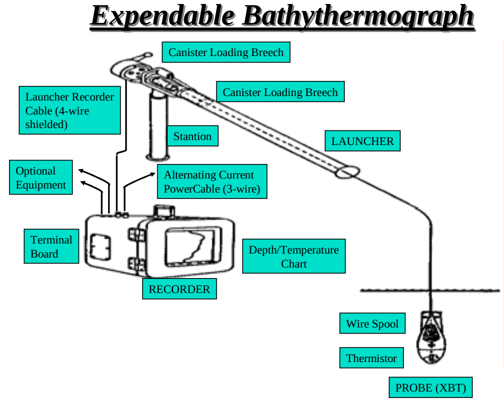
Leroy

$$P = \left[1.0052405\left(1 + 5.28\times10^{-3}\sin\phi\right)z + 2.36\times10^{-6}z^2 + 10.196\right]\times10^4 \text{ Pa}$$

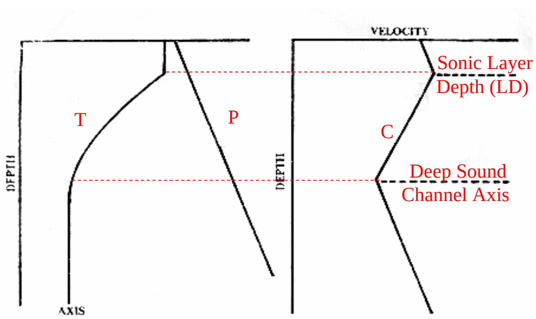
ϕ - latitude in degrees

z - depth in meters

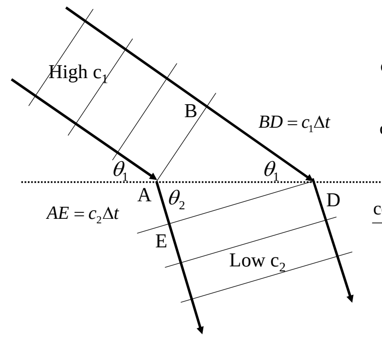
Expendable Bathythermograph



Typical Deep Ocean Sound Velocity Profile (SVP)



Refraction



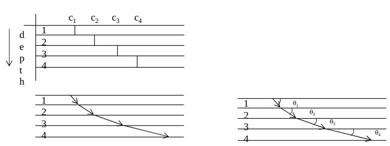
$$\cos(\theta_1) = \frac{BD}{AD} = \frac{c_1\Delta t}{AD}$$

$$\cos(\theta_2) = \frac{AE}{AD} = \frac{c_2\Delta t}{AD}$$

$$\frac{\cos(\theta_1)}{c_1\Delta t} = \frac{\cos(\theta_2)}{c_2\Delta t} = \frac{1}{AD}$$

$$\frac{\cos(\theta_1)}{c_1} = \frac{\cos(\theta_2)}{c_2}$$

Multiple Boundary Layers



where $c_1 < c_2 < c_3 < c_4$ and $\theta_1 > \theta_2 > \theta_3 > \theta_4$

$$\frac{\cos(\theta_1)}{c_1} = \frac{\cos(\theta_2)}{c_2} = \frac{\cos(\theta_3)}{c_3} = \ldots = \frac{\cos(\theta_n)}{c_n} = \text{constant}$$

Simple Ray Theory

$$\text{gradient} = g = \frac{\Delta c}{\Delta z} = \frac{c - c_1}{z - z_1}$$

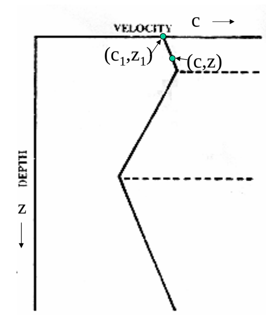
$$c = c_1 + gz$$

Snell's Law

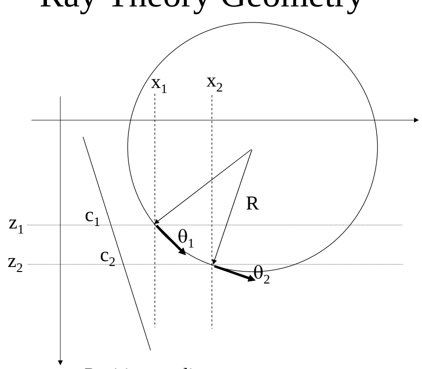
$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c}$$

$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c_1 + gz}$$

$$z = R\left(\cos\theta - \cos\theta_1\right) \qquad R = \frac{c_1}{g\cos\theta_1}$$

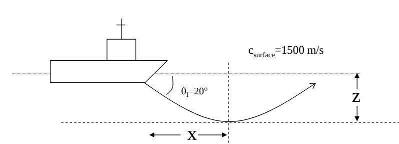


Ray Theory Geometry



Positive gradient, g

The **z (Depth)** and x (Range) Directions



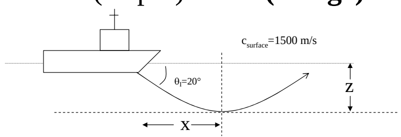
$$R = \frac{c_1}{g\cos\theta_1}$$

$$z = R\left(\cos\theta - \cos\theta_1\right) \qquad dz = -R\sin\theta d\theta$$

$$\int_{z_1}^{z} dz = -R\int_{\theta_1}^{\theta} \sin\theta d\theta$$

$$\boxed{z - z_1 = R\left(\cos\theta - \cos\theta_1\right)}$$

The z (Depth) and **x (Range)** Directions



$$R = \frac{c_1}{g\cos\theta_1}$$

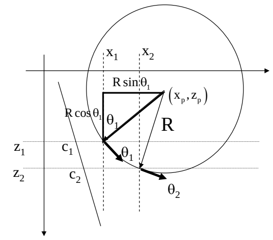
$$z = R\left(\cos\theta - \cos\theta_1\right) \qquad dz = -R\sin\theta d\theta$$

$$\tan\theta = \frac{dz}{dx}$$

$$\int_{x_1}^{x} dx = \int_{z_1}^{z} \frac{dz}{\tan\theta} = -R\int_{\theta_1}^{\theta} \frac{\sin\theta d\theta}{\tan\theta} = -R\int_{\theta_1}^{\theta} \cos\theta d\theta$$

$$x - x_1 = -R\left(\sin\theta - \sin\theta_1\right)$$

Why is R = Radius?



Positive gradient, g

$$x - x_1 = -R\left(\sin\theta - \sin\theta_1\right)$$
$$z - z_1 = R\left(\cos\theta - \cos\theta_1\right)$$

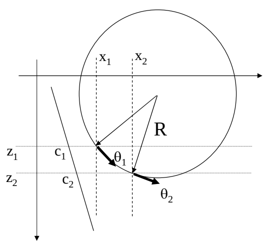
$$x - x_p = -R\sin\theta$$
$$z - z_p = R\cos\theta$$

$$x_p = x_1 + R\sin\theta_1$$
$$z_p = z_1 - R\cos\theta_1$$

$$\left(x - x_p\right)^2 + \left(z - z_p\right)^2 = \left(-R\right)^2 \sin^2\theta + R^2\cos^2\theta$$

$$\left(x - x_p\right)^2 + \left(z - z_p\right)^2 = R^2$$

Summary



Positive gradient, g

$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c}$$

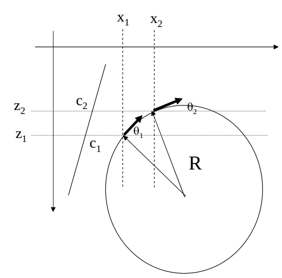
$$x - x_1 = -R\left(\sin\theta - \sin\theta_1\right)$$

$$z - z_1 = R\left(\cos\theta - \cos\theta_1\right)$$

$$R = \frac{c_1}{g\cos\theta_1}$$

$$g = \frac{\Delta c}{\Delta z} = \frac{c - c_1}{z - z_1}$$

Negative Gradient



Negative gradient, g

$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c}$$

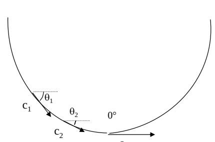
$$x - x_1 = -R\left(\sin\theta - \sin\theta_1\right)$$

$$z - z_1 = R\left(\cos\theta - \cos\theta_1\right)$$

$$R = \frac{c_1}{g\cos\theta_1}$$

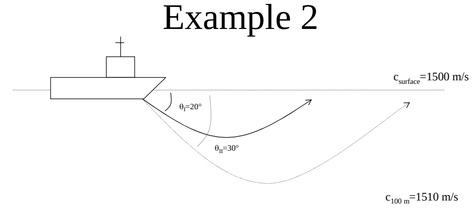
$$g = \frac{\Delta c}{\Delta z} = \frac{c - c_1}{z - z_1}$$

Example 1



- Given: c_1 = 964 m/s, c_2 = 1299 m/s, θ_2 = 30°
  Δz(between 1 and 0) = 3000m
- Find: θ_1, c_o, g (between pt 1 and 0), R

Example 2



- Find gradient, g
- Find Radius of Curvature, R, for each ray.
- Skip distance – i.e. the distance until the ray hits the surface again
- Max depth reached by each ray

Backups

$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c}$$

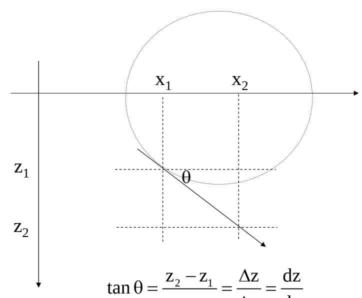
$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c_1 + gz}$$

$$c_1\cos\theta_1 + gz\cos\theta_1 = c_1\cos\theta$$

$$z = \frac{c_1}{g\cos\theta_1}\left(\cos\theta - \cos\theta_1\right)$$

$$\boxed{z = R\left(\cos\theta - \cos\theta_1\right)} \qquad \boxed{R = \frac{c_1}{g\cos\theta_1}}$$

Slope = tanθ

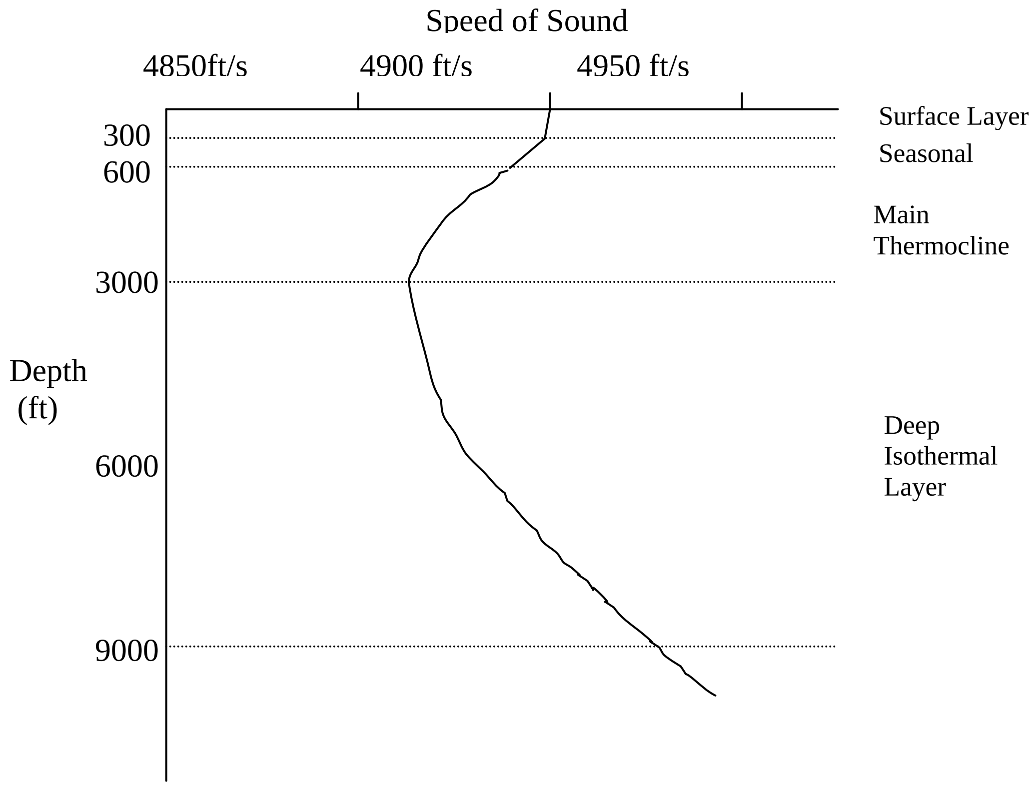


$$\tan\theta = \frac{z_2 - z_1}{x_2 - x_1} = \frac{\Delta z}{\Delta x} = \frac{dz}{dx}$$

Regions of the Sound Velocity Profile

When we plot the speed of sound as a function of depth in the ocean, it is called the sound velocity profile or SVP. An SVP is a very useful tool for being able to predict the path of propagation of sound in the ocean. A nominal SVP is shown in the following figure.

This example shows that the oceans waters are divided into three main layers. First, the surface or seasonal layer has varying depth and profiles depending on many conditions such as



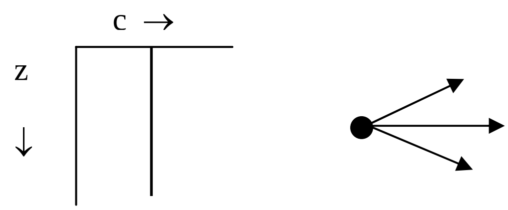
season, time of day, currents, latitude, etc. It is the most variable layer. The second layer, the Main Thermocline connects the seasonal layer to the deep isothermal layer. The deep isothermal layer, below about 500 to 1000 meters, is at a temperature of about 34°F and the speed of sound only increases due to the increase in pressure.

The layer of most interest is the surface layer because it is the one that varies most. During the warm summer months, the water near the oceans surface is warmer than the water below and there is a sharp negative gradient in the speed of sound. In the winter, the water is not heated as much because the air is cooler and the warmer water from below tends to rise and create more mixing in the upper layer. Additionally, there is more mixing of the surface layer due to effects of strong winter storms and larger waves.

Ray Tracing

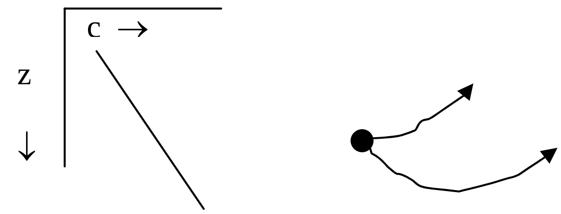
Now that we know how the speed of sound varies as a function of depth, we can begin to predict the path of sound propagating through the ocean. As we said before, sound rays tend to bend toward layers of water that are cooler (where speed of sound in the water is lower.) Let's develop some thumbrules for picturing the path of the sound rays. Knowing the gradient gives us insight into the direction the sound ray will refract or bend.

Constant Velocity Profile, $g = 0$

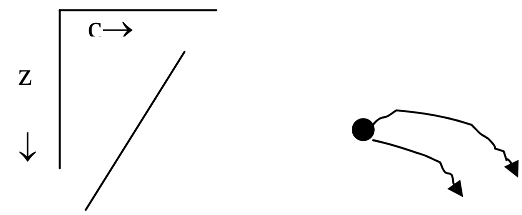


Acoustic rays are rectilinear, meaning rays travel in straight paths. This is not an isothermal layer. An isothermal layer would have a slight positive gradient due to the effect of the increase in pressure with increasing depth on the speed of sound.

g = "+"



Typical for the deep isothermal layer or surface layer in the winter. Temperature is constant or increasing. For isothermal case, c increases due to pressure with $g = 0.017\ s^{-1}$. Most often the water is isothermal because of the mixing effect of wind near the surface. Because of this isothermal layers are called “mixed layers.” + g causes acoustic rays to be refracted upward and can result in a Surface Channel.



g = "-"

This is a typical SVP for the surface layer during the summer months. Results from temperature decreasing faster than pressure effect increases. A negative gradient produces shadow zones.

Note that when sound is generated by a source in the ocean, the sound is radiated in all directions spherically around the source. Thus sound may travel several different paths away from the source and may travel into other layers. Some common propagation modes of sound are shown below:

Surface Duct

If the surface layer has a positive gradient and that layer is deep enough, the sound may be bent back towards the surface then reflected back into the layer. After it is reflected back downward, it is bent back towards the surface again only to be reflected at the surface again. This effectively traps the sound in the surface layer.

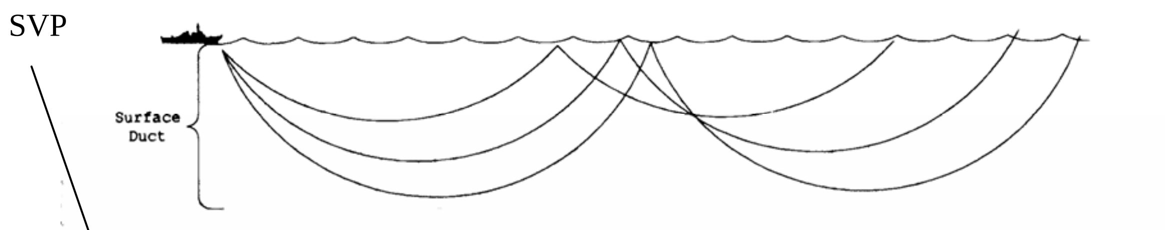
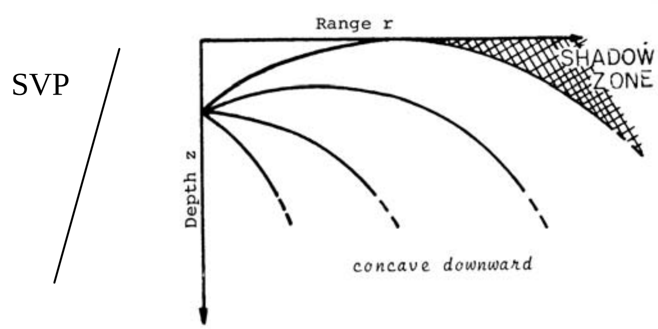
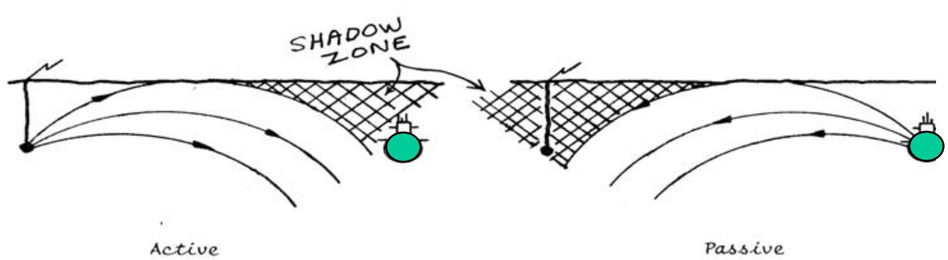


Figure 15-11. Surface duct.

If, on the other hand, a negative SVP exists near the surface, surface shadow zones are created that provide safe havens for submerged platforms.



Originally referred to as the afternoon effect, the below sketch shows how a submarine cannot be detected by either a passive or an active sonobuoy.



Convergence Zone

The phenomenon of sound bending back towards the surface in a surface duct should not be confused with sound that is bent back towards the surface due to the positive sound speed gradient in the **deep isothermal layer**. If the ocean depth is great enough, sound rays that travel into the deep isothermal layer will also get bent back towards the surface. These rays travel great distances though before being bent back up to the surface.

The main difference between a surface duct and a convergence zone (CZ) is that in the latter case, all sound rays return to the surface in a small concentrated area called a convergence zone. These zones can be at distances of up to 60 km or more from the source (typically 40 km in Mediterranean Sea and 60 km in Atlantic Ocean). In other words, the rays all follow more or less the same path and undergo very small geometric spreading, therefore a contact may only be detected when it is within the annulus of the convergence zone. Additionally, after the sound reflects from the surface at the first convergence zones, additional CZs may also occur at integer multiples of the first convergence zone distance with a widening of the beam causing a blurring of the CZ.

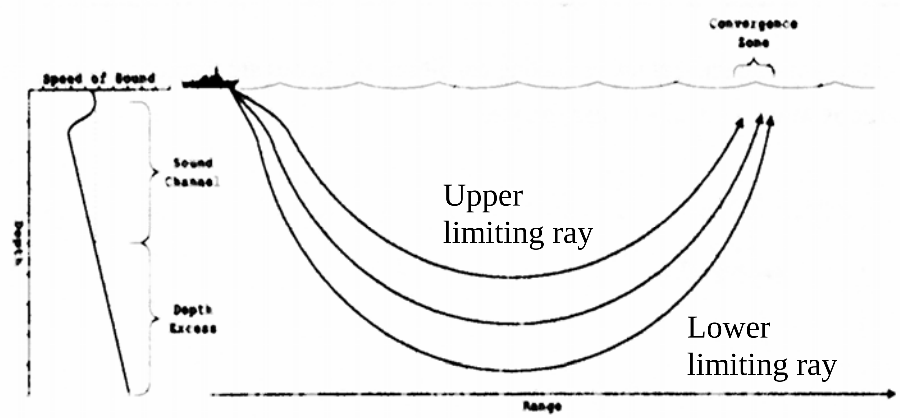
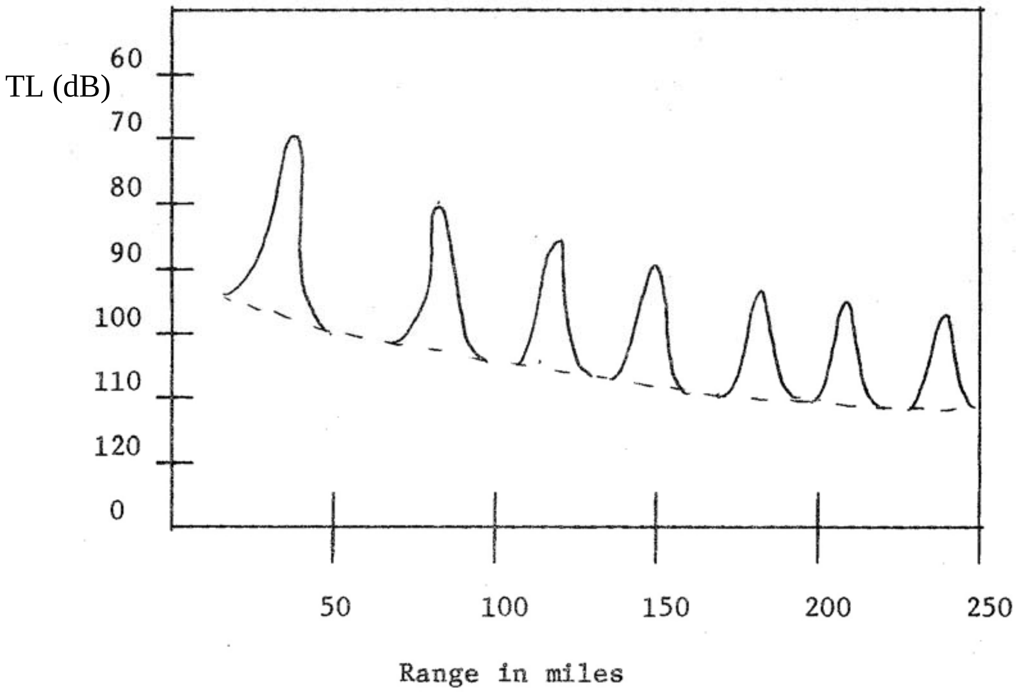


Figure 15-14. Convergence zone.

In the above figure, notice that there is a ray leaving the surface nearly horizontal that must be horizontal again when the sound speed is the same as that on the surface. This ray is termed the “upper limiting ray.” Additionally, there is another ray leaving the surface at a downward angle that just barely grazes the bottom before being bent back to the surface. This ray is termed the “lower limiting ray. All sound energy leaving the source between these too limiting rays must return to the surface in the annulus of the convergence zone.

This lower limiting ray becomes horizontal at a higher sound speed than at the surface or horizontal depth of the upper limiting ray. Since it is the increasing pressure with depth that causes this higher sound speed to bring the lower limiting ray horizontal, there must be an adequate “depth excess” in order for the two limiting rays to bracket most of the energy leaving the source. As a practical rule of thumb, depth excess is generally considered adequate when it is greater than 300 fathoms. This will provide adequate velocity excess for the lower limiting ray to bracket most of the energy leaving the source. Seasonal decreases in the surface sound speed may cause adequate depth excess to exist in the winter in locations that do not support convergence zones in the summer when the surface sound speed is higher.

The significance of convergence zones is that they affect the transmission loss experienced by sound leaving a source. During our discussion of the passive sonar equation we pointed out that transmission loss is due principally to geometric spreading (we will soon develop equations to quantify this loss due to spreading). The dashed line in the below figure represents a nominal transmission loss as a function of distance from a source. Convergence zones modify this spreading effect by significantly reducing transmission loss in the areas where the sound is all focused back to the surface. This is termed “transmission gain” and is shown in the below diagram as solid spikes at multiples of the distance to the first convergence zone.



Deep Shadow Zone

Note that where the positive gradient changes to a negative gradient can create a sonic layer. Some of the rays from the surface or near the surface will travel into the negative gradient region but get bent downward more sharply. This will create a shadow zone where a receiver may not be able to detect a submerged source. Submariners have always sought to hide in these shadow zones.

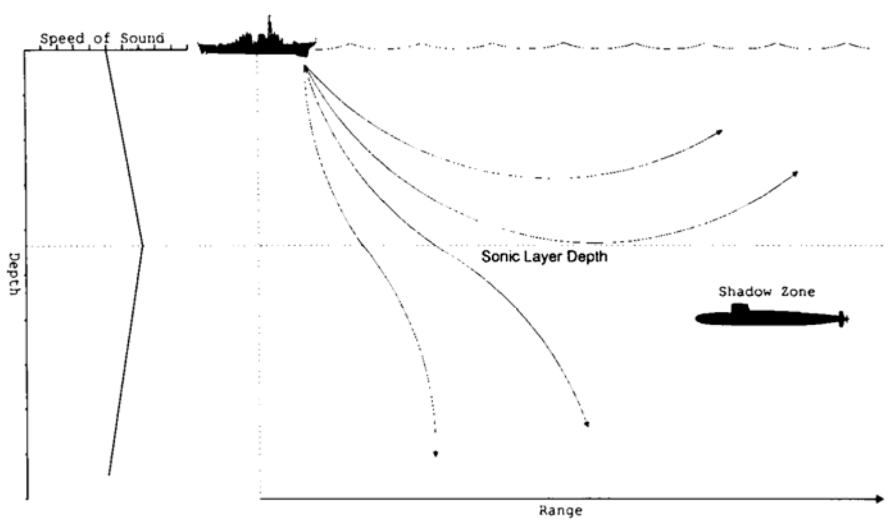


Figure 15-12. Sonic layer.

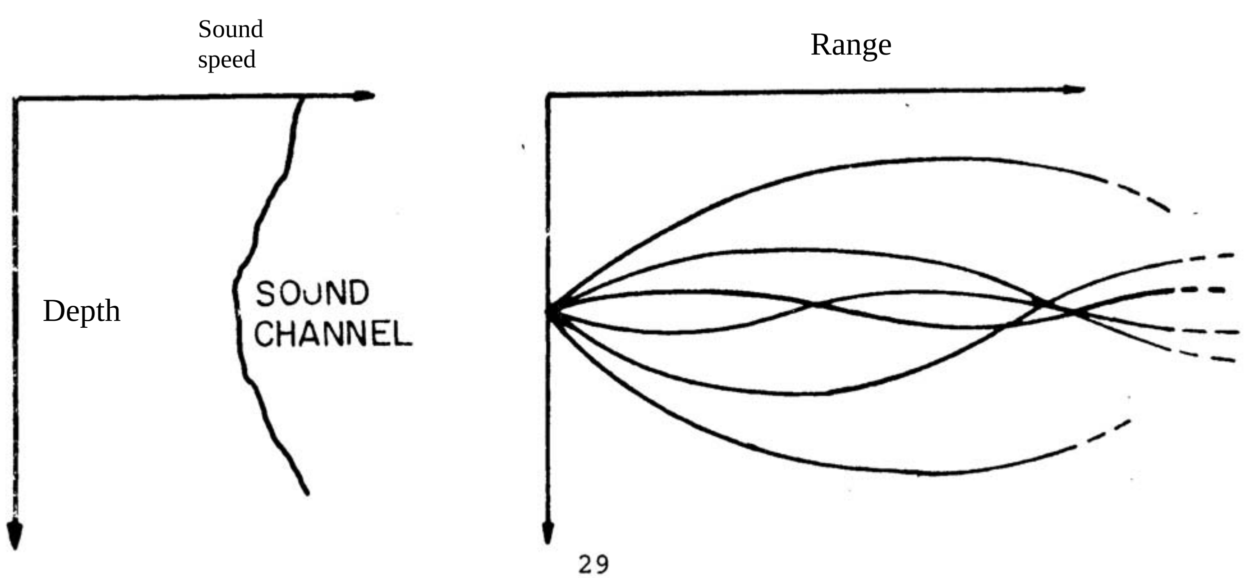
A thumbrule to select a best keel depth (BD) which keeps the entire submarine safely below the layer depth is called Amos' Rule:

$$BD = 17(LD)^{1/2} \quad \text{if } LD < 60\text{m.}$$
$$BD = LD + 60\text{m} \quad \text{if } LD > 60\text{m.}$$

Deep Sound Channel

Where the negative gradient of the main thermocline meets the positive gradient of the deep isothermal layer, a sound channel called the "deep sound channel" is created. Deep Sound Channels occur in most deep ocean basins. In the deep sound channel, sound traveling near the deep sound channel axis is continually bent back towards the axis and can travel down the axis for many miles. Above the depth of the axis of the deep sound channel, the temperature of the water has the most significant affect on sound velocity. As the water gets deeper, it reaches a steady temperature of 2-4° C at deep depths. Since the temperature is not changing, below the sound channel axis, pressure has the dominant affect on sound velocity. Thus as you go further down in depth, pressure increases and so does sound velocity. Hence the positive gradient of the sound velocity below the deep sound channel axis.

If the sound source is close to the channel axis (minimum *c*), acoustic rays are successively refracted by the two gradients without interacting with the interfaces. This type of propagation is called SOFAR (Sound Fixing And Ranging). It allows for very large transmission ranges because of the absence of energy loss by reflection at the interfaces and concentration of a large number of multiple paths, thereby minimizing geometric spreading. We can achieve ranges of several thousands of kilometers by using low frequencies.

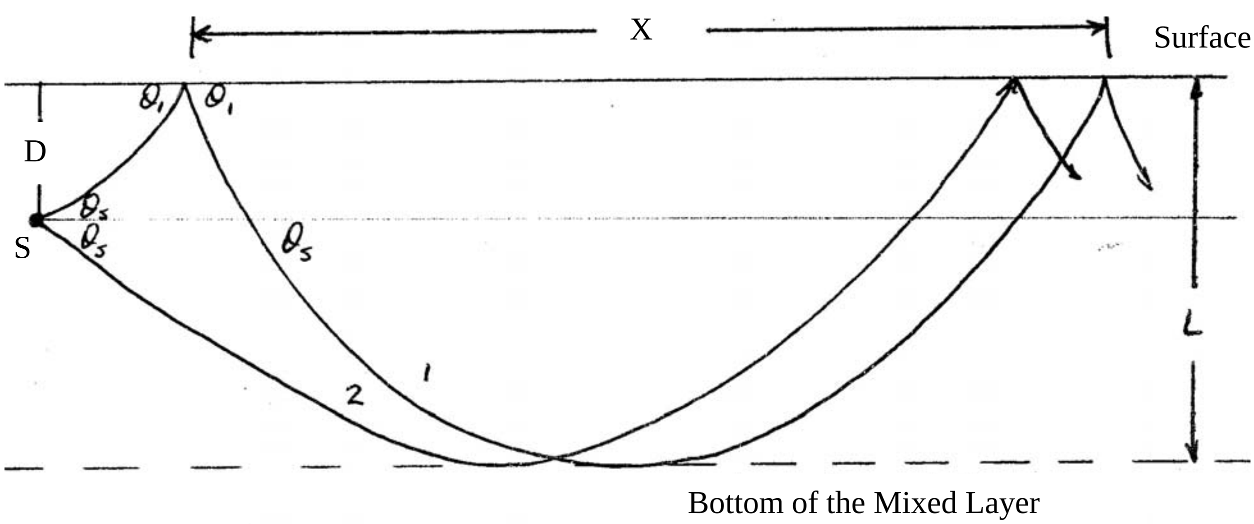


Downed pilots used this phenomenon during WW II. They carried small explosive charges that they would release if they were forced down into the water. These charges then sank down and exploded. The sound from these explosions would then travel many miles across the oceans in the deep sound channel and would be picked up by several receivers to give an approximate location of the downed pilots.

An Example in the Surface Mixed Layer

Wind driven surface waves mix the water near the surface and can produce an isothermal layer of water. The depth of this layer can vary from zero to more than 100 m. As we pointed out earlier, the increase in pressure with depth causes a positive gradient of 0.017 s^{-1}. A mixed layer is capable of producing a surface duct discussed earlier. In this example we will describe the limiting rays of the trapped sound in the duct.

In the below figure, two sound rays from a source, S, at a depth, D, are is a mixed layer of depth, L. The two rays shown are the limiting rays in that any ray leaving S at an angle greater than θ_s will pass out the bottom of the mixed layer, creating the deep shadow zone discussed previously. These rays designated 1 and 2, leave at equal angles above and below the surface and can be seen in the following way. The angle θ_1 at which ray 1 strikes the surface is related to θ_s by Snell's law. The angle of reflection from the surface is also θ_1 since the angle of reflection is equal to the angle of incidence. The angle of reflection, $\theta_{1,}$ is also related by Snell's Law to the angle at which ray 1 crosses the depth of S, so this angle is also θ_s. Since both rays become horizontal at the bottom of the layer, again, by Snell's Law, they must have the same angle at the depth of the source. Thus we have shown that the two limiting rays just remaining in the surface duct start at S with angles of $\pm\theta_s$.



In this example, start with 3 items that are normally known:

Source depth = D = 40 m

Layer depth = L = 100 m

Sound speed at the surface = c_1 = 1500 m/s

a. First find the speed of sound at the source (denoted by c_s) and at the bottom of the mixed layer (denoted by c_o).
Since the gradient is a constant, g = 0.017 s^{-1},

$$c_s = c_1 + 0.017D = 1500\frac{m}{s} + 0.017s^{-1}x40m = 1500.68\frac{m}{s}$$

and

$$c_o = c_1 + 0.017L = 1500\frac{m}{s} + 0.017s^{-1}x100m = 1501.7\frac{m}{s}$$

b. Find the angle of the rays at the source. This can be done using Snell's Law, applied between S and the bottom of the layer.

$$\frac{\cos\theta_s}{1500.68\frac{m}{s}} = \frac{1}{1501.7\frac{m}{s}}$$
$$\theta_s = 2.11^o$$

c. Find the angle of the rays at the surface. This can be done by applying Snell's Law between the surface and the layer bottom.

$$\frac{\cos\theta_1}{1500\frac{m}{s}} = \frac{1}{1501.7\frac{m}{s}}$$
$$\theta_1 = 2.73^o$$

d. Find the Radius of curvature of the ray.

$$R = \frac{c_o}{g} = \frac{1501.7\frac{m}{s}}{0.017s^{-1}} = 88335.3m$$

e. Find the skip distance, X. this is the distance between two successive reflections off the surface.

$$X = x - x_1 = R\left(\sin\theta - \sin\theta_1\right) = 88335.3m\left[\sin\left(2.73\right) - \sin\left(-2.73\right)\right] = 8404m$$

Definitions:

Surface Layer: Heated daily by the sun and mixed by the wind. Usually isothermal. Depth proportional to the winds. Daytime heating effects. Also known as the Mixed Layer.

Seasonal Thermocline: Temperature decreases with depth. During the winter, it may not exist due to deep mixing of the surface layer.

Main (Permanent) Thermocline: Start of the layering not affected by mixing. Characterized by decreasing temperature and sound speed. Minimum temperature is 4°C (39°F). Associated with the minimum sound speed. (At higher latitudes the depth of the minimum, sound speed is much shallower.

Deep Isothermal Layer: Deep ocean. Constant water temperature: 4°C (39°F). Sound speed increases as depth increases due to increase in pressure.

Thermocline: A constant temperature variation with depth, most often a negative change (temperature decreasing with depth). Can be seasonal or permanent.

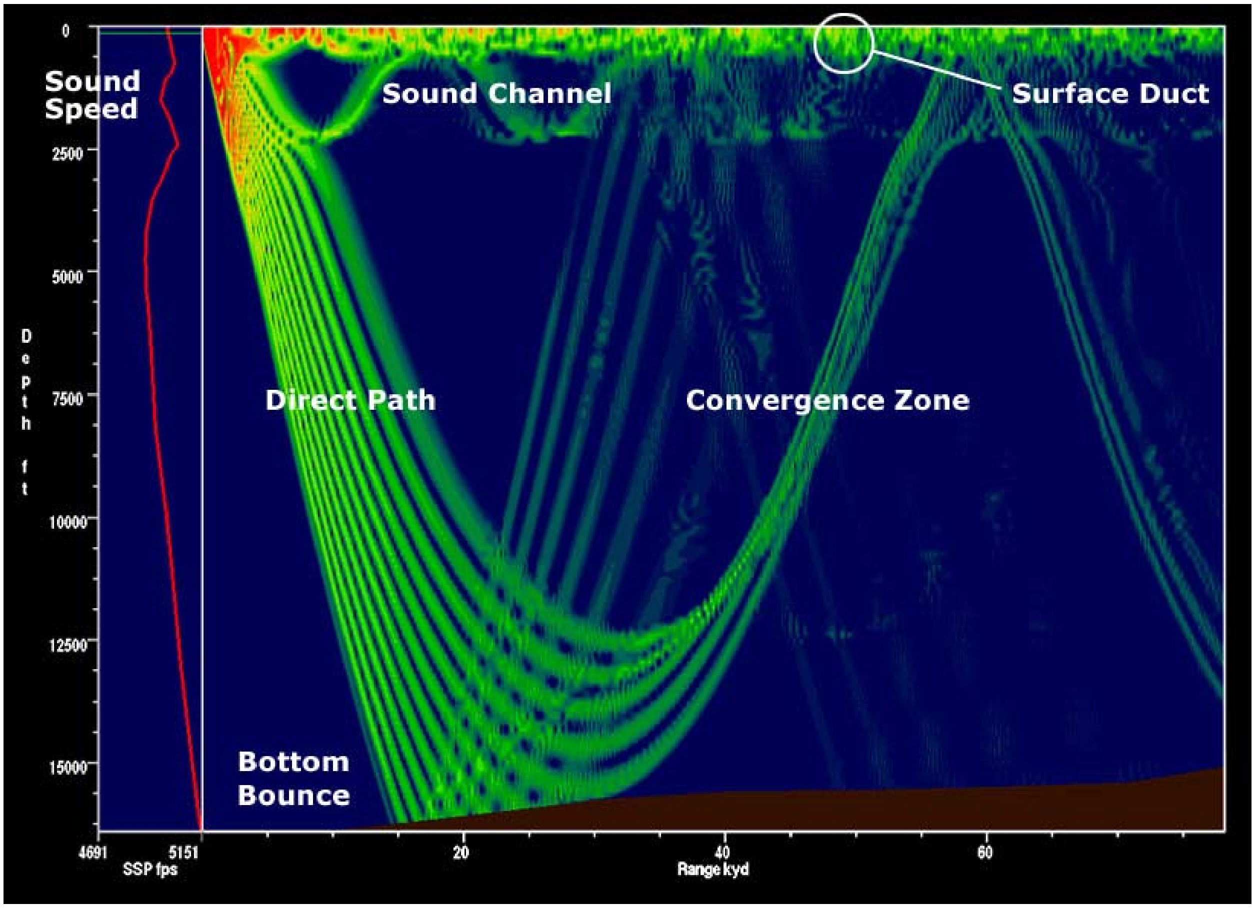
Surface Limited Ray: A ray that has a zero angle at the surface ($\theta_{surface} = 0°$) where the ray is refracted and not reflected at the surface. Any ray with an initial angle greater than that of the Surface Limited Ray will reflect off the surface and will not be refracted. Any ray with an initial angle less than the Surface Limited Ray will not reflect off the surface.

Bottom Limited Ray: A ray that will refract back upwards just as it reaches the bottom ($\theta_{bottom} = 0°$). Any ray with an initial angle greater than the Bottom Limited Ray will reflect off the bottom, not refract.

Surface Channel: Corresponds to a layer with sound velocity increasing from the surface down. Caused by a shallow isothermal layer appearing during winter, can also be caused by very cold water at the surface (melting ice or river influx)

Deep Sound Channel Axis: The depth where the speed of sound is a minimum. This depth marks the end of the Main Thermocline and the beginning of the Deep Isothermal Layer. SVP of large ocean basins has Deep Channel Axis between @200 m and 2000 m.

The following image provided courtesy of NAVOCEANO provides a good visual representation of the effects we have discussed in this section.

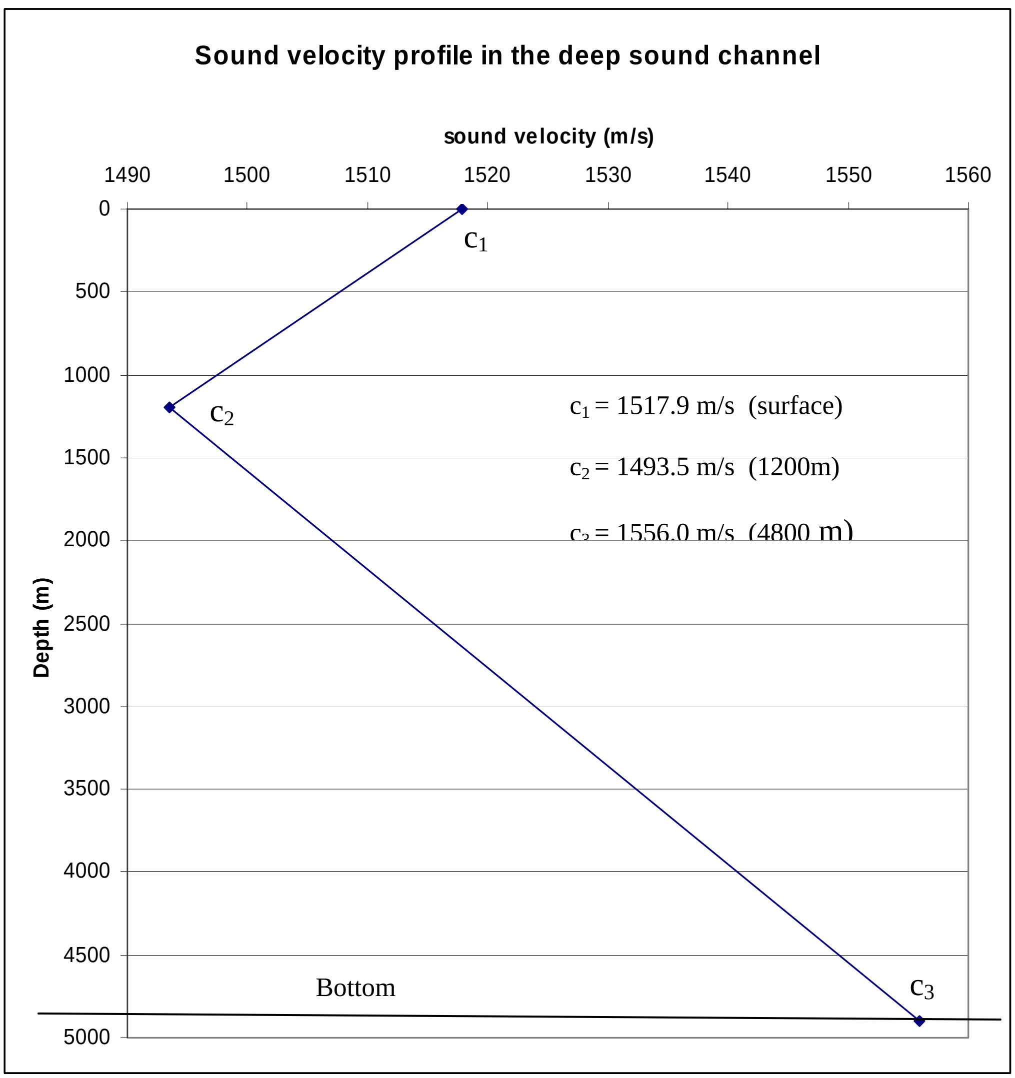


Problems:

1. State whether the gradient for each of the following layers is positive, negative or zero and state why?
   a) Surface layer,
   b) Seasonal Thermocline
   c) Main Thermocline
   d) Deep Isothermal Layer.

2. What happens to the range at which the shadow zone occurs when the source (or receiver) is moved from the top to the bottom of the mixed layer?

3. Since the speed of sound increases as depth increases solely due to increasing pressure in the deep isothermal layer, what is the gradient in the deep isothermal layer?



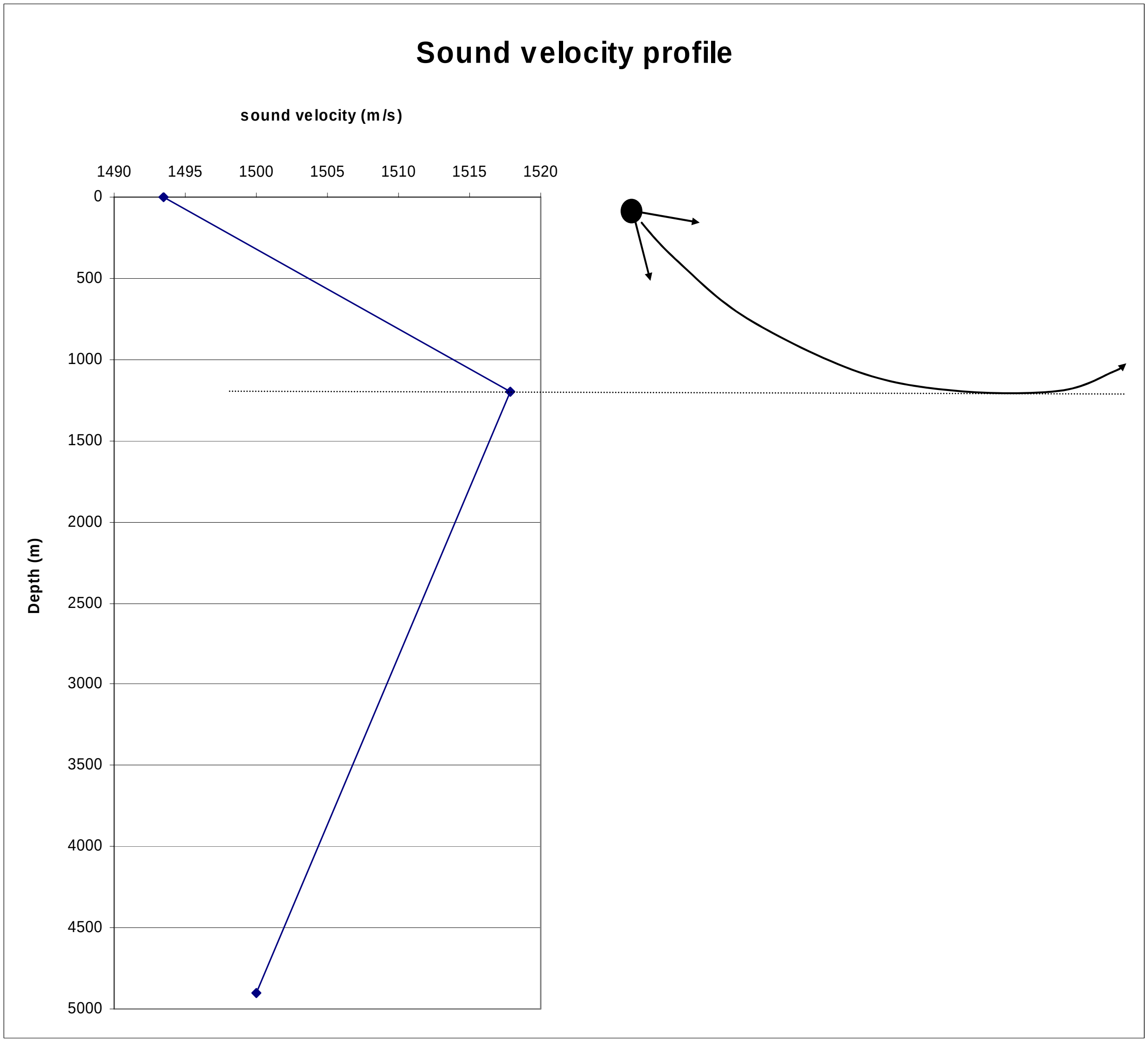
4. A sound source is at a depth of 0 m (just below the surface). There are two primarily important rays that form the boundaries of a convergence zone. One is called the "upper limiting ray." The other is called the "lower limiting ray."
   a) Draw a ray from the source making an angle of 0 degrees with respect to the horizontal and draw it until it reaches the surface again (the upper limiting ray)
   b) Draw a ray from the source making an angle of θ with respect to the horizontal. The angle, θ is chosen such that the ray just grazes the bottom and can return to the surface again. Draw the ray until it reaches the surface again. Compute the angle, θ below.

   <u>Upper Limiting Ray</u>
   c) What is the radius of the upper limiting ray between the surface and 1200 m?
   d) At what angle does this ray reach 1200 m?
   e) What is the horizontal distance traveled by this ray between the surface and 1200m?
   f) What is the radius of the upper limiting ray below 1200 m?
   g) At what depth does the ray become horizontal?
   h) What is the horizontal distance traveled by the ray between the point where the ray is at the depth of 1200 m and the point where it is horizontal?
   i) What is the horizontal distance traveled by the ray between the source and the point where it is horizontal?

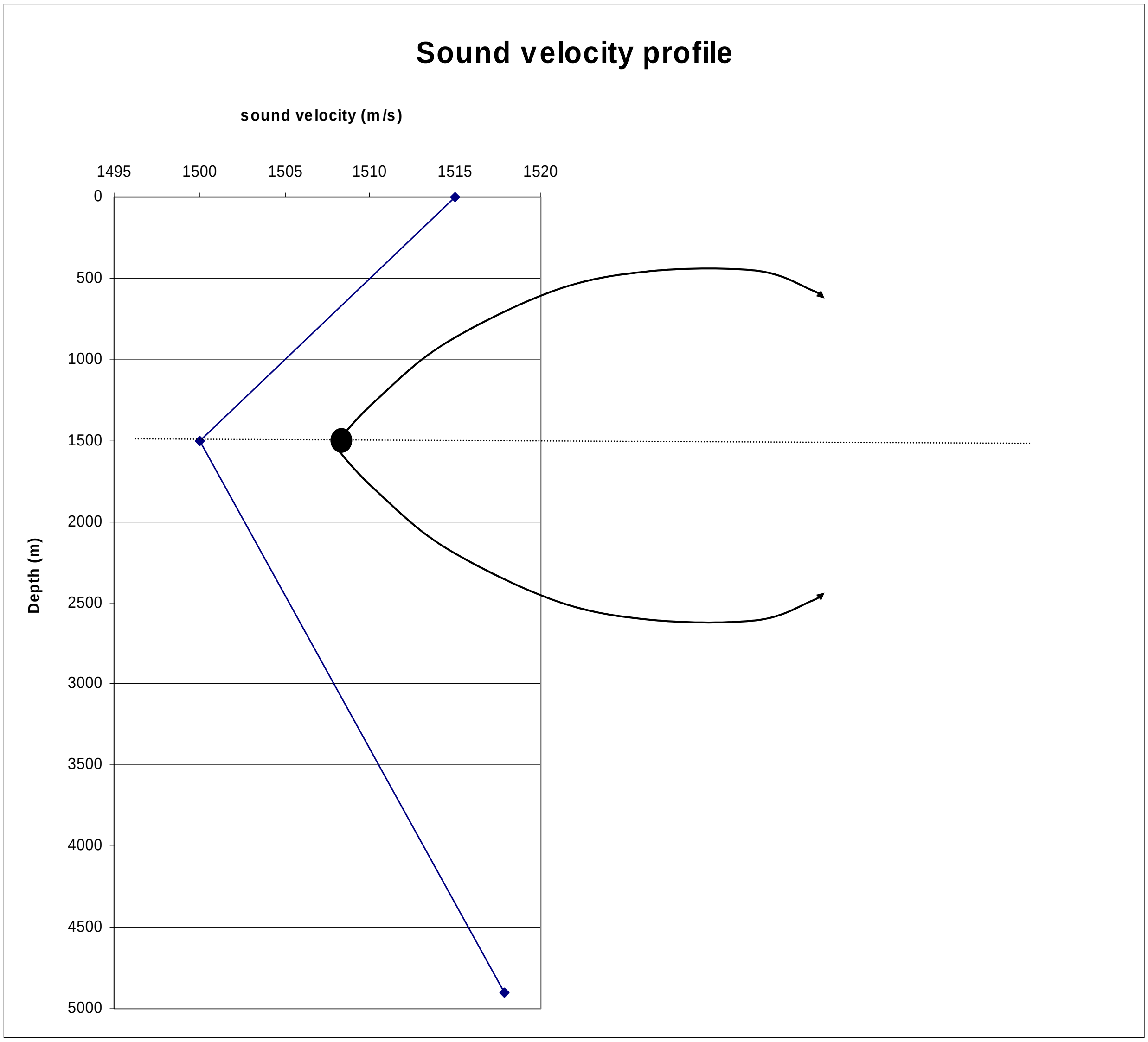
   <u>Lower Limiting Ray</u>
   j) At what angle does the lower limiting ray leave the surface?
   k) At what angle does this ray reach 1200 m?
   l) What is the radius of the upper limiting ray between the surface and 1200 m?
   m) At what horizontal range does the ray reach 1200 m?
   n) What is the radius of the lower limiting ray below 1200 m?
   o) What is the horizontal distance traveled by the ray between the point where the ray is at a depth of 1200 m and the point where the ray grazes the bottom.
   p) What is the total horizontal distance traveled by the ray from the surface until it grazes the bottom?

   <u>Convergence Zone</u>
   q) What is the distance to the first convergence zone?

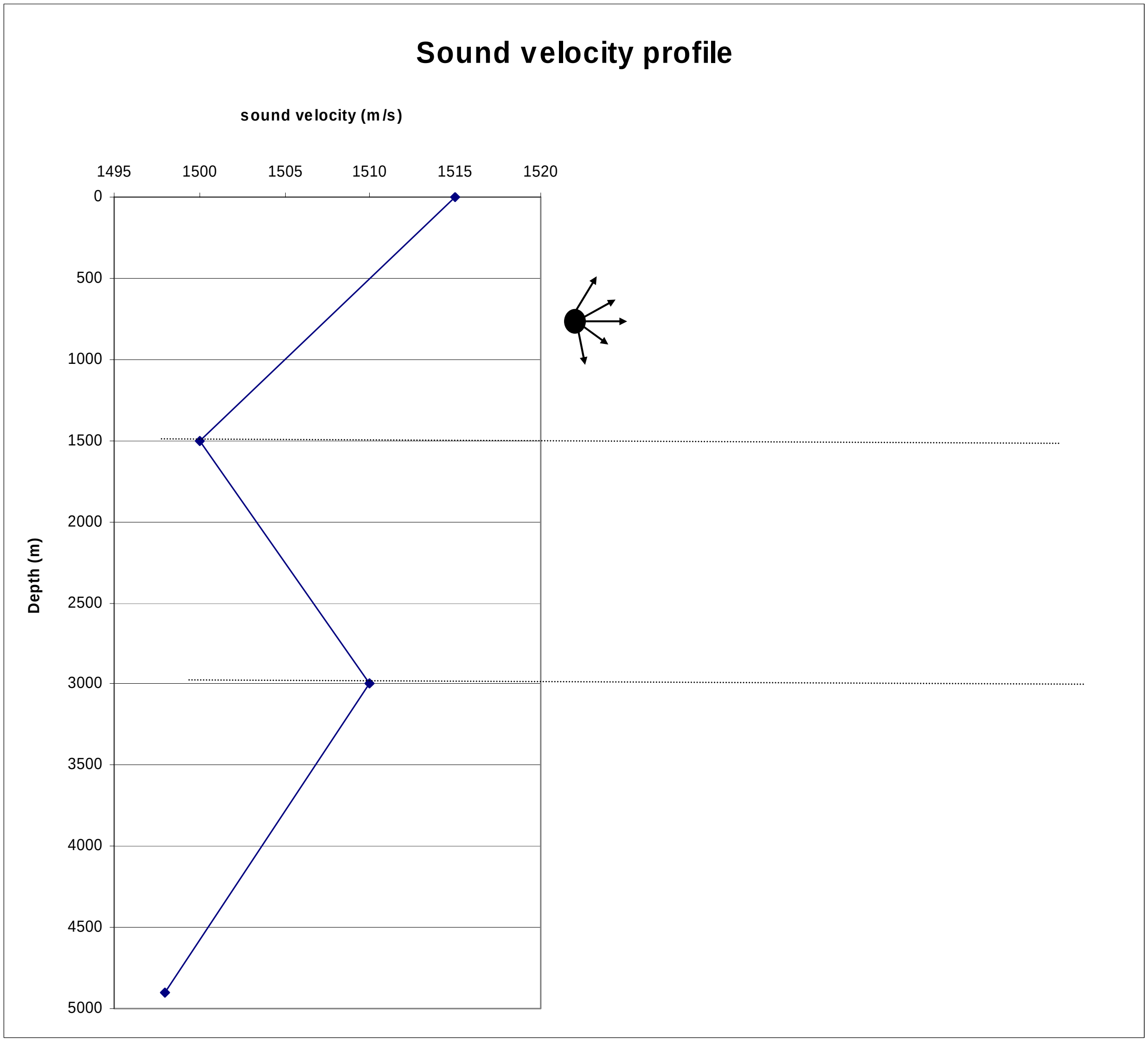
5. On the SVP below, draw the continuation of the rays from the source until they hit the bottom, surface or reach the right hand side of the page. Draw and label any limiting rays and any shadow zones.



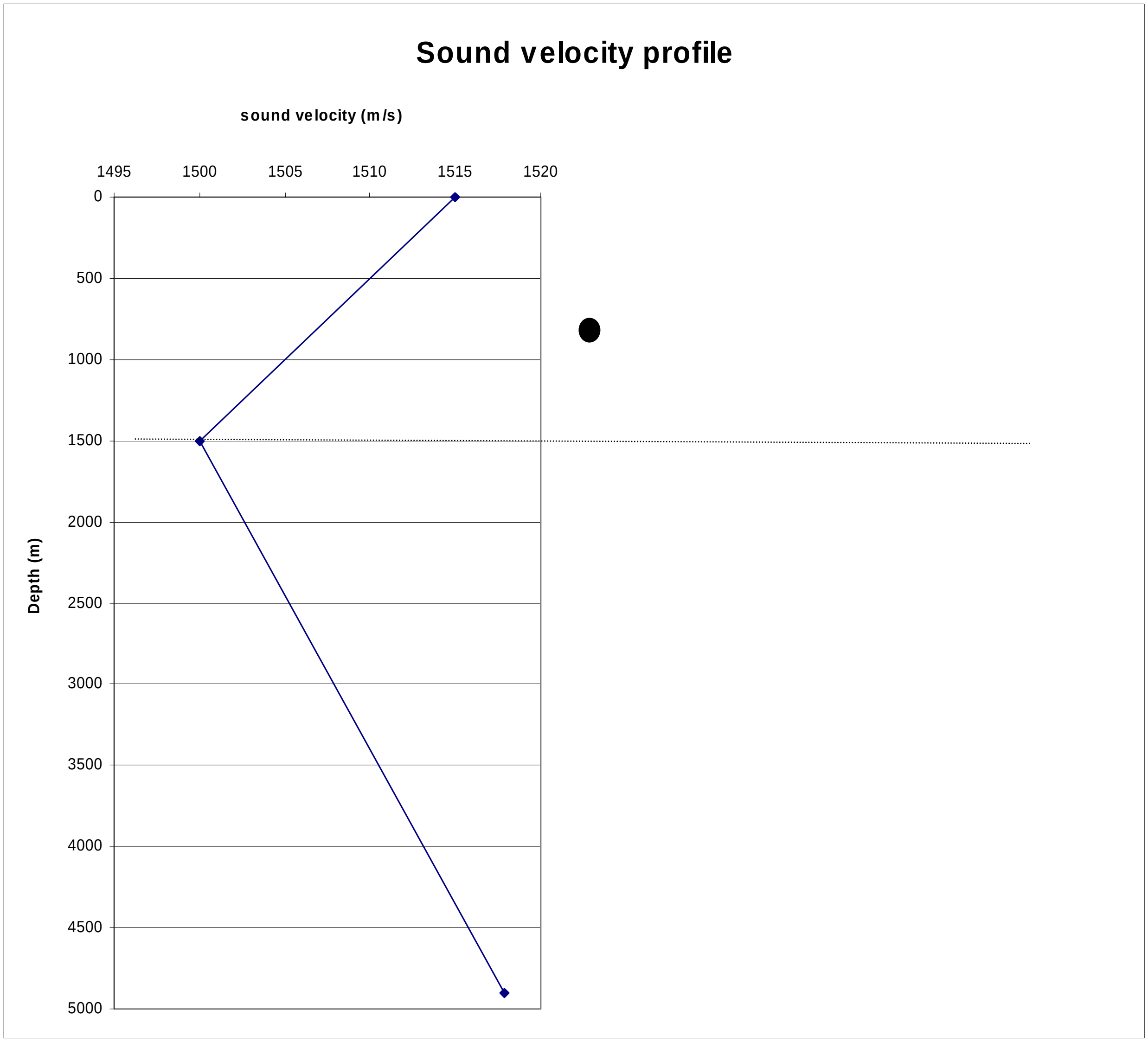
6. Continue these rays until they hit the surface, the bottom or reach the right hand side of the page. Draw other rays to show how a sound channel is formed.



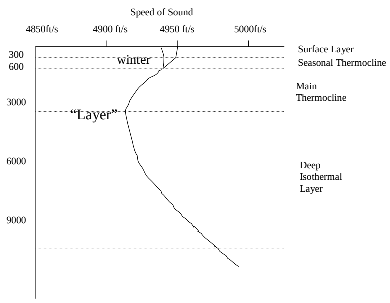
7. Continue these rays until they hit the surface, the bottom or reach the right hand side of the page.



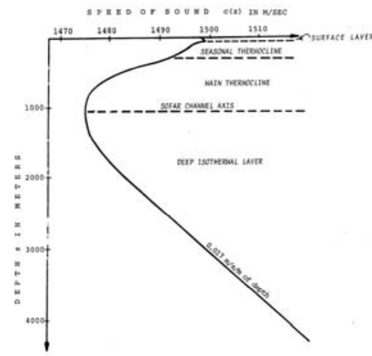
8. In the following sound velocity profile draw rays leaving the source towards the right hand side of the page at a large variety of angles with respect to the horizontal to show any:
   Surface duct ray paths
   Bottom bounce ray paths
   Shadow zones
   Convergence zones
   Limiting rays
   The depth excess
   Or other phenomena



Nominal SVP (Sound Velocity Profile)



Typical SVP

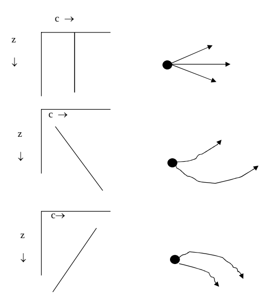


Gradients and Bending

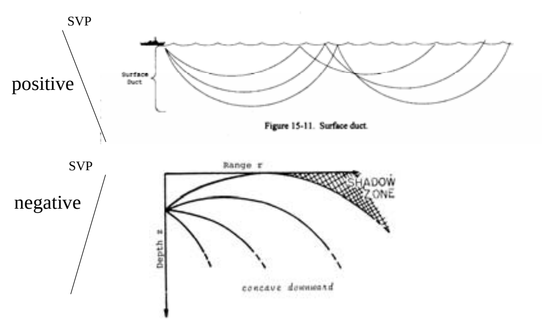
Iso-speed

positive

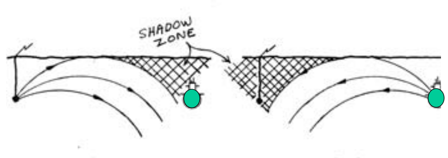
negative



Surface Duct – Surface Shadow Zone



Surface Shadow Zone Implications



Originally called the “afternoon effect”

Convergence Zone

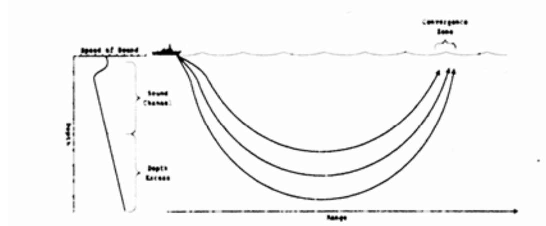
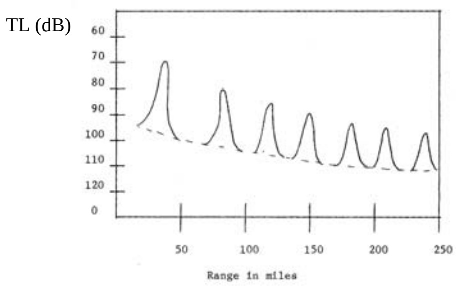


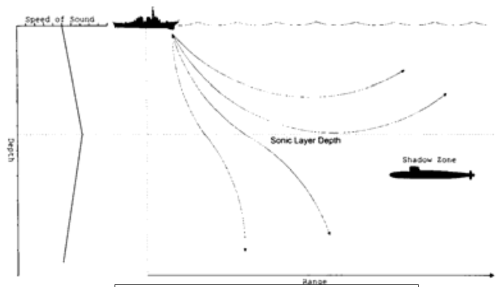
Figure 15-14. Convergence zone

Depth Excess > 300 fathoms

CZ Effect on Transmission Loss (TL)

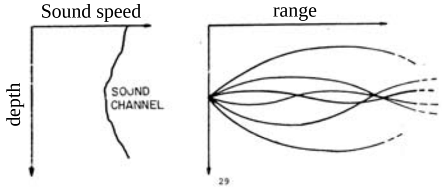


Deep Shadow Zones

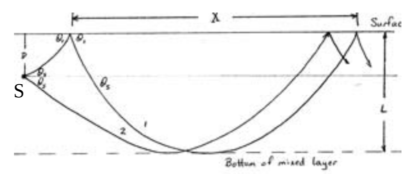


BD = 17(LD)1/2 if LD < 60m.
BD = LD + 60m if LD > 60m.

Deep sound channel



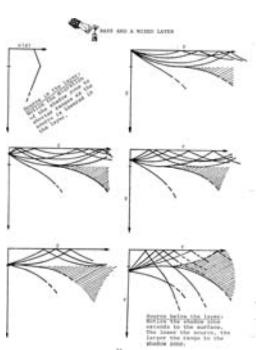
Mixed Layer Example



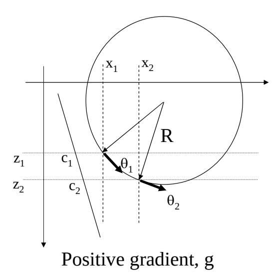
D = 40 m
L = 100 m
C_1 = 1500 m/s
g = 0.017 s^{-1}

- Find sound speed at S and at bottom of layer
- Angle of the rays at the source
- Angle of the rays at the surface
- Radius of curvature of the ray
- Skip distance , X

Mixed Layer



Summary



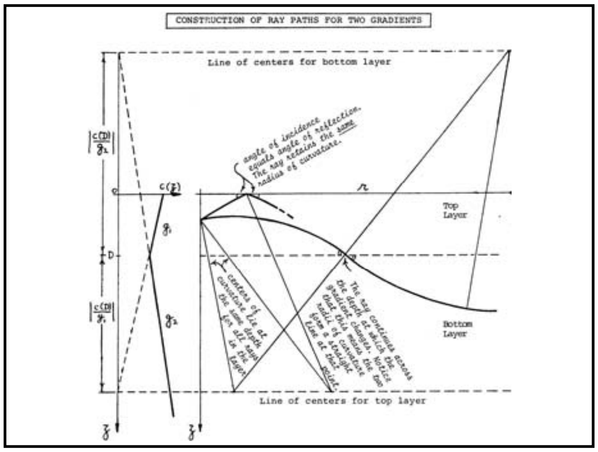
$$\frac{\cos\theta_1}{c_1} = \frac{\cos\theta}{c}$$

$$x - x_1 = -R\left(\sin\theta - \sin\theta_1\right)$$

$$z - z_1 = R\left(\cos\theta - \cos\theta_1\right)$$

$$R = \frac{c_1}{g\cos\theta_1}$$

$$g = \frac{\Delta c}{\Delta z} = \frac{c - c_1}{z - z_1}$$

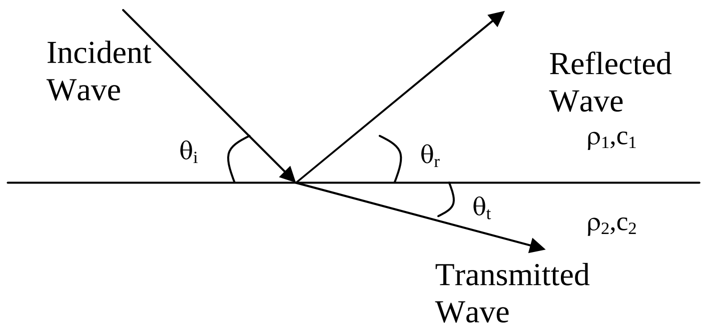

CONSTRUCTION OF RAY PATHS FOR TWO GRADIENTS
Line of centers for bottom layer
angle of incidence equals angle of reflection. The ray retains the same radius of curvature.
$\left|\frac{c(D)}{g_2}\right|$
$\left|\frac{c(D)}{g_1}\right|$
0
c(z)
D
g_1
g_2
r
Top Layer
Bottom Layer
centers of curvature lie at the same depth for all rays in the layer
The ray continues across the depth at which the gradient changes. Notice that this means the two radii of curvature form a straight line at that point.
Line of centers for top layer
z
z

Boundary Losses

Let's revisit Snell's Law and investigate what happens to a sound wave incident upon a boundary.

Using the figure below, we will try to determine how much of the sound energy of an incident wave is actually reflected at the boundary. According to Snell's Law,

$$\frac{\cos\theta_i}{c_1} = \frac{\cos\theta_t}{c_2}$$



Additionally, we expect that the incident angle and the reflected angle are the same. This follows logically from Snell's Law as well since the speed of sound for the incident and reflected waves are the same.

$$\frac{\cos\theta_i}{c_1} = \frac{\cos\theta_r}{c_1} \Rightarrow \theta_i = \theta_r$$

Reflection Coefficient

The reflection coefficient expresses the ratio of the intensity of the reflected wave to the intensity of the incident wave ($I_r = RI_i$). In all cases we are referring to the time average of the acoustic intensities and the rms acoustic pressures and particle velocities. The equation for the reflection coefficient would be:

$$R = \frac{I_r}{I_i} = \frac{\frac{p_r^2}{\rho_1 c_1}}{\frac{p_i^2}{\rho_1 c_1}} = \frac{p_r^2}{p_i^2}$$

To figure out how much energy is reflected, we must examine the following boundary conditions:

1) The pressure at the boundary is continuous.
2) The normal component of the velocity must also be continuous at the boundary.

To say that a property is continuous means that it is the same on both sides of the boundary. Let's look at the first condition concerning the pressure. To state this condition in equation format:

$$p_i + p_r = p_t$$

Since both the incident and reflected wave are on the same side of the boundary, their rms acoustic pressures added together must equal the rms acoustic pressure of the transmitted wave.

To satisfy the second condition, the component of the velocity normal to the boundary must also be continuous at the boundary. An equation for this would be:

$$|u_i|\sin\theta_i - |u_r|\sin\theta_r = |u_t|\sin\theta_t$$

The negative sign in the reflected term is because it is moving in the opposite direction as the other two velocities.

We can relate the rms particle velocity to the rms acoustic pressure using the following relationship:

$$p_a = zu = \rho cu$$

where z=ρc is the acoustic impedance of the medium. Thus:

$$\frac{p_i}{\rho_1 c_1}\sin\theta_i - \frac{p_r}{\rho_1 c_1}\sin\theta_r = \frac{p_t}{\rho_2 c_2}\sin\theta_t$$

If we substitute in the value of transmitted acoustic pressure from the pressure boundary condition we have,

$$\frac{p_i}{\rho_1 c_1}\sin\theta_i - \frac{p_r}{\rho_1 c_1}\sin\theta_r = \frac{p_i + p_r}{\rho_2 c_2}\sin\theta_t = \frac{p_i}{\rho_2 c_2}\sin\theta_t + \frac{p_r}{\rho_2 c_2}\sin\theta_t$$

Remembering that the incident and reflected angles are the same, we will rearrange to bring terms with reflected pressure and incident pressure on opposite sides of the equation.

$$\frac{p_i}{\rho_1 c_1}\sin\theta_i - \frac{p_i}{\rho_2 c_2}\sin\theta_t = \frac{p_r}{\rho_1 c_1}\sin\theta_i + \frac{p_r}{\rho_2 c_2}\sin\theta_t$$

Rearranging,

$$p_i\left(\rho_2 c_2 \sin\theta_i - \rho_1 c_1 \sin\theta_t\right) = p_r\left(\rho_2 c_2 \sin\theta_i + \rho_1 c_1 \sin\theta_t\right)$$

Or,

$$\frac{p_r}{p_i} = \frac{\left(\rho_2 c_2 \sin\theta_i - \rho_1 c_1 \sin\theta_t\right)}{\left(\rho_2 c_2 \sin\theta_i + \rho_1 c_1 \sin\theta_t\right)} = \frac{\left(\frac{\rho_2}{\rho_1}\sin\theta_i - \frac{c_1}{c_2}\sin\theta_t\right)}{\left(\frac{\rho_2}{\rho_1}\sin\theta_i + \frac{c_1}{c_2}\sin\theta_t\right)}$$

Using this result, we can easily establish an expression for the Reflection Coefficient, R.

$$R\left(\theta_i, \theta_t\right) = \frac{p_r^2}{p_i^2} = \left[\frac{m\sin\theta_i - n\sin\theta_t}{m\sin\theta_i + n\sin\theta_t}\right]^2$$

$$\text{where } m = \frac{\rho_2}{\rho_1} \;\;\&\;\; n = \frac{c_1}{c_2}$$

Notice that the subscripts are reversed in the equations for m and n. From this equation we can see that the Reflection Coefficient is dependent upon not only the mediums on each side of the boundary, but also the angle of incidence and the transmitted angle of the wave.

Further, we can express θ_t in terms of θ_i using Snell's Law and some trigonometric identities.

$$\frac{\cos\theta_i}{c_1} = \frac{\cos\theta_t}{c_2}$$

$$\cos\theta_t = \frac{c_2}{c_1}\cos\theta_i = \frac{\cos\theta_i}{n}$$

$$\sin\theta_t = \sqrt{1-\cos^2\theta_t} = \sqrt{1-\frac{\cos^2\theta_i}{n^2}}$$

A more useful expression for R then becomes:

$$R\left(\theta_i\right) = \left[\frac{m\sin\theta_i - \sqrt{n^2-\cos^2\theta_i}}{m\sin\theta_i + \sqrt{n^2-\cos^2\theta_i}}\right]^2$$

where m and n are as expressed earlier.

Normal Incidence

A useful case to study is when the incident wave arrives at an angle of 90° or normal to the boundary surface. Substituting θ_i = 90°, we get the following for the reflection coefficient:

$$R = \left(\frac{m-n}{m+n} \right)^2$$

dB Loss

A logical definition for the loss at a boundary is to subtract the reflected level from the incident level in dB. Applying the definition of the decibel level and the rules for subtraction,

$$dB_{loss} = L_{in} - L_{ref} = 10\log\left(\frac{I_{in}}{I_o}\right) - 10\log\left(\frac{I_{ref}}{I_o}\right) = 10\log\left(\frac{I_{in}}{I_{ref}}\right) = -10\log\left(\frac{I_{ref}}{I_{in}}\right) = -10\log\left(R\right)$$

Total Reflection

One special case is when there is total reflection (R=1). This occurs when the incident angle is less than a special angle called the critical angle. For there to be a critical angle, the speed of sound in the incident medium MUST BE less than the speed of sound in the second medium or:

$$\frac{c_1}{c_2} < 1$$

If this condition exists, the critical incident angle can be calculated using Snell's Law and letting the transmitted angle go to its minimum possible value of zero:

$$\theta_c = \cos^{-1}\left(\frac{c_1}{c_2}\right)$$

Transmission Coefficient

We will define the Transmission Coefficient in a manner consistent with the Reflection Coefficient.

$$T = \frac{I_t}{I_i} = \frac{\dfrac{p_t^2}{\rho_2 c_2}}{\dfrac{p_i^2}{\rho_1 c_1}} = \frac{\rho_1 c_1}{\rho_2 c_2}\frac{p_t^2}{p_i^2} = \frac{n}{m}\frac{p_t^2}{p_i^2}$$

The Transmission Coefficient can be easily derived if we take a look at the rate at which energy of the wave crosses the boundary. Since the energy of the incident wave must be conserved, it must equal the energy in the reflected plus transmitted wave. To express this in terms of an equation:

$$
\begin{aligned}
&I_i = I_t + I_r \quad \text{or,} \\
&1 = \frac{I_t}{I_i} + \frac{I_r}{I_i} \\
&1 = T + R \\
&\text{thus,} \\
&T(\theta_i) = 1 - R(\theta_i)
\end{aligned}
$$

Rather than establishing a separate equation for the transmission coefficient, we will generally first calculate the reflection coefficient using the equation above and then solve for the transmission coefficient by subtracting the reflection coefficient from one.

dB loss on transmission across a boundary would be defined similar to that for the dB loss on reflection.

$$
dB_{loss} = L_{in} - L_{trans} = 10\log\left(\frac{I_{in}}{I_o}\right) - 10\log\left(\frac{I_{trans}}{I_o}\right) = 10\log\left(\frac{I_{in}}{I_{trans}}\right) = -10\log\left(\frac{I_{trans}}{I_{in}}\right) = -10\log(T)
$$

Angle of Intromission

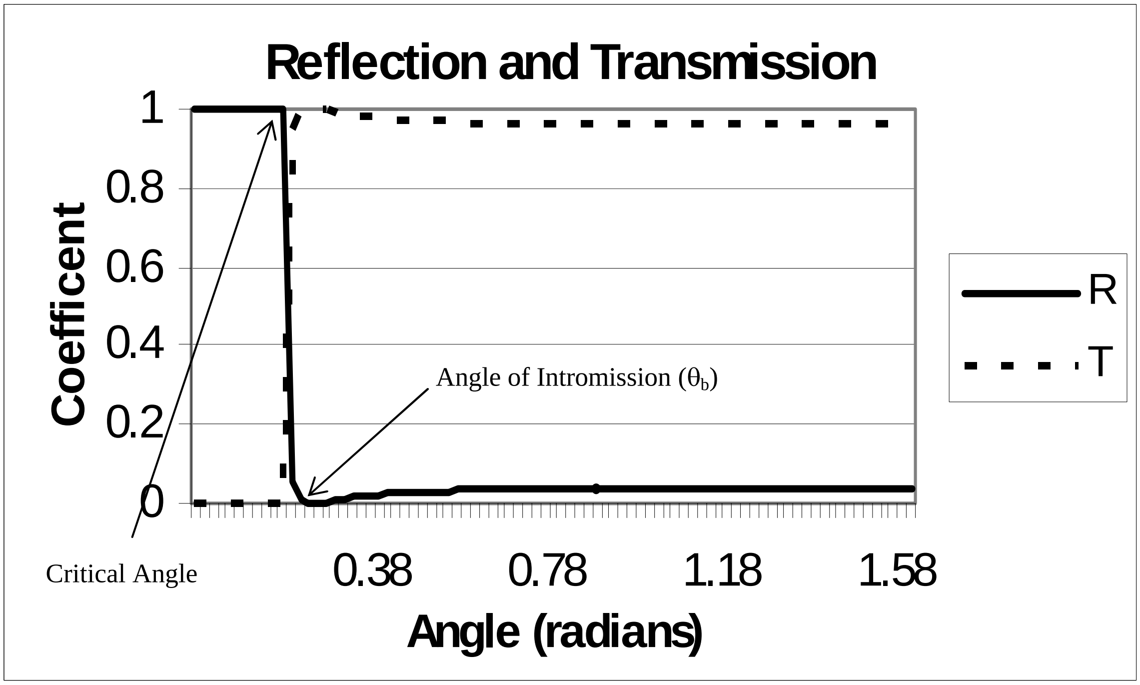
One special case for the Transmission Coefficient is when the Transmission Coefficient equals one ($T(\theta_i)=1$) and there is complete transmission of the incident wave and none of the energy is reflected. This occurs only at one angle (if it occurs at all) and that angle is referred to as the angle of intromission, θ_b. Using the equation for Reflection Coefficient when R = 0, and solving for the angle, we find:

$$
\theta_b = \cos^{-1}\left[\sqrt{\frac{(m^2 - n^2)}{(m^2 - 1)}}\right]
$$

(Note that the quantity $\frac{(m^2 - n^2)}{(m^2 - 1)}$ must be positive and less than 1 for the angle to exist. This is a rare case for most acoustics problems.)

Example

An example to illustrate how each of the coefficients vary as a function of angle is shown. For this example we will use m = 0.65, n = 0.98 and plot the Reflection Coefficient and Transmission Coefficient as a function of the angle of incidence (θ_i).



If we solve for the critical angle and angle of intromission we find:

$$\theta_c = \cos^{-1}(n)$$
$$\theta_c = 0.20 \text{ radians or,}$$
$$\theta_c = 11.5°$$

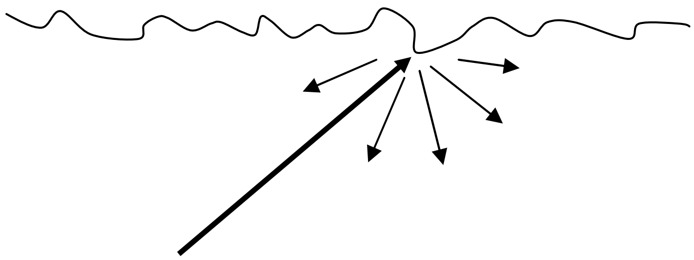
and

$$\theta_b = \cos^{-1}\left[\sqrt{\frac{(m^2 - n^2)}{(m^2 - 1)}}\right]$$
$$\theta_b = 0.26 \text{ radians or,}$$
$$\theta_b = 15.2°$$

as seen on the plot above. Also of interest to note is that the Reflection Coefficient is equal to 1 below the critical angle, θ_c, but the Transmission Coefficient is equal to 1 **only** at the angle of intromission.

Reflection from a Rough Surface

This entire discussion has assumed reflection from a sound ray bouncing off a smooth surface. This is called "specular reflection." Often the boundary is not smooth as in the case of a coral filled or rocky ocean floor, or a wind blown wave filled surface. In this case sound comes of the surface at various angles and the result is referred to as scattering.

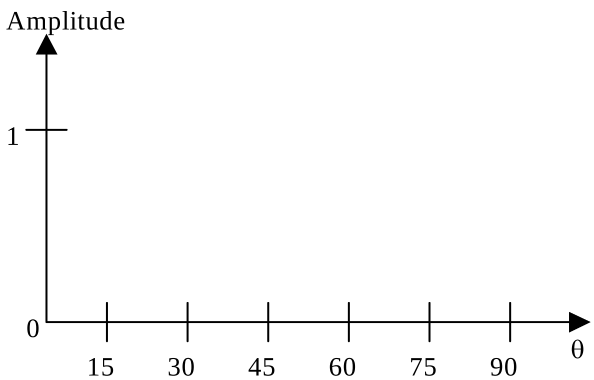


Some of the energy comes back in the direction toward the source of the incident sound and is called backscattering. In the operation of an active sonar system this backscattering results in the reception of unwanted sound which tends to mask the target echo. This unwanted sound is called "surface reverberation."

Problems

1. Given m = 0.5 and n = 0.7, determine:

   a) The angle of intromission if it exists.
   b) The critical angle if it exists.
   c) Sketch a plot of the reflection (dashed line) and transmission coefficients (solid line) as functions of angle from $0^o \le \theta \le 90^o$. Also compute R(90^o) and T(90^o).



   d) Using your graph above, if I_i = 0.2 W/m^2, determine I_r and I_t if:
      i) $\theta_1 \le \theta_c$?
      ii) $\theta_1 = \theta_b$?
      iii) $\theta_1 = 90^o$?

2. An SH-60F produces noise with an intensity of 750 KW/m^2 in a hover just above a glassy smooth sea. Given c_1 = 343 m/s, ρ_1=1.2 kg/m^3, c_2 = 1500 m/s, and ρ_2 = 1000 kg/m^3, determine the level of sound transmitted underwater in dB re 1 μPa (strikes sea surface θ_1 = 90^o).

3. If the transmitter is positioned at an angle where the reflection coefficient is 0.57, determine the intensity of a sound wave immediately below the surface of the sand if the incident intensity is 75 W/m^2.

4. For a sound wave in water incident onto a specially coated material,

$$m = 0.85$$
$$n = 0.95$$

Sketch a plot of R = I_r/I_i and T = I_t/I_i as a function of θ from 0^o to 90^o. (note that T = 1 – I)

5. Given the following data for the sediments in the Arctic Ocean bottom and for sea water near the bottom:

	Density	Sound Speed
Sea Water	1050 kg/m^3	1520 m/s
Artic Bottom	1300 kg/m^3	1440 m/s

a) For a plane acoustic wave incident on the bottom from the water, is there a critical angle? If so, calculate it.
b) Is there an angle of intromission? If so, calculate it.
c) Express the reflection loss in dB (assume normal incidence). The db loss would be found from 10 log (R).
d) Sketch a plot of $R = I_r/I_i$ and $T = I_t/I_i$ as a function of θ from 0^o to 90^o.
(note that T = 1 – I)

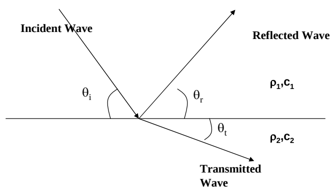
6. A plane sound wave is incident normally from air onto a smoth ocean surface. Given the following data:

	Density	Sound Speed
Air	1.20 kg/m^3	350 m/s
Artic Bottom	1000 kg/m^3	1500 m/s

a) If the intensity of sound in air is 10^{-2} W/m^2 incident normal to the air-water interface, what is the intensity of the sound in the water just below the surface?
b) What is the level in dB re 1 μPa below the surface?

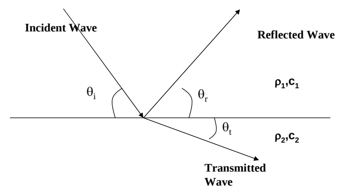
Transmission and Reflection

$$\theta_{\mathrm{i}} = \theta_{\mathrm{r}}$$



$$\frac{\cos\theta_{\mathrm{i}}}{\mathrm{c}_1} = \frac{\cos\theta_{\mathrm{t}}}{\mathrm{c}_2}$$

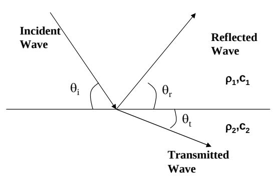
Reflection Coefficient



$$\mathrm{R} = \frac{\mathrm{I_r}}{\mathrm{I_i}} = \frac{\dfrac{\mathrm{p_r^2}}{\rho_1 \mathrm{c_1}}}{\dfrac{\mathrm{p_i^2}}{\rho_1 \mathrm{c_1}}} = \frac{\mathrm{p_r^2}}{\mathrm{p_i^2}}$$

Boundary Conditions

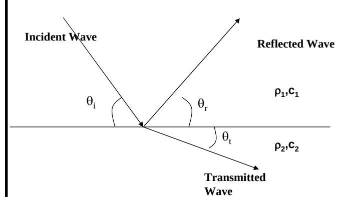
- **The pressure at the boundary is continuous.**
- **The normal component of the velocity must also be continuous at the boundary.**



$$p_i + p_r = p_t$$

$$|u_i|\sin\theta_i - |u_r|\sin\theta_r = |u_t|\sin\theta_t$$

Reflection Coefficient



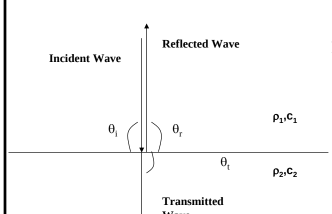
$$\mathrm{R}(\theta_{\mathrm{i}}, \theta_{\mathrm{t}}) = \frac{\mathrm{p_r^2}}{\mathrm{p_i^2}} = \left[\frac{\mathrm{m}\sin\theta_{\mathrm{i}} - \mathrm{n}\sin\theta_{\mathrm{t}}}{\mathrm{m}\sin\theta_{\mathrm{i}} + \mathrm{n}\sin\theta_{\mathrm{t}}}\right]^2$$

where $\mathrm{m} = \dfrac{\rho_2}{\rho_1}$ & $\mathrm{n} = \dfrac{\mathrm{c_1}}{\mathrm{c_2}}$

Snell's Law

$$\mathrm{R}(\theta_{\mathrm{i}}) = \left[\frac{\mathrm{m}\sin\theta_{\mathrm{i}} - \sqrt{\mathrm{n}^2 - \cos^2\theta_{\mathrm{i}}}}{\mathrm{m}\sin\theta_{\mathrm{i}} + \sqrt{\mathrm{n}^2 - \cos^2\theta_{\mathrm{i}}}}\right]^2$$

Normal Incidence

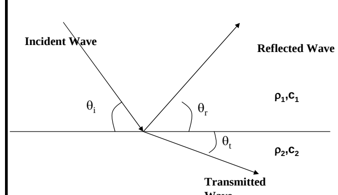


$$\mathrm{R}(\theta_{\mathrm{i}}) = \left[\frac{\mathrm{m}\sin\theta_{\mathrm{i}} - \sqrt{\mathrm{n}^2 - \cos^2\theta_{\mathrm{i}}}}{\mathrm{m}\sin\theta_{\mathrm{i}} + \sqrt{\mathrm{n}^2 - \cos^2\theta_{\mathrm{i}}}}\right]^2$$

$$R = \left(\frac{m-n}{m+n}\right)^2$$

$$\mathrm{m} = \frac{\rho_2}{\rho_1} \quad \& \quad \mathrm{n} = \frac{\mathrm{c_1}}{\mathrm{c_2}}$$

Critical Angle

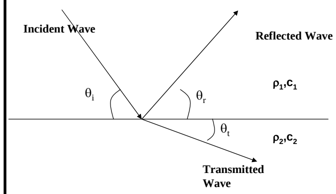


$$\frac{c_1}{c_2} < 1$$

$$\theta_c = \cos^{-1}\left(\frac{c_1}{c_2}\right)$$

Incident Angles less than the critical angle cannot have a transmitted wave

Transmission Coefficient



$$T=\frac{I_t}{I_i}=\frac{\frac{p_t^2}{\rho_2 c_2}}{\frac{p_i^2}{\rho_1 c_1}}=\frac{\rho_1 c_1}{\rho_2 c_2}\frac{p_t^2}{p_i^2}=\frac{n}{m}\frac{p_t^2}{p_i^2}$$

$$I_i = I_t + I_r \quad \text{or,}$$

$$1=\frac{I_t}{I_i}+\frac{I_r}{I_i}$$

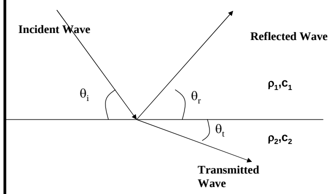
$$1 = T + R$$

thus,

$$T(\theta_i)=1-R(\theta_i)$$

$$m=\frac{\rho_2}{\rho_1} \quad \& \quad n=\frac{c_1}{c_2}$$

Angle of Intromission



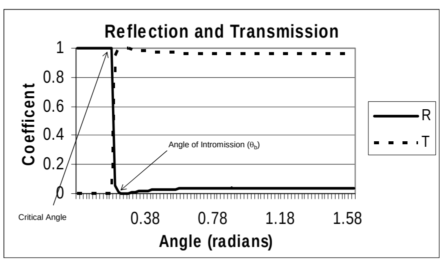
$$T = 1$$

$$R(\theta_i)=\left[\frac{m\sin\theta_i-\sqrt{n^2-\cos^2\theta_i}}{m\sin\theta_i+\sqrt{n^2-\cos^2\theta_i}}\right]^2=0$$

$$\theta_b=\cos^{-1}\left[\sqrt{\frac{(m^2-n^2)}{(m^2-1)}}\right]$$

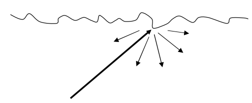
$$m=\frac{\rho_2}{\rho_1} \quad \& \quad n=\frac{c_1}{c_2}$$

Example: m = 0.65, n = 0.98



Find the critical angle and the angle of intromission

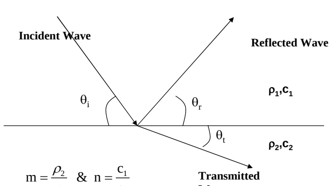
Non-specular Reflection



Backscattering
Reverberation

Summary

$$\theta_i = \theta_r$$



$$m=\frac{\rho_2}{\rho_1} \quad \& \quad n=\frac{c_1}{c_2}$$

$$\frac{\cos\theta_i}{c_1}=\frac{\cos\theta_t}{c_2}$$

$$R(\theta_i)=\left[\frac{m\sin\theta_i-\sqrt{n^2-\cos^2\theta_i}}{m\sin\theta_i+\sqrt{n^2-\cos^2\theta_i}}\right]^2$$

$$R=\left(\frac{m-n}{m+n}\right)^2$$

$$1 = T + R$$

$$\theta_c=\cos^{-1}\left(\frac{c_1}{c_2}\right)$$

$$\theta_b=\cos^{-1}\left[\sqrt{\frac{(m^2-n^2)}{(m^2-1)}}\right]$$

Backup Slides

$$p_a = zu = \rho c u$$

$$\frac{p_i}{\rho_1 c_1}\sin\theta_i - \frac{p_r}{\rho_1 c_1}\sin\theta_r = \frac{p_t}{\rho_2 c_2}\sin\theta_t$$

$$\frac{p_i}{\rho_1 c_1}\sin\theta_i - \frac{p_r}{\rho_1 c_1}\sin\theta_r = \frac{p_i + p_r}{\rho_2 c_2}\sin\theta_t = \frac{p_i}{\rho_2 c_2}\sin\theta_t + \frac{p_r}{\rho_2 c_2}\sin\theta_t$$

$$\frac{p_i}{\rho_1 c_1}\sin\theta_i - \frac{p_i}{\rho_2 c_2}\sin\theta_t = \frac{p_r}{\rho_1 c_1}\sin\theta_i + \frac{p_r}{\rho_2 c_2}\sin\theta_t$$

$$p_i\left(\rho_2 c_2 \sin\theta_i - \rho_1 c_1 \sin\theta_t\right) = p_r\left(\rho_2 c_2 \sin\theta_i + \rho_1 c_1 \sin\theta_t\right)$$

$$\frac{p_r}{p_i} = \frac{\left(\rho_2 c_2 \sin\theta_i - \rho_1 c_1 \sin\theta_t\right)}{\left(\rho_2 c_2 \sin\theta_i + \rho_1 c_1 \sin\theta_t\right)} = \frac{\left(\frac{\rho_2}{\rho_1}\sin\theta_i - \frac{c_1}{c_2}\sin\theta_t\right)}{\left(\frac{\rho_2}{\rho_1}\sin\theta_i + \frac{c_1}{c_2}\sin\theta_t\right)}$$

$$\frac{\cos\theta_i}{c_1} = \frac{\cos\theta_t}{c_2}$$

$$\cos\theta_t = \frac{c_2}{c_1}\cos\theta_i = \frac{\cos\theta_i}{n}$$

$$\sin\theta_t = \sqrt{1-\cos^2\theta_t} = \sqrt{1-\frac{\cos^2\theta_i}{n^2}}$$

$$R(\theta_i) = \left[\frac{m\sin\theta_i - \sqrt{n^2 - \cos^2\theta_i}}{m\sin\theta_i + \sqrt{n^2 - \cos^2\theta_i}}\right]^2 = 0$$

$$m\sin\theta_i - \sqrt{n^2 - \cos^2\theta_i} = 0$$

$$m^2\sin^2\theta_i - n^2 + \cos^2\theta_i = m^2\left(1-\cos^2\theta_i\right) - n^2 + \cos^2\theta_i = 0$$

$$m^2 - m^2\cos^2\theta_i - n^2 + \cos^2\theta_i = m^2 - n^2 + \left(1-m^2\right)\cos^2\theta_i = 0$$

$$m^2 - n^2 = \left(m^2 - 1\right)\cos^2\theta_i$$

$$\cos^2\theta_i = \frac{m^2 - n^2}{m^2 - 1}$$

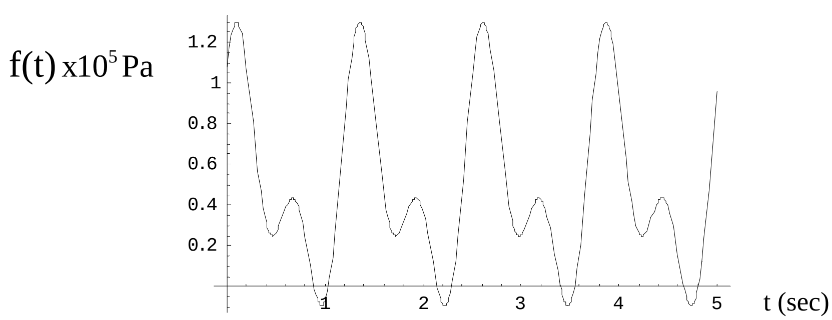
Fourier Analysis

In our Mathematics classes, we have been taught that complicated functions can often be represented as a long series of terms whose sum closely approximates the actual function. Taylor series is one very powerful application of this idea. In the case of Taylor series, the function is approximated by a constant value of the function at a particular point added to successive derivatives evaluated at that same point and multiplied by specific constants or coefficients. Another type of series is the Fourier series. Here specific constants are multiplied by sine and cosine terms to generate the series that approximates the function.

As an example, consider the following series of five terms that represent the oscillating pressure sensed by a hypothetical detector as a sound passes by:

$$f\left(t\right)=\left\{\frac{1}{2}+\frac{1}{3}\cos[5\,t]+\frac{1}{4}\cos[10\,t]+\frac{1}{3}\sin[5\,t]+\frac{1}{4}\sin[10\,t]\right\}\mathrm{x}10^{5}\,\mathrm{Pa}$$

Notice some things about this series. The first term is a constant, sometimes called the "DC" term using an analogy to electrical voltages and currents. The second and third are cosine terms. The angular frequency of the second term is 5 rad/sec and the amplitude is $1/3\,\mathrm{x}10^{5}\,\mathrm{Pa}$. The third term has twice the angular frequency so it oscillates twice as fast, but has and amplitude of only $1/4\,\mathrm{x}10^{5}\,\mathrm{Pa}$. The fourth and fifth terms have the same frequency and amplitude as the second and third but are shifted in phase by 90 degrees relative to the cosines. When plotted for 5 sec, this series looks like this:



Most often in experimental acoustics, we have a detector to receive a signal like this one and it is our purpose to work backwards and determine the frequencies and the amplitudes of the tones (terms in the series) that make up the periodic signal. The method of finding these tones is called "Fourier Analysis." Finding the frequencies is simply a matter of determining the overall period of the repeating signal. The fundamental frequency, or frequency of the first sine or cosine term in the series (in Hertz), is simply the reciprocal of that frequency. Higher frequency terms are just multiples or harmonics of the fundamental frequency. Generally the frequency is given in rad/sec instead of Hz.

Finding the coefficients or amplitude of each term occurs using a very clever bit of mathematics discovered by Fourier. This method is sometimes called "Fourier's Hammer" because it is used to hammer out each of the coefficients (amplitudes) in the series. We'll study this method in some detail below.

In fact, many sounds are combinations of discrete frequency components that we hear as one sound. In class, we will use spectrum analyzers and digital oscilloscopes which use digital signal processing algorithms to find the magnitude (proportional to the Fourier Series Coefficient) and frequency of each component of a signal.

Calculating Coefficients

Starting with a ***periodic function*** (such as a sound wave), we can breakdown this function into separate frequency components by using Fourier Analysis. Note that we must KNOW the period of the wave and BE ABLE TO DEFINE the function, f(t), over that period to be able to use Fourier Analysis. Often the function will be zero, a constant, or a straight line with constant slope. Whatever it is, we must be able to write a math expression (or a good approximation) for the function over the entire period.

First let us be very specific about the frequency in rad/sec. Once we have identified the period over witch the function repeats, the angular frequency is:

$$\omega = \frac{2\pi}{T}$$

In the example plot of the periodic function above, the period is approximately 1.25 sec by inspection of the time scale. This is consistent with the equation we plotted since

$$\omega = \frac{2\pi}{T} = \frac{2\pi \mathrm{rad}}{1.25\,\mathrm{sec}} \approx 5\mathrm{rad}/\mathrm{sec}.$$

Other terms in the Fourier series will have frequencies that are multiples of 5 rad/sec, e.g. 10 rad/sec, 15 rad/sec, 20 rad/sec,…..

Calculating the amplitudes is somewhat more complicated. First consider the equation we plotted above (where I have dropped the units and constant $10^{5)}$:

$$f(t) = \frac{1}{2} + \frac{1}{3}\cos[5\,t] + \frac{1}{4}\cos[10\,t] + \frac{1}{3}\sin[5\,t] + \frac{1}{4}\sin[10\,t]$$

Even though we know the amplitude of the first cosine term is 1/3, let's try to develop a method to unmask it. First, multiply each term by cos(5t).

$$f(t)\cos(5t) = \frac{1}{2}\cos(5t) + \frac{1}{3}\cos[5\,t]\cos(5t) + \frac{1}{4}\cos[10\,t]\cos(5t)$$
$$+ \frac{1}{3}\sin[5\,t]\cos(5t) + \frac{1}{4}\sin[10\,t]\cos(5t)$$

Next, we find the time average of each term in the series using the normal definition for the time average of a function. This is a reasonable approach because we are looking for a representative value for the amplitude averaged over at least one cycle, not an instantaneous value.

$$\langle f(t)\rangle = \frac{1}{T}\int_0^T f(t)dt$$

The result looks complicated and long but will quickly simplify.

$$\frac{1}{T}\int_0^T f(t)\cos(5t)dt = \frac{1}{T}\int_0^T \frac{1}{2}\cos(5t)\,dt + \frac{1}{T}\int_0^T \frac{1}{3}\cos[5\,t]\cos(5t)\,dt + \frac{1}{T}\int_0^T \frac{1}{4}\cos[10\,t]\cos(5t)\,dt$$

$$+\frac{1}{T}\int_0^T \frac{1}{3}\sin[5\,t]\cos(5t)dt + \frac{1}{T}\frac{1}{4}\int_0^T \sin[10\,t]\cos(5t)dt$$

A quick inspection of the left side of the equal sign reveals that most of the terms integrate to zero. In fact all but one term are zero since,

$$\int_0^T \sin n\omega t\cos m\omega t dt = \int_0^T \sin n\omega t \sin m\omega t dt = \int_0^T \cos n\omega t\cos m\omega t dt = 0$$

unless m=n. In that case, (sine would be identical)

$$\frac{1}{T}\int_0^T \cos n\omega t\cos m\omega t dt = \frac{1}{T}\int_0^T \cos^2 n\omega t dt = \frac{1}{2}$$

This leaves us with the following:

$$\frac{1}{T}\int_0^T f(t)\cos(5t)dt = 0 + \frac{1}{T}\int_0^T \frac{1}{3}\cos^2[5\,t]dt + 0 + 0 + 0 = \frac{1}{3}\left(\frac{1}{2}\right)$$

Rearranging slightly shows that the coefficient we were trying to find, i.e. the 1/3, must be calculated as follows:

$$\frac{1}{3} = \frac{2}{T}\int_0^T f(t)\cos(5t)dt = a_1$$

The name we will give to this coefficient is a_1. We arbitrarily decide to call all the coefficients for cosine terms "a" and for sine terms "b." The subscript tells us which harmonic of the fundamental frequency the coefficient is associated with. In this case, n=1 is the fundamental term.

Hopefully you see that this approach can be used to find **any** coefficient (any value of a_n or b_n). All we have to do is multiply the series by either $\cos n\omega t$ or $\sin n\omega t$ and time average the result. Since most of the terms average to zero, the result can be summarized in the following set of rules. In truth, finding Fourier coefficients can be a very mechanical procedure that you can perform simply by learning these rules.

Let us start with any time varying signal, f(t). If f(t) is periodic over the interval $0 \le t \le T$, it can be broken down into a series of frequency components (coefficients) where:

$$\omega = \frac{2\pi}{T}$$

the coefficients are calculated by:

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt \quad \text{for } n = 0,\ 1,\ 2,\ 3,\ ....$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt \quad \text{for } n = 1,\ 2,\ 3,\ ....$$

Note that n goes from 0 to ∞ for a_n but n goes from 1 to ∞ for b_n. That is because there is no b_0 term. The sin of (nωt) where n=0 is always 0, thus b_0 is always 0.

The coefficents a_0, a_1, a_2, …, and b_1, b_2, b_3, … are the Fourier coefficients of the function, f(t). Now the original function f(t), can be described as the summation of many different sine and cosine functions.

$$f(t) = \frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t) + a_3\cos(3\omega t) + \cdots$$
$$+ b_1\sin(\omega t) + b_2\sin(2\omega t) + b_3\sin(3\omega t) + \cdots$$

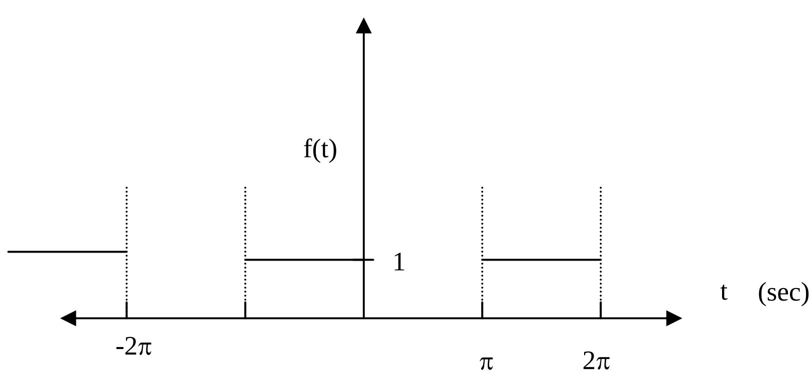
or,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right]$$

Example

Given the periodic function : $f(t) = \begin{cases} 0 & \text{when } 0 < t < \pi \\ 1 & \text{when } \pi < t < 2\pi \end{cases}$

which repeats every 2π seconds. A sketch of the function would look like:



The function can be expanded into a series of sine and cosine terms that when added together, replicate the original function. It is our job to find the coefficients of those terms.

First we must identify the period of the repeating function. Hopefully it is obvious that T = 2π seconds. From this we find the angular frequency, ω.

$$\omega = \frac{2\pi}{T} = \frac{2\pi \text{rad}}{2\pi \sec} = 1 \text{ rad} / \sec.$$

This is a convenient result since the angular frequency of harmonic terms is just nω = n rad/sec.

The coefficients are then found as follows. Notice that we break the integral up into 2 pieces where the function has two different constant values, zero and one.

$$a_n = \frac{2}{T}\int_0^{2\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi}\int_0^{\pi} 0*\cos(nt)\,dt + \frac{1}{\pi}\int_{\pi}^{2\pi} 1*\cos(nt)\,dt$$

$$a_n = \frac{1}{\pi}\int_{\pi}^{2\pi} \cos(nt)\,dt = \frac{1}{n\pi}\sin(nx)\Big|_{\pi}^{2\pi}$$

$$a_n = \frac{1}{n\pi}\left[\sin(n*2\pi) - \sin(n*\pi)\right] = 0$$

and

$$b_n = \frac{2}{T}\int_0^{2\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_0^{\pi} 0*\sin(nt)\,dt + \frac{1}{\pi}\int_{\pi}^{2\pi} 1\sin(nt)\,dt$$

$$b_n = \frac{1}{\pi}\int_{\pi}^{2\pi} \sin(nt)\,dt = -\frac{1}{n\pi}\cos(nx)\Big|_{\pi}^{2\pi}$$

$$b_n = -\frac{1}{n\pi}\left[\cos(n*2\pi) - \cos(n*\pi)\right]$$

$$b_n = -\frac{1}{n\pi}\left(1 - \cos(n\pi)\right) \quad \text{for n = odd numbers otherwise } b_n = 0$$

$$b_n = -\frac{2}{n\pi} \quad \text{for n = odd numbers}$$

and

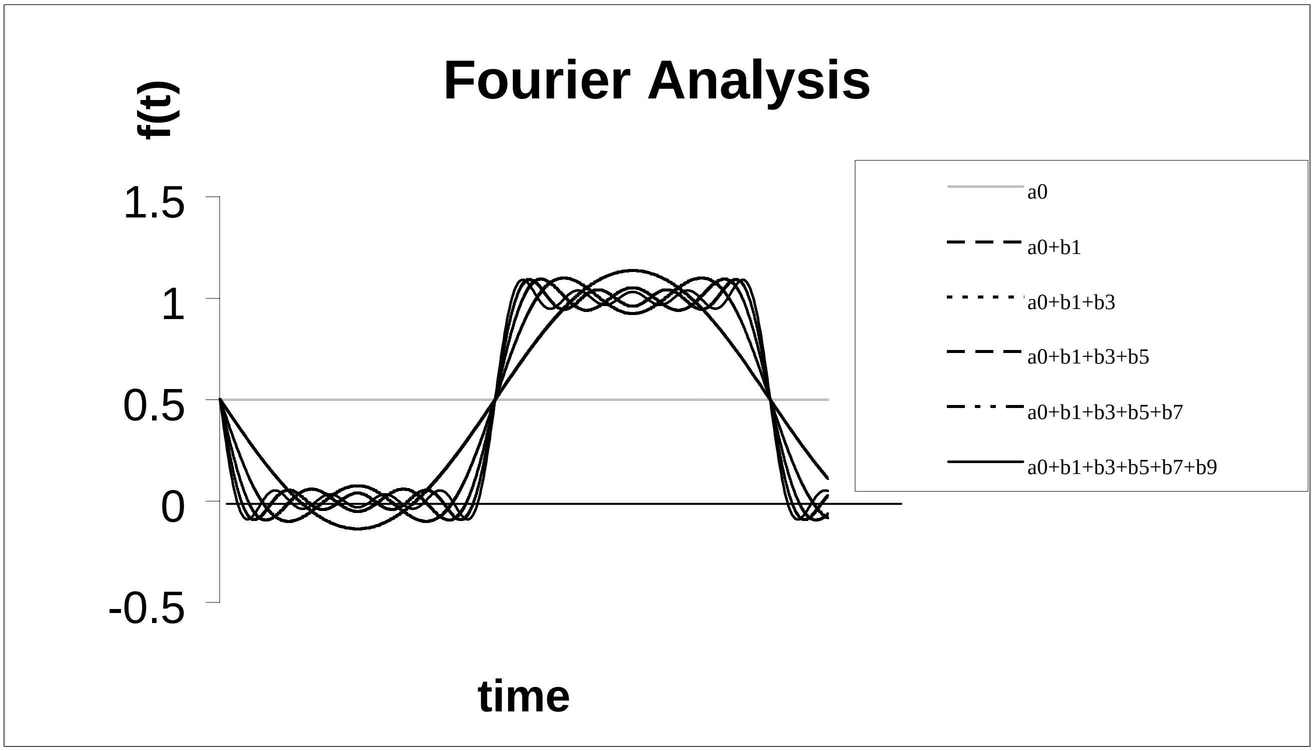
$$a_0 = \frac{2}{T}\int_0^{2\pi} f(t)\cos(0*t)\,dt = \frac{1}{\pi}\int_0^{\pi} 0*dt + \frac{1}{\pi}\int_{\pi}^{2\pi} 1*dt$$

$$a_0 = \frac{1}{\pi}\int_{\pi}^{2\pi} dt = 1$$

thus, the original function can be expanded to:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right] = \frac{1}{2} - \frac{2}{\pi}\left[\frac{\sin(1t)}{1} + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right]$$

If we added up all the terms of the Fourier Expansion, a graphical representation would look like this:



The important thing to note is that the original square wave function can be composed from adding components of multiple sine and cosine functions with frequencies that are multiples of the base frequency. The base frequency of the components is the same as the base frequency of the square wave.

Odd or Even Functions

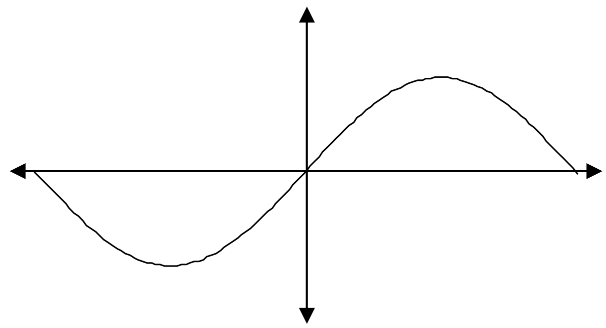
By looking at the form of the input signal, f(t), we can come up with some shortcut rules for deriving the coefficients. If we can determine if the f(t) is an odd or even function, we can determine whether the a or b coefficients are equal to zero as in the last example. A function is odd or even based on the following:

$$\text{Even Function}: \; f(-t) = f(t)$$
$$\text{Odd Function}: \; f(-t) = -f(t)$$

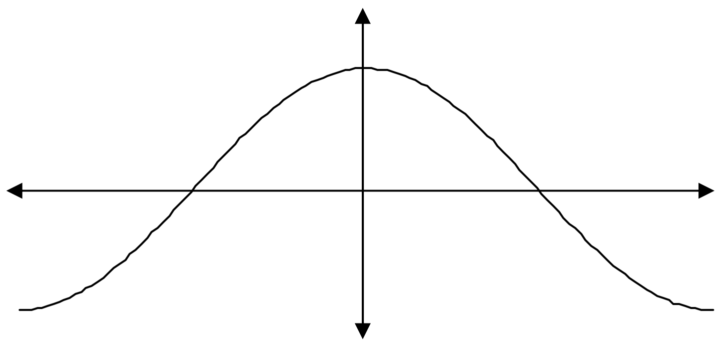
Even functions are thus functions that are symmetric about the y-axis. Odd functions are functions that are symmetric about the x-axis AND are mirror images of each other (symmetric about the origin). Many functions are neither odd nor even, but understanding this characteristic function type lets us anticipate which Fourier coefficients might be zero.

Some samples of even and odd functions.

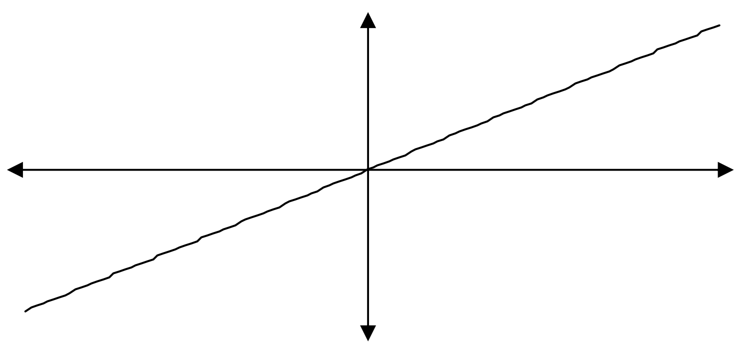
An odd function f(t) = sin (ωt)



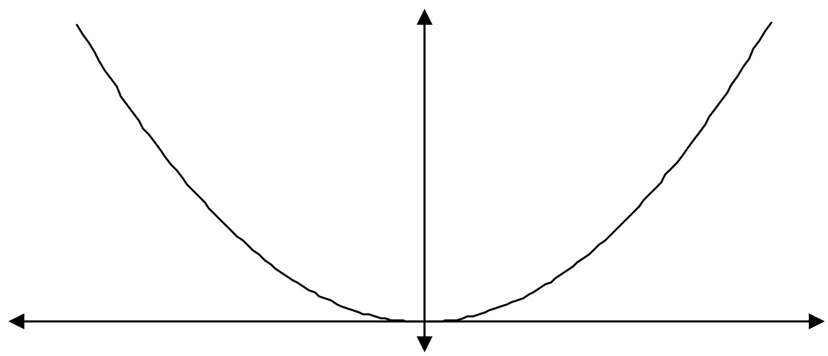
An even function f(t) = cos(ωt)



An odd function f(t) = t



An even function $f(t) = t^2$



Since cosines are even, other even functions are made up only of cosines. On the other hand, odd functions are made up only of sines. Thus the coefficients for the different type functions are:

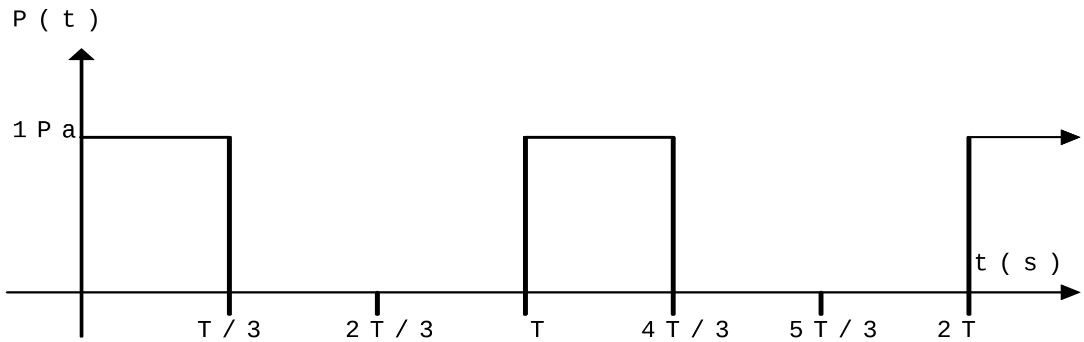
$$\text{If } f(x) \text{ is odd then } \begin{cases} a_n = 0 \\ b_n = \dfrac{2}{T}\displaystyle\int_0^T f(t)\sin(n\omega t)\,dt \quad \text{for } n = 1, 2, 3, \ldots. \end{cases}$$

$$\text{If } f(t) \text{ is even then } \begin{cases} a_n = \dfrac{2}{T}\displaystyle\int_0^T f(t)\cos(n\omega t)\,dt \quad \text{for } n = 0, 1, 2, 3, \ldots. \\ b_n = 0 \end{cases}$$

Remember, some functions are neither even or odd in which case you must simply calculate all the Fourier coefficients and see what results are obtained.

Problems

1. Given the following pressure function, ***p(t),*** which can be described as a square wave of ***1 Pa*** for ***T/3 sec***, and ***0 Pa*** for ***2T/3 sec*** shown below where ***T = 1 sec*** :



$$p(t) = \begin{Bmatrix} 1\,Pa, & 0 < t < \dfrac{T}{3} \\ 0\,Pa, & \dfrac{T}{3} < t < T \end{Bmatrix}$$

a) Is this function odd, even, both, or neither? How do you know?

b) What is the base or fundamental frequency of the square wave?

c) Perform the integrations to calculate the coefficient, "a_o".

d) Perform the integrations to calculate the coefficient, "a_n" coefficients.

e) Perform the integrations to determine the "b_n" coefficients.

f) Fill out the following table for $0 \le n \le 9$:

n	a_n (Pa)	b_n (Pa)	$T_n = T / n$ (sec)	$f_n = 1 / Tn$ (Hz)
0		N/A	N/A	N/A
1				
2				
3				
4				
5				
6				
7				
8				
9				

g) What is the pattern here? List the frequencies of the first nine non-zero harmonics of the fundamental that go make up the first nine terms of the Fourier Expansion.

Lesson 7

Fourier Series – Periodic Functions

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right] \text{ or,}$$

$$f(t) = \frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t) + \cdots$$
$$+ b_1 \sin(\omega t) + b_2 \sin 2(\omega t) + \cdots$$

for a function $f(t)$ where :

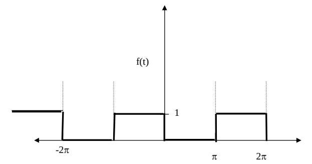
$\omega = \frac{2\pi}{T}$ the coefficients are calculated by :

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt \quad \text{for } n = 1, 2, 3, \ldots.$$

Example

$$f(t) = \begin{cases} 0 & \text{when } 0 < t < \pi \\ 1 & \text{when } \pi < t < 2\pi \end{cases}$$



Note: $T = 2\pi \text{ sec} \Rightarrow \omega = \frac{2\pi}{T} = 1\frac{\text{rad}}{\text{sec}}$

Coefficients

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)dt = \frac{1}{\pi}\int_\pi^{2\pi} 1\cos(nt)dt + \frac{1}{\pi}\int_0^{\pi} 0 * \cos(nt)dt$$

$$a_n = \frac{1}{\pi}\int_\pi^{2\pi}\cos(nt)dt = \frac{1}{n\pi}\sin(nx)\Big|_\pi^{2\pi}$$

$$a_n = \frac{1}{n\pi}\left[\sin(n*2\pi) - \sin(n*\pi)\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)dt = \frac{1}{\pi}\int_\pi^{2\pi} 1\sin(nt)dt + \frac{1}{\pi}\int_0^{\pi} 0 * \sin(nt)dt$$

$$b_n = \frac{1}{\pi}\int_\pi^{2\pi}\sin(nt)dt = -\frac{1}{n\pi}\cos(nx)\Big|_\pi^{2\pi}$$

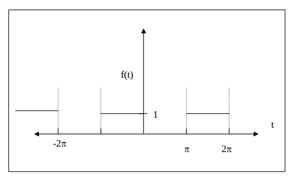
$$b_n = -\frac{1}{n\pi}\left[\cos(n*2\pi) - \cos(n*\pi)\right]$$

$$b_n = -\frac{1}{n\pi}(1 - \cos(n\pi)) \quad \text{for n = odd numbers otherwise } b_n = 0$$

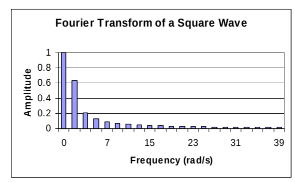
$$b_n = -\frac{2}{n\pi} \quad \text{for n = odd numbers}$$

Example

Time Domain



Frequency Domain

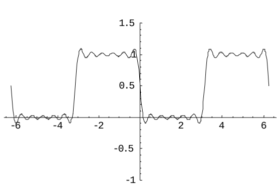


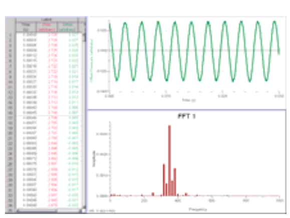
$$f(t) = \begin{cases} 0 & \text{when } 0 < t < \pi \\ 1 & \text{when } \pi < t < 2\pi \end{cases}$$

$$f(t) = \frac{1}{2} - \frac{2}{\pi}\left[\frac{\sin(1t)}{1} + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right]$$

Demos

- Mathematica
- Logger Pro

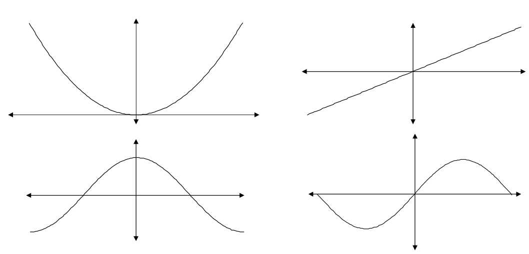




Odd and Even Functions

Even Function : $f(-t) = f(t)$

Odd Function : $f(-t) = -f(t)$



Even

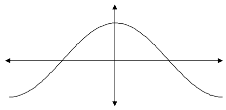
Odd

Odd and Even Functions

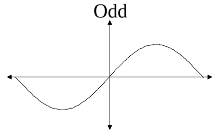
Even Function : $f(-t) = f(t)$

Odd Function : $f(-t) = -f(t)$

Even



Odd



If $f(t)$ is even then $\begin{cases} a_n = \dfrac{2}{T}\displaystyle\int_0^T f(t)\cos(n\omega t)\,dt \quad \text{for } n = 0,1,2,3,.... \\ b_n = 0 \end{cases}$

If $f(t)$ is odd then $\begin{cases} a_n = 0 \\ b_n = \dfrac{2}{T}\displaystyle\int_0^T f(t)\sin(n\omega t)\,dt \quad \text{for } n = 1,2,3,.... \end{cases}$

Spectrum Level and Band Level

Intensity, Intensity Level, and Intensity Spectrum Level

As a review, earlier we talked about the intensity of a sound wave. We related the **intensity** of a sound wave to the acoustic pressure where:

$$\langle I \rangle = \frac{p_{a\,max}^2}{2\rho c} \quad \text{or,}$$

$$\langle I \rangle = \frac{\langle p_a^2 \rangle}{\rho c} = \frac{p_{a\ rms}^2}{\rho c}$$

Next we defined the **Intensity Level**, L, as the decibel quantity defined to be consistent with the fact that our ears registered intensity on a logarithmic vice linear scale.

$$L \equiv 10 \log \frac{\langle I \rangle}{I_{ref}}$$

In this definition, I_{ref} was determined by a standard convention. Intensity Levels in water were most usually referenced to 1 μPa pressure which is equivalent to an intensity of 6.67 x 10^{-19} W/m^2. In air the intensity reference is most usually 1 x 10^{-12} W/m^2 which is equivalent to a reference pressure of 20 μPa.

We will now define a new quantity, the spectrum level or intensity spectrum level (ISL).

The intensity spectrum level (ISL) is the intensity level of the sound wave within a 1 Hz band.

This is accomplished by comparing the intensity in a 1 Hz band to the reference level in a 1 Hz band. The equation for the ISL is:

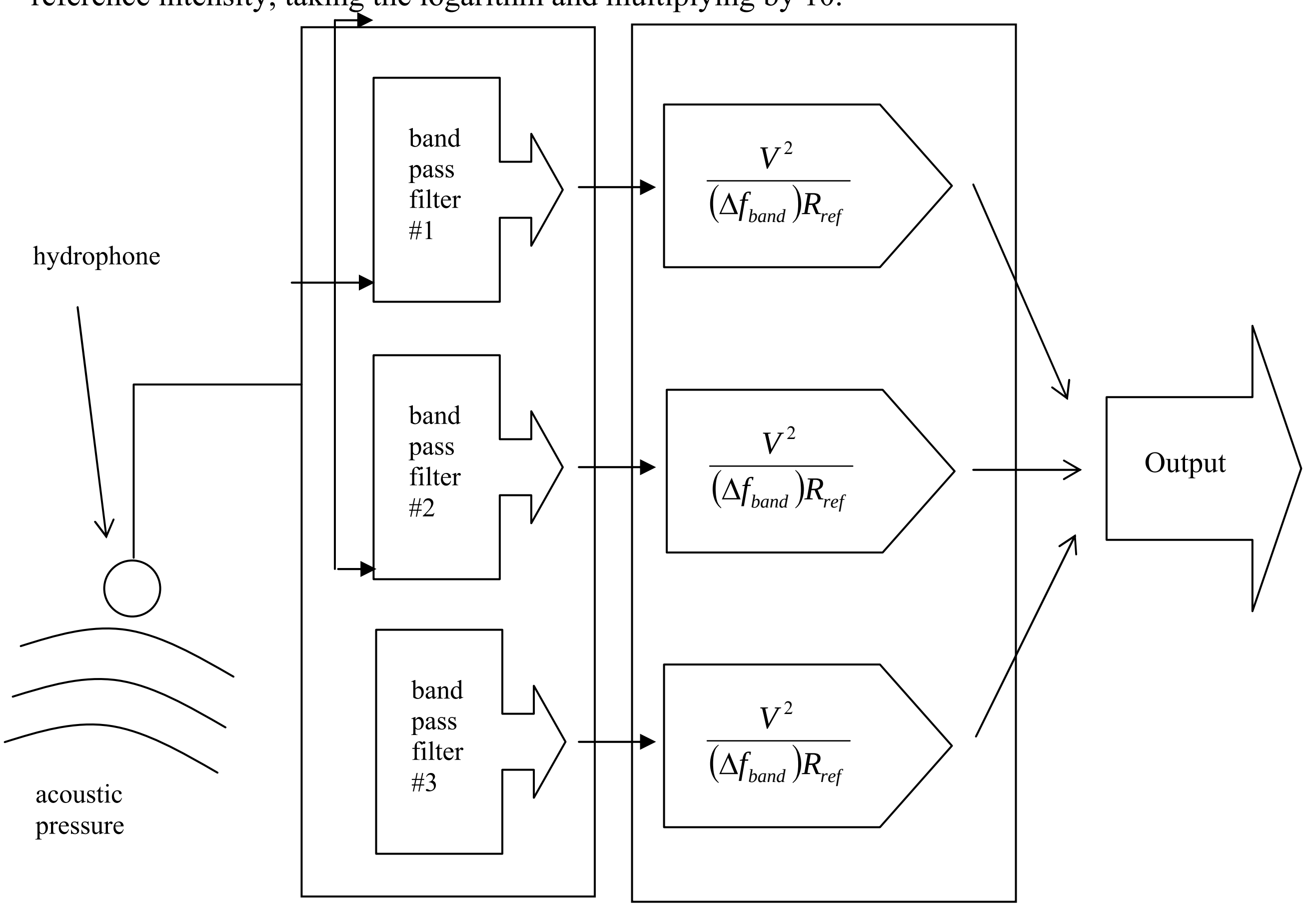
$$ISL = 10 \log \frac{I(\text{in 1 Hz band})}{I_{ref}(\text{in 1 Hz band})} = 10 \log \frac{I(\text{in 1 Hz band}) * 1\text{Hz}}{I_{ref}}$$

While this might seem a needless distinction, we can easily show in the lab that a pure tone with an intensity of 0.01 W/m2 is painfully loud. On the other hand, the same intensity spread as noise over the entire audible frequency bandwidth (20 Hz to 20 kHz) is nowhere near as loud. The ISL gives us the intensity in the 1 Hz band compared to the reference level (normally 1 μPa corresponding to 6.67 x 10^{-19} W/m^2 in water). This allows us to truly compare apples to apples. We saw in our brief study of Fourier analysis that most sound waves are made up of the combination of many different frequencies of sound waves. To compare one level to another, we must compare both levels within the <u>same</u> 1 Hz band. But what if the bandwidth of our equipment is different than 1 Hz?

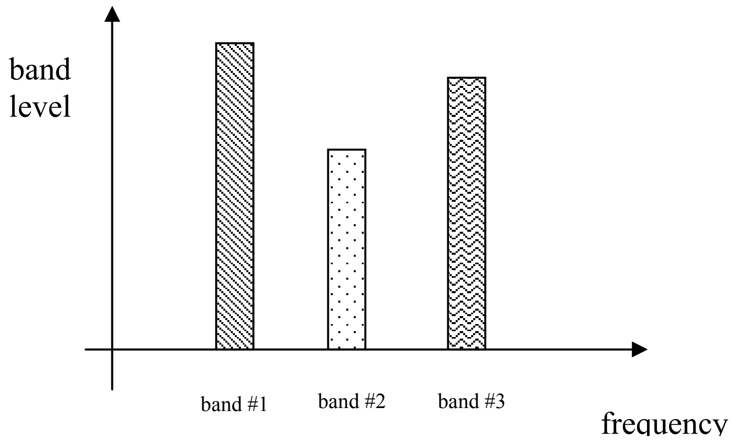
Band Level

Let's describe the processing within a typical sonar system. In this system, a sound wave is incident upon a transducer or hydrophone, which converts the varying acoustic pressure to a voltage level using a piezoelectric material such a naturally occurring quartz crystals or man made ceramic materials like PZT. Piezoelectric materials respond to external applied stresses by building up charge on their surfaces. This charge redistribution is sensed as a voltage by electrodes attached to the surfaces of the piezoelectric material.

This voltage is then passed through a set of parallel band pass filters to separate the voltage signal into the different frequency bands. After passing through the band pass filters, all frequencies outside the particular band are eliminated. The voltage representing pressure in the band is then converted to an rms power level by squaring the voltage and taking the average over a period of time called the integration time. An intensity level is created by dividing by the reference intensity, taking the logarithm and multiplying by 10.



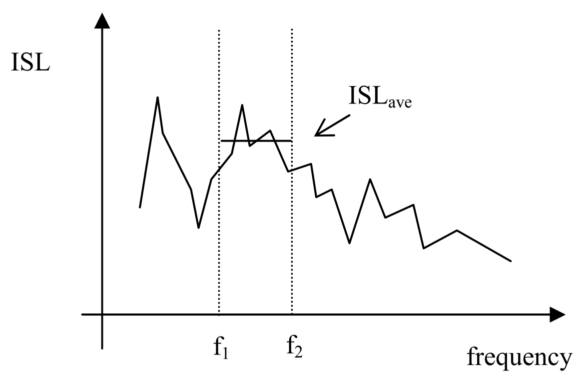
The result is then an average intensity level in each of the bands that we have split the signal into, as shown on the following graph. This is called a "Band Level" and given the symbol, BL. In this simple example system, there are only 3 bands. Real systems have many more bands. After the display is created as a plot of average intensity level in a band versus the frequency of the band, we see we have essentially created the Fourier Transform of the time domain signal. Devices capable of measuring and displaying frequency components of a signal in this manner are called spectrum analyzers.



The Band Level is the intensity level over a band other than 1 Hz.

Below is a plot representing the spectrum level of environmental noise within an imaginary environment. To determine the band level in the frequency band shown on the plot, we can use the following equation:

$$BL = 10\log\frac{I_{tot}}{I_{ref}} = 10\log\frac{I(\text{in a 1 Hz band})\Delta f}{I_{ref}}$$



The Intensity in a 1 Hz band is often called the Intensity Spectral Density. Using the multiplication rule for logarithms,

$$BL = 10\log\frac{I(\text{in a 1 Hz band})\text{x1Hz}}{I_{ref}} + 10\log\frac{\Delta f}{1\text{Hz}}$$

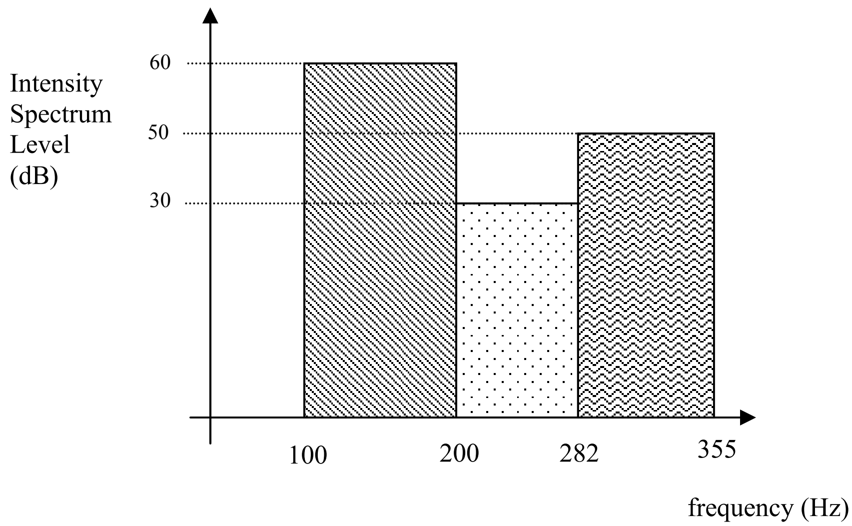
$$BL = ISL_{ave} + 10\log(\Delta f)$$

$$\text{where } \Delta f = f_2 - f_1$$

This equation can be used to compare the energy in a band other than a 1 Hz band. It might appear in the second term that we are taking the logarithm of quantity that has units. This is only because it is conventional to drop the 1 Hz in the denominator when writing the equation.

Example

Using the plot of ISL as a function of frequency shown below, calculate a) the band level of every band and b) the total band level.



a) To calculate the band levels:

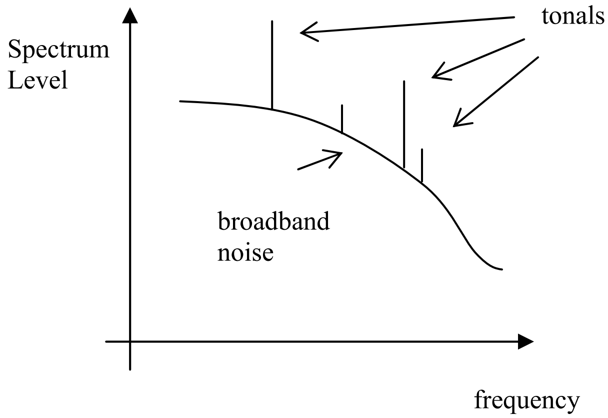
$$\mathrm{BL_1 = ISL_{ave} + 10\log(\Delta f)}$$
$$\mathrm{BL_1 = 60 + 10\log(100)}$$
$$\mathrm{BL_1 = 80\ dB}$$
$$\mathrm{BL_2 = 49.1\ dB}$$
$$\mathrm{BL_3 = 68.6\ dB}$$

b) To calculate the total band level:

$$BL_{tot} = 10\log\left(10^{BL_1/10} + 10^{BL_2/10} + 10^{BL_3/10}\right)$$
$$BL_{tot} = 80.3\ \mathrm{dB}$$

Types of Spectrums

A broadband spectrum is one where the sound pressure levels are spread continuously across a spectrum. A tonal spectrum is one where there is a discrete non-continuous spectrum with different frequency components. Sound pressure level measurements may contain components of broadband sounds as well as tonals as shown below.

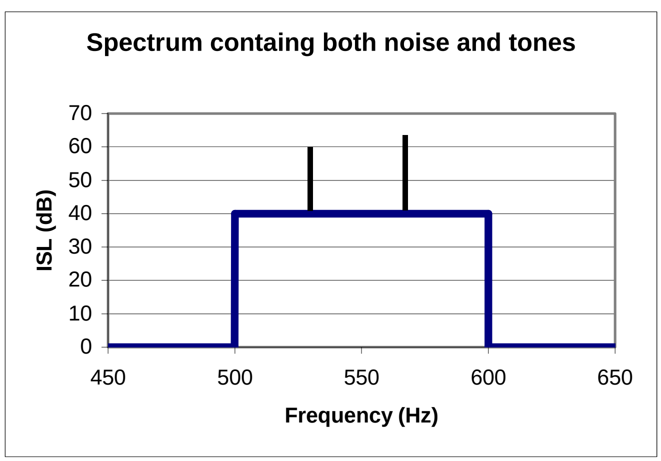


To calculate the total band level of broadband and tonal spectrums, we must add in the band level (or ISL if they have a 1 Hz bandwidth) of each tonal separately with the band level of the broadband noise. An equation would be:

$$BL_{tot} = \left[ISL_{ave} + 10\log\left(\Delta f_{BB}\right)\right] \oplus L_{tonal\ \#1} \oplus L_{tonal\ \#2} \oplus \ldots$$

Example

In the following figure, the background noise is constant (40 dB) over the band width from 500 to 600 Hz. There are two tones with levels of 60 dB and 63 dB respectively. What is the total Band Level?



$$BL_{tot} = \left[ISL_{ave} + 10\log\left(\Delta f_{BB}\right)\right] \oplus L_{tonal\ \#1} \oplus L_{tonal\ \#2}$$

$$BL_{tot} = 40\text{dB} + 10\log\left(100\text{Hz}\right) \oplus 60\text{dB} \oplus 63\text{dB} = 66\text{dB}$$

Bandwidth and Common Bandwidths

Using the example above, let's describe some features of the frequency bands. First, it is often easier to describe a frequency band by stating the center frequency and the bandwidth versus stating the bottom and top frequency of the band. The center frequency or "average" frequency (though it is not a true average) of a frequency band can be found by using the following definition:

$$f_c = \sqrt{f_1 f_2}$$

Mathematically this frequency is the geometric mean of the upper and lower frequencies. The bandwidth is simply:

$$\Delta f = f_2 - f_1$$

Constant Bandwidth

Where all bands are the same number bandwidth, i.e. all bands are 10 Hz wide.

Proportional Bandwidth

Where the ratio of the upper frequency to the lower frequency are constant.

One-octave bandwidth

The first band from 100 Hz to 200 Hz, used in the previous example is an example of a one-octave bandwidth. A one-octave bandwidth is where:

$$f_2 = 2^1 f_1$$

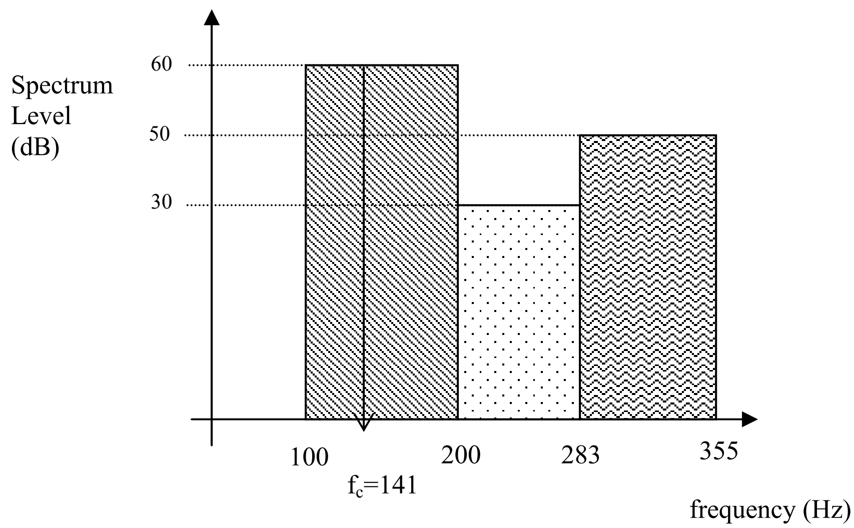
Also, using the definition to calculate the center frequency we find it is 141 Hz.

Half-octave bandwidth

The second band is an example of a half-octave bandwidth. A half-octave bandwidth is where:

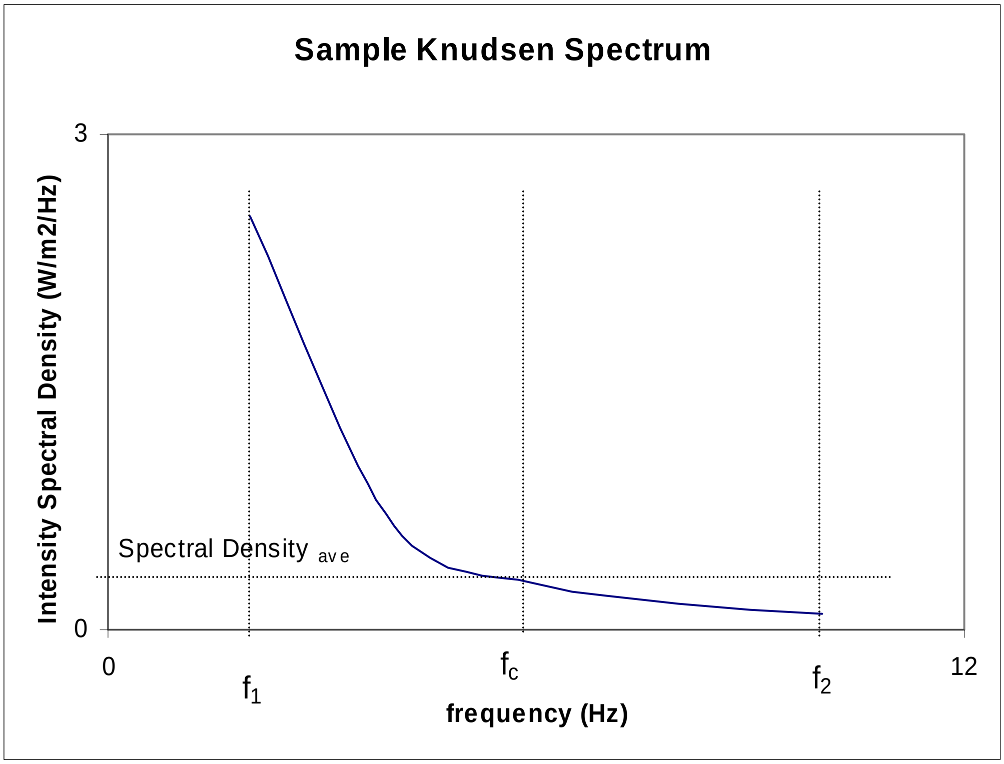
$$f_2 = 2^{1/2} f_1$$

Note also that the center frequency of the one-octave band above is where the octave is split into two half-octave bands. An octave band with a 200 Hz lower frequency has an upper frequency of 400 Hz. The center frequency is $\sqrt{200\text{Hz} * 400\text{Hz}} = 283\text{Hz}$, exactly the same number calculated from the definition of the half octave band. It may have seemed odd that the center frequency was not the simple average of the upper and lower frequencies. Hopefully this observation explains the use of the geometric mean for calculating the center frequency.



Appendix A – Why is center frequency a geometric mean vice a simple arithmetic average?

It has been observed that noise in the sea from the wind driven surface is not flat across all frequency bands. Instead we see that noise decreases with increasing frequency. The exact shape of non constant noise is called a Knudsen spectrum and Intensity Spectral Density is proportional to $1/f^2$.



The exact mathematical description of a Knudsen spectrum is:

$$dI = \frac{A}{f^2}df$$

where A is a constant. The intensity in a band from f_1 to f_2 is then:

$$I_{Band} = \int_{f_1}^{f_2} \frac{A}{f^2}df = A\left(\frac{1}{f_1} - \frac{1}{f_2}\right) = \frac{A}{f_1 f_2}\left(f_2 - f_1\right) = \frac{A}{f_1 f_2}\Delta f$$

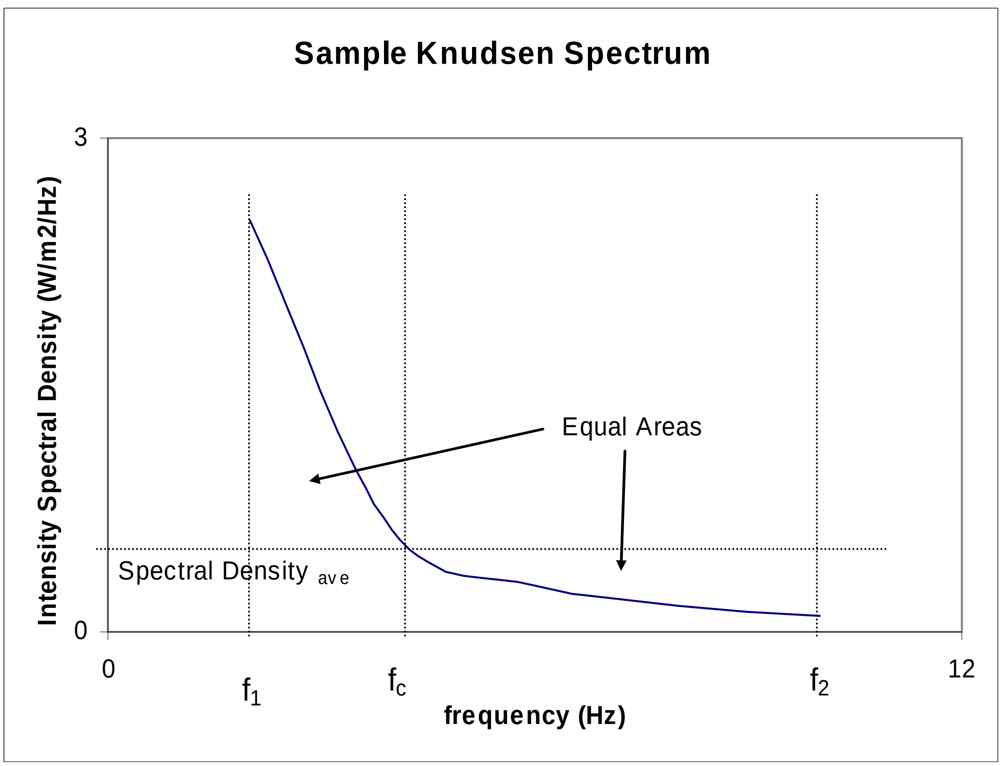
If the noise spectrum where constant, the total intensity in the band is just the Intensity Spectral Density x bandwidth. In the case of the non-constant spectral density, the term $\frac{A}{f_1 f_2}$ represents the best average value of the Intensity spectral density. It is evaluated at the center frequency.

$$\frac{A}{f_1 f_2} = \text{Spectral Density}_{ave} = \left.\frac{dI}{df}\right|_{ave} = \frac{A}{f_c^2}$$

As such, we see that the center frequency must be:

$$f_c = \sqrt{f_1 f_2}$$

Simply looking at the Knudsen spectrum above shows us the problem with using the arithmetic average for the center frequency. Our knowledge of approximate integration tells us the area above and below the average Intensity Spectral Density must be approximately equal. This is clearly not the case. Instead, a lower average value must be to balance the equal areas.



Problems

1. Given the following FFT for pressure p(t) {where T = 1 sec, p_o = 1 μPa, ρ = 1000 Kg/m^3, and c = 1500 m/s}:

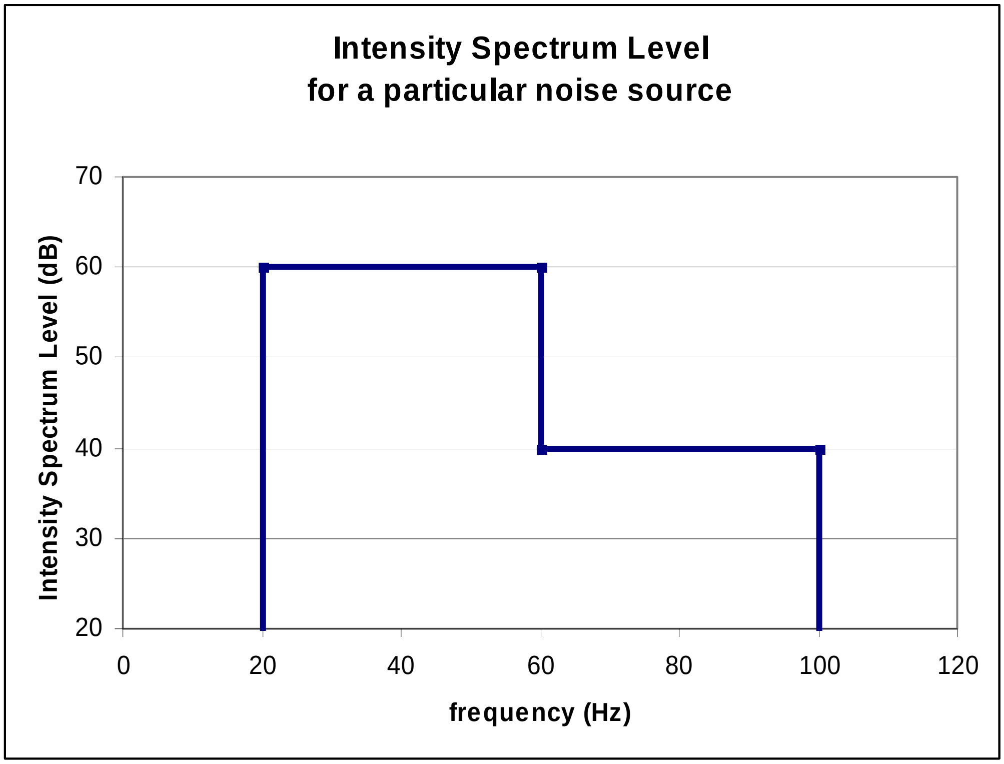
$$p(t) = [1 + 4\cos\left(2\pi 1t/T\right) - 3\cos(2\pi 2t/T) + 6\cos(2\pi 5t/T) +$$
$$2\sin(2\pi 1t/T) + 5\sin(2\pi 3t/T) + 8\sin(2\pi 5t/T)]\ Pa$$

a) Complete the table below for $0 \le n \le 6$:

n	a_n (Pa)	b_n (Pa)	T_n (sec)	f_n (Hz)	P^2_{rms} (Pa)	$\langle I_n \rangle$W/ m^2	BL_n (dB)
0							
1							
2							
3							
4							
5							
6							

b) Plot the Cosine Amplitude spectrum ***a_n* (Pa) vs f (Hz)**. Use your own graph paper
c) Plot the Sine Amplitude spectrum ***b_n* (Pa) vs f (Hz)**. Use your own graph paper.
d) Plot the time averaged Intensity spectrum, ***$\langle I_n \rangle$* (W/m^2) vs f (Hz)**. Use your own graph paper.
e) Plot the discrete Band Level spectrum, ***BL_n* (dB) vs f (Hz)**. Use your own graph paper.
f) Determine the total Band Level, ***BL_{TOT}*** for $0Hz \le f \le 6Hz$ and the average Intensity Spectrum Level, ***ISL_{AVE}*** using the information above.

2. The intensity Spectrum Level (ISL) is given below for a source of noise:



a) Over what frequency interval is the Intensity Spectrum Level a constant 60 dB?
b) What is the Intensity Spectrum Level in the range 60 Hz $<f<$ 100 Hz?
c) What is the Band Level (BL) for the noise in the frequency range 20 Hz $<f<$ 60 Hz?
d) What is the Band Level (BL) for the noise in the frequency range 60 Hz $<f<$ 100 Hz?
e) What is the Band Level (BL) for the noise in the frequency range 20 Hz $<f<$ 100 Hz?

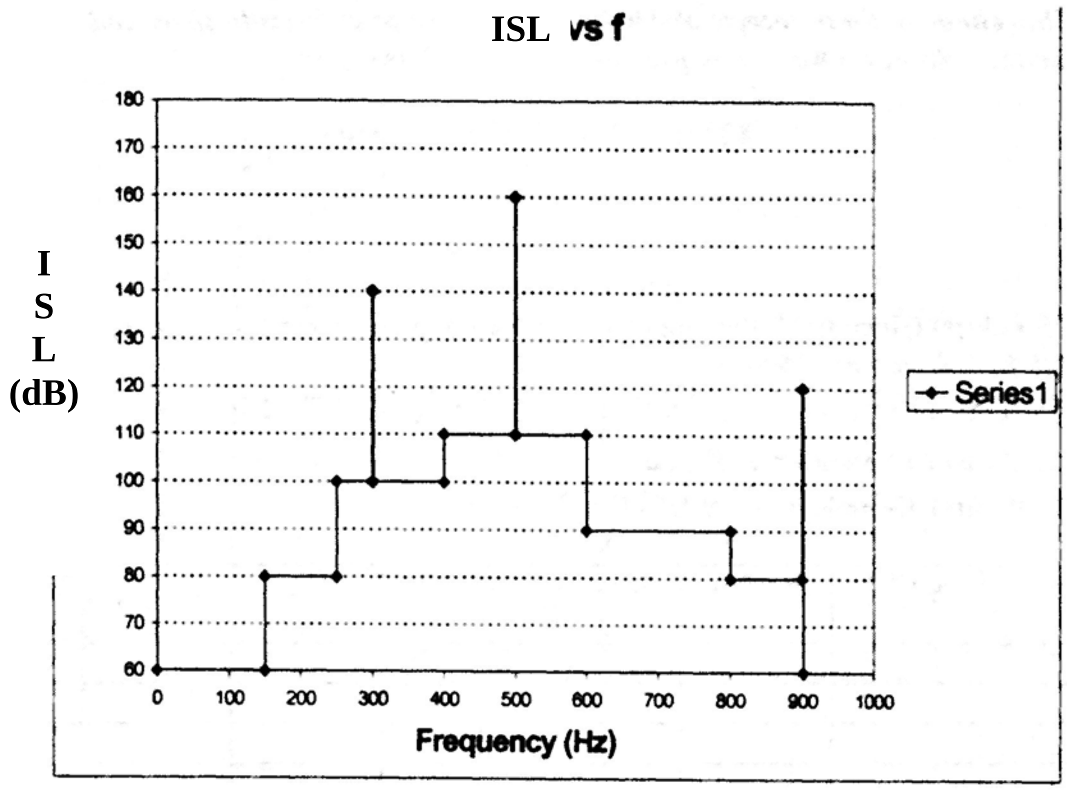
3. If the Band Level is 100 dB in a white noise bandwidth of 50 Hz, what is the (average) Intensity Spectrum Level?

4. For aone-third octave band centered on a frequency of 1000 Hz, calculate the lower and upper frequencies and the bandwidth.

5. The lower frequency of a one-third octave band is 200 Hz. Find the upper frequency, the center frequency, and the bandwidth.

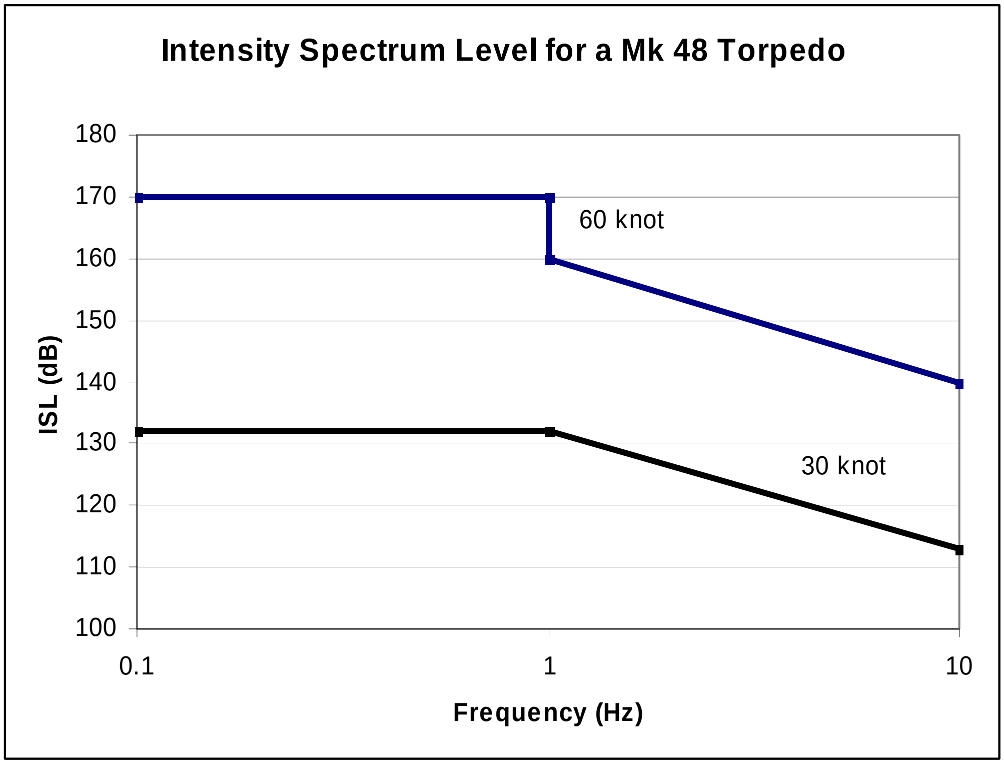
6. a) For a one-octave bandwidth, show that the bandwidth is about 71% of the center frequency.
b) For a half-octave bandwidth, show that the bandwidth is about 35% of the center frequency.
c) For a third-octave bandwidth, show that the bandwidth is about 23% of the center frequency.

7. Given the following graph of “white noise” and tonals:



a) Compute the total band level BL_{tot}, for a receiver having a one octave bandwidth centered around 637.4 Hz. Assume each tonal has a Δf=1 Hz.
b) Compute the average “white noise” intensity spectrum level, ISL_{ave}.

8. Using the figure below, estimate the noise level at 1 m from a U.S. Mark 48 torpedo traveling at 30 knots in the frequency band ranging 200 Hz to 10 kHz.



a) Repeat for a torpedo traveling at 30 knots.
b) Repeat for a torpedo traveling at 30 knots but for a sonar receiver with a band ranging from 100 Hz to 10 kHz.

Intensity, Intensity **Level**, and Intensity **Spectrum Level**

$$\langle I \rangle = \frac{p^2_{a\,max}}{2\rho c} \quad \text{or,}$$

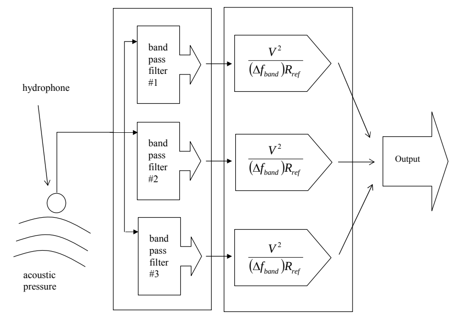
$$\langle I \rangle = \frac{\langle p_a^2 \rangle}{\rho c} = \frac{p^2_{a\ rms}}{\rho c}$$

$$L \equiv 10 \log \frac{\langle I \rangle}{I_{ref}}$$

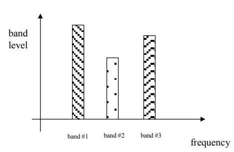
The spectrum level is the intensity level of the sound wave **within a 1 Hz band**.

$$ISL = 10 \log \frac{I(\text{in 1 Hz band})}{I_{ref}(\text{in 1 Hz band})} = 10 \log \frac{I(\text{in 1 Hz band}) \bullet 1\text{Hz}}{I_{ref}}$$

A Sonar System



System Output

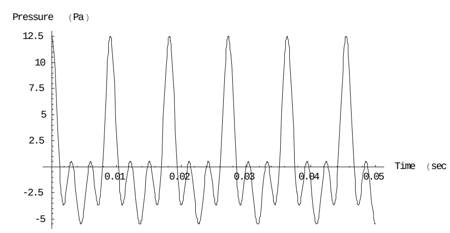


- Fourier Transform of the Time Domain Signal
- Frequency Analyzer

Example

$$p(t) = \left\{\frac{1}{2} + 5\cos\left[2\pi(110\text{Hz})t\right] + 3\cos\left[(2)2\pi(110\text{Hz})t\right] + 4\cos\left[(3)2\pi(110\text{Hz})t\right]\right\} \text{Pa}$$

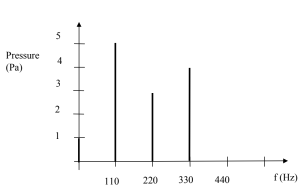
$$f(t) = \frac{1}{2}a_0 + a_1\cos(\omega t) + a_2\cos(2\omega t) + \cdots$$
$$+ b_1\sin(\omega t) + b_2\sin 2(\omega t) + \cdots$$



Fourier Coefficients

n	f_n (Hz)	P_{max} (Pa)	P^2_{rms} (Pa)	I (W/m²)	L (dB)
0	0	1	0.5	3.3×10^{-7}	117
1	110	5	12.5	8.3×10^{-6}	131
2	220	3			
3	330	4			

$$p(t) = \left\{\frac{1}{2} + 5\cos\left[2\pi(110\text{Hz})t\right] + 3\cos\left[(2)2\pi(110\text{Hz})t\right] + 4\cos\left[(3)2\pi(110\text{Hz})t\right]\right\} \text{Pa}$$



$$P^2_{max} = a_n^2 + b_n^2$$

$$P^2_{rms} = \frac{P^2_{max}}{2}$$

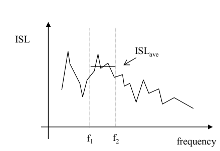
$$\langle I \rangle = \frac{P^2_{rms}}{\rho c} = \frac{P^2_{max}}{2\rho c}$$

Band Level

$$BL = 10\log\frac{I_{tot}}{I_{ref}} = 10\log\frac{I(\text{in a 1 Hz band})\Delta f}{I_{ref}}$$

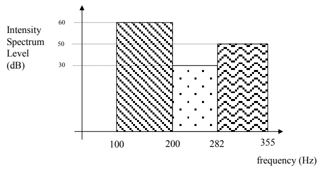
$$BL = 10\log\frac{I(\text{in a 1 Hz band})1\text{Hz}}{I_{ref}} + 10\log\frac{\Delta f}{1\text{Hz}}$$

$BL = ISL_{ave} + 10\log(\Delta f)$
where $\Delta f = f_2 - f_1$



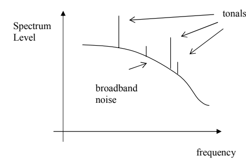
The Band Level is the intensity level over a band other than 1 Hz.

Example



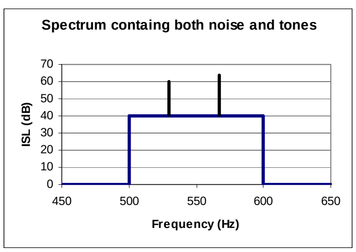
- Using the plot of ISL as a function of frequency shown, calculate
  - the band level of every band
  - the total band level
  - the total band level in a band from 150 Hz to 300 Hz

Types of Spectrums



$$BL_{tot} = \left[ISL_{ave} + 10\log\left(\Delta f_{BB}\right)\right] \oplus L_{tonal\ \#1} \oplus L_{tonal\ \#2} \oplus \ldots$$

Example



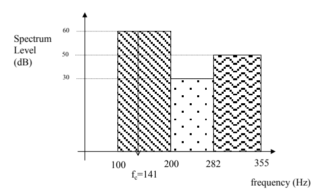
What is the total BL?

Common Bandwidths

- Constant Bandwidth

  $\Delta f = \text{constant}$
- Proportional Bandwidth
  - Octave Bandwidth
    - $f_2 = 2^1 f_1$
  - Half Octave Bandwidth
    - $f_2 = 2^{1/2} f_1$
- Center Frequency

  $f_c = \sqrt{f_1 f_2}$



Preferred Octave Bands

TABLE 4.2

Comparison of 1-octave and ⅓-octave bands

1 Octave			⅓ Octave		
Lower cutoff frequency (Hz)	Center frequency (Hz)	Upper cutoff frequency (Hz)	Lower cutoff frequency (Hz)	Center frequency (Hz)	Upper cutoff frequency (Hz)
11	16	22	14.1	16	17.8
			17.8	20	22.4
			22.4	25	28.2
22	31.5	44	28.2	31.5	35.5
			35.5	40	44.7
			44.7	50	56.2
44	63	88	56.2	63	70.8
			70.8	80	89.1
			89.1	100	112
88	125	177	112	125	141
			141	160	178
			178	200	224
177	250	355	224	250	282
			282	313	355
			355	400	447
355	500	710	447	500	562
			562	630	708
			708	800	891
710	1,000	1,420	891	1,000	1,122
			1,122	1,250	1,413
			1,413	1,600	1,778
1,420	2,000	2,840	1,778	2,000	2,239
			2,239	2,500	2,818
			2,818	3,150	3,548
2,840	4,000	5,680	3,548	4,000	4,467
			4,467	5,000	5,623
			5,623	6,300	7,079
5,680	8,000	11,360	7,079	8,000	8,913
			8,913	10,000	11,220
			11,220	12,220	14,130
11,360	16,000	22,720	14,130	16,000	17,780
			17,780	20,000	22,390

Why do we care?

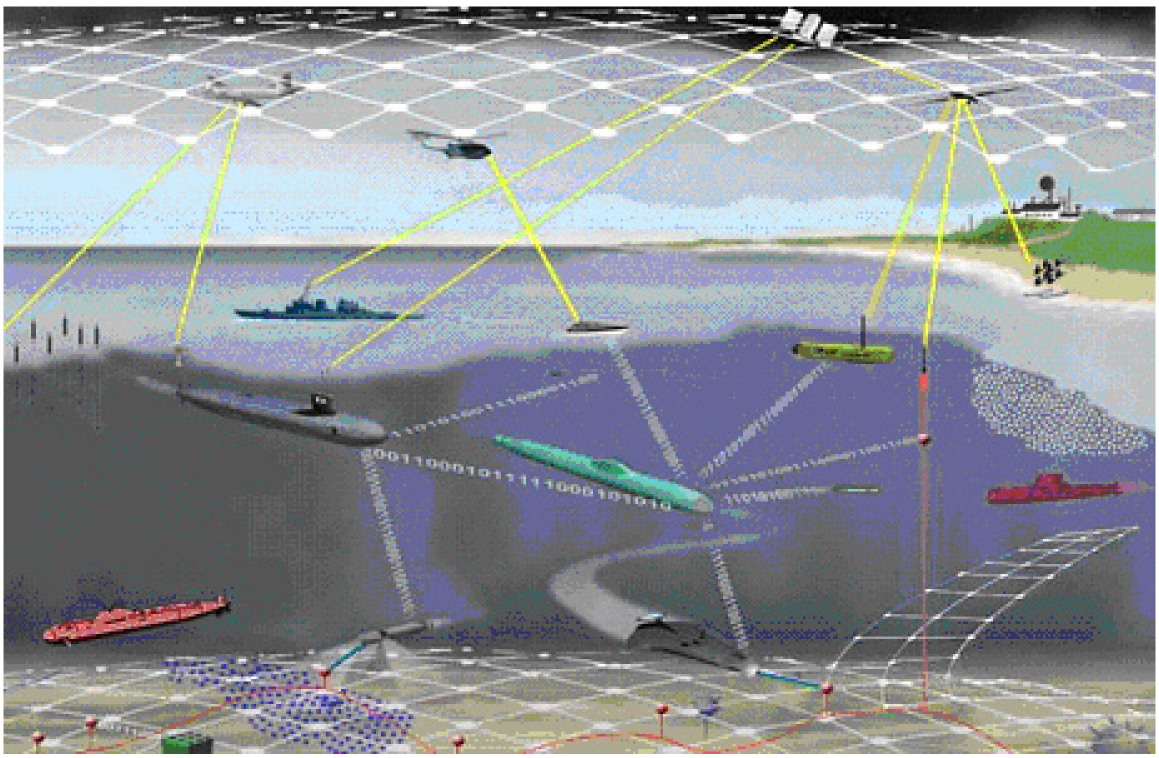
Source Level (SL) and Noise Level (NL) are both examples of Band Levels (BL) where the frequency band(s) are defined by the frequencies of our Sonar System

Passive SONAR Equation Intro

There are many forms of equations that comprise the passive SONAR equation but what is common amongst all forms of the equation is that they try to quantify all the affects on passive detection of sound from a contact or target. The form of the equation that will be presented this semester most often is:

$$L_{S/N} = SL\text{-}TL - (NL\text{-}DI)$$

In the below schematic of the undersea battle space, the sound emitted by one of the submarine platforms is represented by Source Level, SL. Losses sustained during sound propagation are represented by the Transmission Loss, TL. Back ground noise in the battle space are represented by the Noise Level, NL, and receiver characteristic, Directivity Index, DI.



Why are we presenting this now?

For the rest of the semester, we will take a closer look at each of the components of the passive SONAR equation. The goal is for each of you to enhance your understanding about each of the components and understand what actions can minimize or maximize the factors as appropriate to your tactical undersea warfare situation. Undersea warfare is a complex process where those with the best understanding of their craft, survive to fight again.

Parts of the Passive SONAR Equation

Signal to Noise Ratio

The intensity level on the left is the ratio of the signal received by a sonar receiver to the noise. Anyone who has ever tuned a radio station manually has experienced the station's signal in the static noise of the receiver. Signal to noise ratio is an important concept because it represents the degree to which an amplifier can be successfully employed to improve this situation. If signal to noise ratio (S/N or SNR) is two low, the noise is nearly equal to the signal. In this case, amplification will also increase the noise and provide no substantial improvement. For high signal to noise ratios, amplification will improve the magnitude of the signal relative to the noise.

A very good question to ask is, how large a signal to noise ratio is necessary? For consumer electronic audio, listeners demand a very high SNR. If all that is necessary is the identification of information, low SNRs might be tolerated. In fact, some systems adopt the convention that the minimum SNR required is 1.0. Regardless of the exact nature of the detection criteria used, we refer to the criteria as Detection Threshold (DT). Any actual signal above the Detection Threshold is referred to as Signal Excess. Sometimes we set the minimum signal to noise ratio such that a trained sonar operator will be able to pick a target out of noise 50% of the time. We refer to this signal to noise ratio for 50% detection as the "Recognition Differential."

Remember that the passive sonar equation compares "levels" (in dB) vice the actual intensities. As such, $L_{S/N}$ is defined

$$\mathrm{L}_{\mathrm{S/N}} = 10\log\left(\frac{\mathrm{Signal}_{\mathrm{required}}}{\mathrm{Noise}}\right)$$

Detection Threshold and Recognition Differential are also a decibel quantities.

Signal Level Received

The signal level received at the detector is the difference of the first two quantities on the right side of the SONAR equation above. The origin of these two terms is the intensity of the signal that is transmitted to the water from the target. This is called the Source Level (SL).

$$SL = 10\log\frac{I_S}{I_0}$$

$$I_S \equiv \text{ Signal Intensity}$$

$$I_0 \equiv \text{ Reference Intensity}$$

As the signal travels through the water, some of the signal is lost through various mechanisms. The totality of this loss is quantified as the Transmission Loss (TL).

$$TL = 10\log\frac{I_S}{I_R} \qquad \text{(For a plane wave)}$$

$$I_R \equiv \text{ Received signal intensity}$$

The source level minus the transmission loss determines how much signal is received at the detector.

$$L_s = (SL - TL) = \text{Signal Level}$$

As a general rule, Transmission Loss is a dependent on the distance between the source and the receiver. Since this distance is often the tactically significant quantity in an undersea engagement, we often rearrange the passive sonar equation to solve for the Transmission Loss. The loss that can be tolerated and still meet the detection criterion is termed Figure of Merit. This quantity provides a means to estimate the distance at which detection can be achieved.

$$TL_{allowable} = \text{Figure of Merit} = SL - L_{S/N\ Threshold} - (NL - DI)$$

There are several conventions we will adopt in refining these basic definitions. Specific items to pay attention to are the location of the source level. Additionally, the frequency bands that contain the signal and noise must always be considered.

Noise Level Present

The Noise Level (L_n or NL) is the sum of the total effect of background and self-noise hindering our ability to detect the target signal. Background noise much be estimated from a variety of sources including wind and weather, shipping, biologic activity, and industrial activity

$$NL = 10\log\frac{I_n}{I_0}$$

$$I_n \equiv \text{ Noise intensity}$$

The Directivity Index (DI) is a ratio of the noise level detected by the detector, to the total noise level over 360°.

$$DI = 10\log\frac{N_{ND}}{N_D}$$

$$N_{ND} \equiv \text{ Noise power from non-directional receiver}$$

$$N_D \equiv \text{ Noise power generated by actual receiver}$$

When a detector is omni-directional, the power ratio is one, corresponding to 0 dB. If a sonar receiver is an array of elements, beams (directions) are formed where the system is more responsive due the interference of coherent sound. In this case, all isotropic noise does not reach the receiver. Since only the noise in the correct beam reaches the receiver, it is effectively lowered compared to the omni-directional case.

The Noise Level (L_n) is the sum of the total effect of background and self-noise hindering our ability to detect the target signal.

$$L_n = (NL - DI)$$

Nomenclature conventions and the Passive SONAR Equation

There is little standardization in the symbols used in various references. In writing the passive sonar equation above, the nomenclature adopted by various academic sources (Urick) was used. In the Naval Warfare Publications (NWPs), the Navy has a slightly different set of symbols.

An "L" quantity is an absolute level (dB referenced to a standard). An "N" quantity uses level subtraction to compare two intensities or pressures. These two acoustic parameters can be measured at different platforms or different times. The below chart compares some of the terms used in the passive (and active) sonar equation.

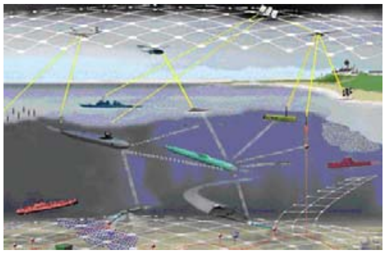
Urick	Description	Navy
SL	Source Level	L_S
TL	Transmission Loss	N_W
NL	Noise Level	L_N
DI	Directivity Index	N_{DI}
DT	Detection Threshold	N_{RD}
$L_{S/N}$ SNR	Signal to Noise Ratio	N_{SN}
RL	Reverberation Level	L_R
TS	Target Strength	N_{TS}

Problems

1. A submarine is conducting a passive barrier patrol against a transiting enemy submarine. The friendly sub has a sonar with a directivity index of 15 dB and a detection threshold of 8 dB. The enemy sub has a source level of 140 dB. Environmental conditions are such that the transmission loss is 60 dB and the equivalent isotropic noise level is 65 dB.
   a) What is the received signal level?
   b) What is the received signal-to-noise ratio in dB?
   c) What is the figure of merit?
   d) Can the enemy sub be detected? Why?

2. A submarine is attempting to detect an aircraft carrier transiting the Straits of Malacca. The aircraft carrier has a source level of 90 dB. The submarine's passive sonar has a directivity index of 20 dB and a detection threshold of 15 dB. Biological noise in the Straits is 54 dB. The submarine's self noise is 50 dB. Given that TL = 10 log r, where r is the range in yards (we will show you where this comes from soon), at what range can the carrier just be detected?

The Passive Sonar Equation



Will the sensor detect the red submarine?

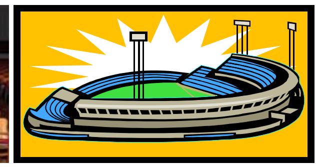
Signal to Noise Ratio

Signal

$$\frac{\text{Signal}}{\text{Noise}} = \frac{\text{Signal Intensity}}{\text{Noise Intensity}}$$

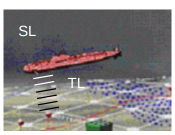
The higher the SNR, the more likely you are to hear (detect) the signal.

Noise (quiet)



Noise (Loud

Source Level and Transmission Loss



$$SL = 10\log\left(\frac{I_S}{I_0}\right)$$

$$TL = 10\log\left(\frac{I_S}{I_R}\right)$$

$$L_R = SL - TL$$

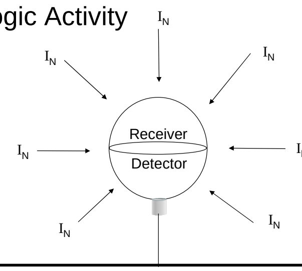
$$L_{S/N} = 10\log\left(\frac{\text{Signal}}{\text{Noise}}\right) = L_R - 10\log\left(\frac{\text{Noise Intensity}}{I_0}\right) = SL - TL - 10\log\left(\frac{\text{Noise Intensity}}{I_0}\right)$$

Sources of Noise

- Shipping Noise
- Wind and Weather
- Marine Life – Biologic Activity
- Self Noise
  - Flow of Water
  - Machinery

Omni directional Noise

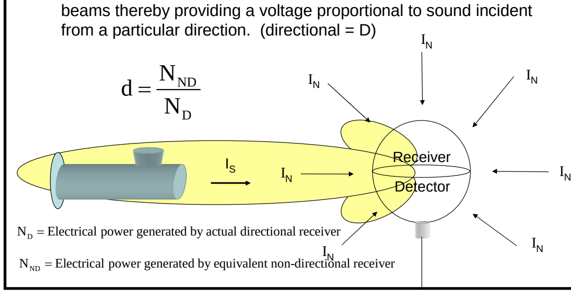
Isotropic Noise



Directivity Factor

- Some detectors are only able to provide a voltage proportional to all incident sound from all directions. (non-directional = ND)
- Other detectors use more sophisticated signal processing and form beams thereby providing a voltage proportional to sound incident from a particular direction. (directional = D)

$$d = \frac{N_{ND}}{N_D}$$



N_D = Electrical power generated by actual directional receiver

N_{ND} = Electrical power generated by equivalent non-directional receiver

The Passive Sonar Equation

$$L_{S/N} = 10\log\left(\frac{\text{Signal}}{\text{Noise}}\right) = L_R - 10\log\left(\frac{\text{Noise Intensity}}{I_0}\right) = SL - TL - 10\log\left(\frac{\text{Noise Intensity}}{I_0}\right)$$

$$L_{S/N} = 10\log\left(\frac{\text{Signal}}{\text{Noise}}\right) = L_R - L_{N\ \text{Received}} = SL - TL - (NL - DI)$$

$$\boxed{L_{S/N} = SL - TL - (NL - DI)}$$

$$SL = 10\log\left(\frac{I_S}{I_0}\right) \qquad NL = 10\log\left(\frac{I_N}{I_0}\right)$$

$$TL = 10\log\left(\frac{I_S}{I_R}\right) \qquad DI = 10\log(d)$$

Figure of Merit

- Often a **detection threshold** is established such that a trained operator should be able to detect targets with that $L_{S/N}$ half of the time he hears them. Called "**Recognition Differential.**" **(RD)**
- Passive sonar equation is then solved for TL allowable at that threshold. Called "**Figure of Merit.**" **(FOM)**

$$TL_{allowable} = \text{Figure of Merit} = SL - L_{S/N\ Threshold} - (NL-DI)$$

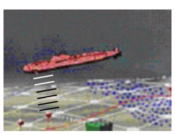
- Since TL logically depends on range, this could provide an estimate of range at which a target is likely to be detected. Called "**Range of the Day.**" **(ROD)**
- Any $L_{S/N}$ above the Recognition Differential is termed "**Signal Excess**." **(SE)** Signal Excess allows detection of targets beyond the Range of the Day.

Example

- A hostile submarine with a Source Level, SL = 130 dB re 1 μPa is near a friendly submarine in a part of the ocean where the Noise Level from all sources, NL = 70 dB re 1 μPa. The directivity factor is 3000 for the friendly submarine's sonar. If the Recognition Differential for the friendly submarine is 20 dB, what is the Figure of Merit?
- If the actual Transmission Loss is 50 dB, what is the Signal Excess.

Signal to Noise Level

$$L_{S/N} = 10\log\left(\frac{\text{Signal Intensity}}{\text{Noise Intensity}}\right) = 10\log\left(\frac{\text{Signal Intensity}}{I_0}\right) - 10\log\left(\frac{\text{Noise Intensity}}{I_0}\right)$$



But we will be measuring the signal intensity level at the receiver/detector, I_R (in the frequency band of the detector)

This is different from the signal intensity level leaving the target, I_S (in the frequency band of the detector)

$$\frac{I_R}{I_0} = \frac{I_S}{I_0}\frac{I_R}{I_S} = \frac{\frac{I_S}{I_0}}{\frac{I_S}{I_R}} \quad \text{OR} \quad 10\log\left(\frac{I_R}{I_0}\right) = 10\log\left(\frac{I_S}{I_0}\right) - 10\log\left(\frac{I_S}{I_R}\right)$$

$$L_R = SL - TL$$

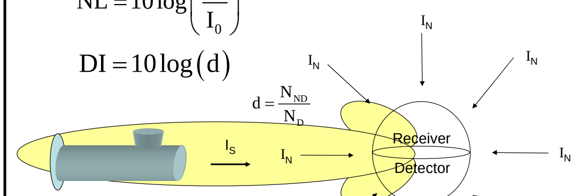
Fraction of emitted intensity reaching receiver

Noise Level and Directivity Index

$$\frac{I_{N\ Received}}{I_0} = \frac{I_N}{I_0}\frac{1}{d} \quad \text{OR} \quad L_{N\ Received} = 10\log\left(\frac{I_{N\ Received}}{I_0}\right) = 10\log\left(\frac{I_N}{I_0}\right) - 10\log(d)$$

$$NL = 10\log\left(\frac{I_N}{I_0}\right) \qquad L_{N\ Received} = NL - DI$$

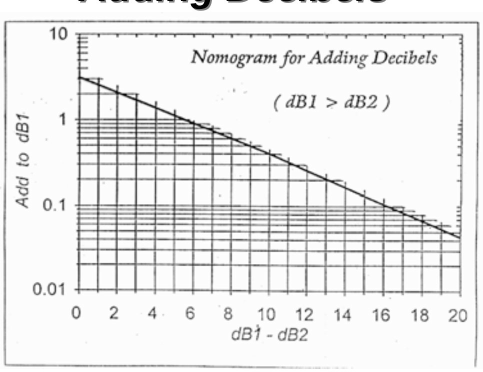
$$DI = 10\log(d) \qquad d = \frac{N_{ND}}{N_D}$$



N_D = Electrical power generated by actual directional receiver

N_{ND} = Electrical power generated by equivalent non-directional receiver

Adding Decibels



Transmission Loss

The oceans form a very complex medium. Sound in the medium does many things except follow a straight path as we would want it to. We will study how the ocean affects the propagation of sound.

To this point we have introduced the passive SONAR equations and provided some mathematical definitions. From this section on, we will break the equation into the individual terms and discuss all the factors that determine the value for each.

Our first parameter is Transmission Loss, TL. Transmission Loss is the parameter that compares the amount of intensity of the signal at a specific range from the source to the source intensity at one yard. The equation for this would be:

$$TL \equiv 10\log\frac{I(1\text{ yd})}{I(r)}$$

Note: Due to early work in transmission loss being done referencing the intensity at one yard from the source, all quantities for the transmission loss equations typically are IN YARDS! Be careful.

Why is transmission loss defined as it is?

As you should remember, levels are expressed in decibels and are just ratios of one quantity to another. This allows us to express terms with large variations (several orders of magnitude or so) to each other rather easily. It also makes calculating quantities in the SONAR equations much easier.

For example, let us look at the combination of the source level and transmission loss, which defines the quantity, the received source level, L_s. If we substitute the definition of source level and transmission loss into the equation for the received source level, we get:

$$L_s = SL - TL = 10\log\frac{I(1\text{ yd})}{I_{ref}} - 10\log\frac{I(1\text{ yd})}{I(r)}$$

$$L_s = 10\log\frac{I(1\text{ yd})}{I_{ref}} + 10\log\frac{I(r)}{I(1\text{ yd})}$$

$$L_s = 10\log\frac{I(r)}{I_{ref}}$$

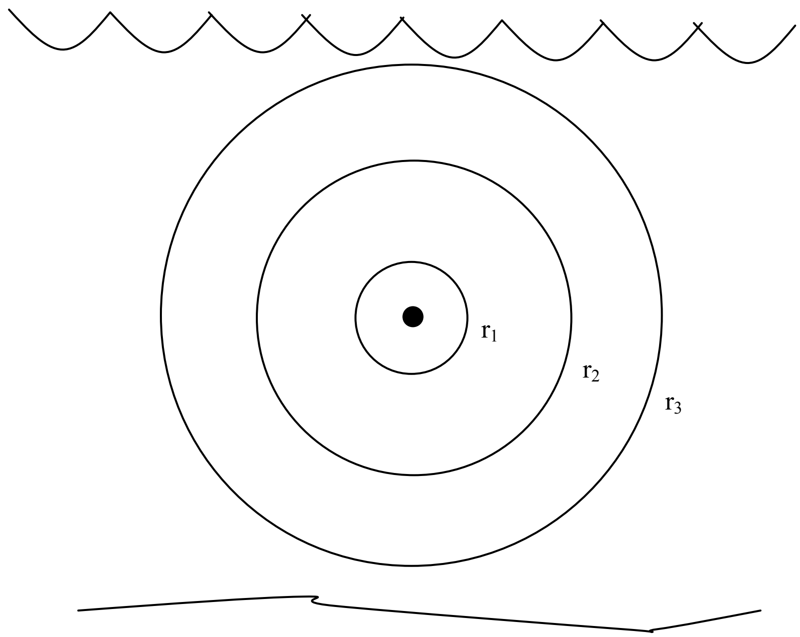
We can now see that the resultant, the received source level, is the ratio (in dB) of the actual intensity at range, r, compared to the intensity of a reference signal with an rms pressure of 1 μPa.

Transmission Loss results from:

1. **Geometric losses due to one of two types of spreading, spherical or cylindrical.**
2. **Attenuation due to absorption, scattering, viscosity, and thermal losses. This will be discussed more, later in the section.**

Spreading Loss

Let's assume we have a point source, which emits a signal in all directions (that is in three dimensions.) The source would produce wave fronts that were spheres that would grow in size as the wave propagates away from the source. Note that the power in each wave front



would be a constant, even though the size of the wave front would grow. (This assumes no power is lost from the wave due to attenuation.) The power of a wave front or sphere can be expressed in terms of the intensity of the wave where:

$$P = IA = I\left(4\pi r^2\right)$$

But again, we said the power of each wave front is a constant as the wave propagates so:

$$P_1 = P_2$$
$$I_1 4\pi r_1^2 = I_2 4\pi r_2^2$$
$$\frac{I_1}{I_2} = \frac{4\pi r_2^2}{4\pi r_1^2} = \left(\frac{r_2}{r_1}\right)^2$$

Note the reversal of the subscripts on the right side of the last equation compared to the left side. As we see the intensity decreases as $1/r^2$.

Now using the definition of Transmission Loss:

$$TL = 10\log\frac{I(1\text{ yd})}{I(r)}$$

where

$$\frac{I(1\text{ yd})}{I(r)} = \frac{4\pi(r)^2}{4\pi(1\text{ yd})^2} = \frac{r^2}{(1\text{ yd})^2}$$

thus

$$TL = 10\log\frac{r^2}{1^2}$$

$$TL = 20\log r$$

NOTE: The range, r, is the range in yards since the definition references the intensity at range r, to the intensity at 1 yard.

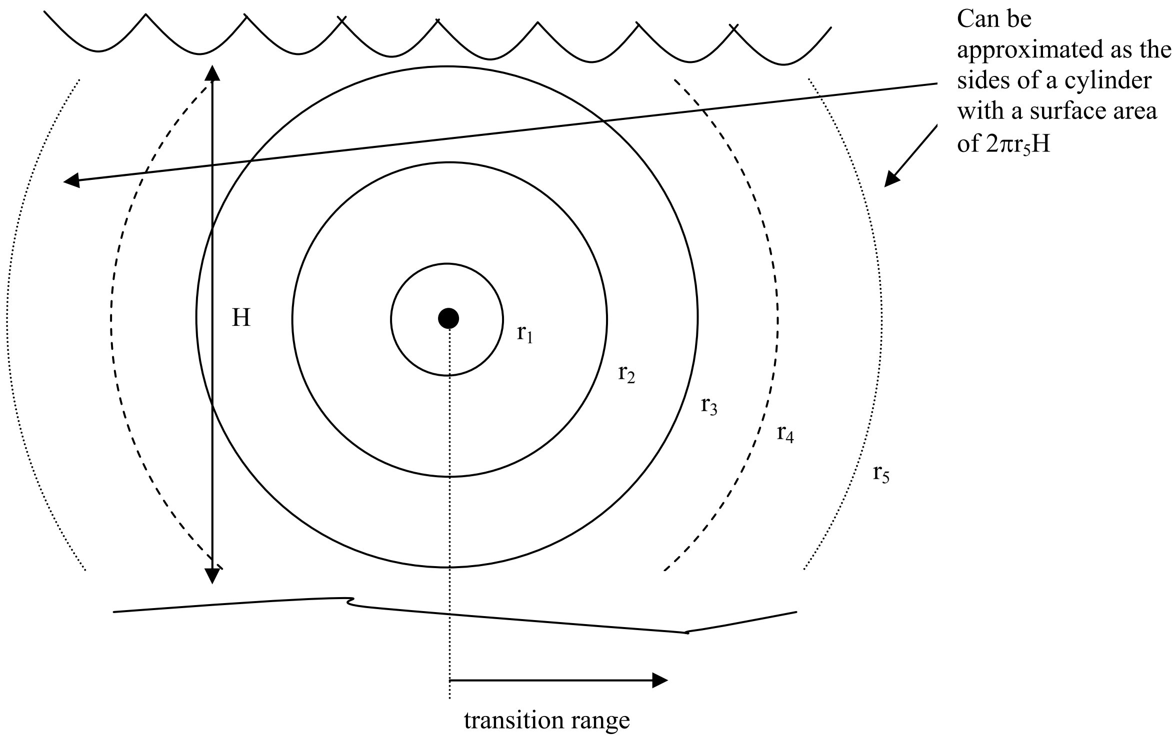
The equation above is for transmission loss only due to spherical spreading. Spherical spreading is the **most dominant factor** in the transmission loss portion of the passive SONAR equation but there are other factors that must be considered.

Cylindrical Spreading

What if the sound wave from a source does not spread out in 3-dimension but gets trapped within some boundaries such as in a surface duct? If we expand the picture that we previously used to show the wave fronts spreading out from a source, we see that the fronts at some point no longer form concentric spheres.

When the wave fronts hit the surface and the bottom, as in the case shown below, the power in the wave fronts is reflected back into the ocean and power is not lost out of the "top" of the wave fronts. All the power of the wave front is resident in the sides of the "cylinders" formed by the wave fronts. We can think of the wave front as being a concentric cylinder that spreads out from the source. Again, the power of each wave front is a constant but now the area of the cylinder where the power is resident is:

$$A = 2\pi rH$$



Note that we can neglect the surface area of the top and bottom of the cylinder because the power is reflected back at those surfaces and we assume no power is lost in the up or down direction.

Using the same mathematical process as we used for the spherical spreading case, we can determine the transmission loss if we only consider cylindrical spreading where:

$$\frac{I(1\text{ yd})}{I(r)} = \frac{2\pi r H}{2\pi(1\text{ yd})H} = \frac{r}{(1\text{ yd})}$$

$$TL = 10\log\frac{I(1\text{ yd})}{I(r)}$$

$$TL = 10\log r$$

The only limitation of this equation is that it does not take into account the spreading of the wave spherically until it reaches the "transition range" where the wave starts to spread cylindrically. To derive an equation that takes both factors into account, we can use one of the properties of logarithms where:

$$TL = 10\log\frac{I(1\text{ yd})}{I(r)} = 10\log\frac{I(1\text{ yd})}{I(r_0)} + 10\log\frac{I(r_0)}{I(r)}$$

where r_0 is the transition range. If the first term accounts for the spherical spreading and the second term accounts for the cylindrical spreading, substituting in our previous equations for each, the sum becomes:

$$TL = 20\log r_0 + 10\log\frac{r}{r_0}$$

Thus, if there is a transition range, r_0, we can calculate the total transmission loss due to both forms of spreading losses.

Transition Range

The range at which spreading losses switch from spherical to cylindrical spreading is not easily determined. The best method to determine the transition range would be to use a complex computer model of the ocean and determine the transition range. One approximation that can be used is presented in Urick, **Principles of Underwater Sound** on page 153. In the book, the author presents one formula where:

$$r_0 = \sqrt{\frac{RH}{8}}\sqrt{\frac{H}{H-d}}$$

where the terms in the equation are defined as:

$$H = \text{mixed layer thickness}$$
$$d = \text{depth of the source}$$
$$R = \frac{c_n}{g\cos\theta_n} = \text{radius of curvature of sound ray}$$

This equation has some **severe limitations**. One, it is only good for the case of a well-defined surface layer (the most likely case where cylindrical spreading will occur) of depth, H, with a constant gradient, g. Two, this is only valid for **one particular ray** of sound from the source. Any other ray that leaves the source at a different angle θ will have a different transition range. Thus, at best this equation can only be used as a rough approximation.

In summary, the student may assume that only spherical spreading occurs unless told otherwise in a problem. Further, the student should not expect to calculate the transition range but should be given a value when one is required.

Attenuation

Attenuation is the lessening of the intensity per unit distance the sound travels. Attenuation losses occur from both **absorption** losses and **scattering** losses. Seawater is a dissipative medium, it absorbs part of the energy of the transmitted wave.

Absorption is the conversion of acoustic energy to heat in the fluid. There are three main causes of absorption losses:

1. Viscosity – Shear and Volume viscosity where as the molecules “rub” together, acoustic energy is converted to thermal energy heats up the fluid.
2. Change in Molecular Structure – Molecules in the seawater are disassociated or broken down into component ions which then recombine after the sound wave travels over the molecule. Above 100 kHz this involves the relaxation of $MgSO_4$ and above 1kHz the relaxation of $(B(OH)_3)$.
3. Heat Conduction – This process is negligible and we will not present it here.

This attenuation causes a decrease in the amplitude of the wave and an exponential decrease in the acoustics pressure resulting in more spreading loss. To account for attenuation in the transmission loss equation, we must define a new term, α, the attenuation coefficient. Using this new term, we can calculate the transmission loss using the equation:

$$TL_{attenuation} = \alpha\left(r - (1yd)\right) \times 10^{-3} \text{ dB}$$

where r is in yards.

Generally, since the range, r, is usually much greater than 1 yard, we can ignore the -1 yard term in the equation. Thus the transmission loss can be expressed as:

$$TL_{attenuation} = \alpha\left(r \times 10^{-3}\right)\text{dB}$$

The student should also note that α will be expressed in dB/kyd or dB per thousand yards. In the equation, that is accounted for by the x10^{-3} to convert the range, r, from yards to kiloyards (kyd).

The most difficult problem will be to determine a correct value for α, the attenuation coefficient.

Viscosity

The viscosity losses are due to two distinct effects. Each of these effects are dependent on not only how the molecules act together in the medium as defined by the coefficients of both shear and volume viscosity but also the frequency of the sound waves.

The first, shear viscosity must be accounted for due to the movement between layers in the medium itself or the shearing of the medium. A theoretical prediction for how shear viscosity affects the attenuation coefficient, α, is given by the equation:

$$\alpha_{shear} = \frac{16\pi^2}{3\rho c^3}\mu_s f^2$$

where μ_s is the shear viscosity coefficient, which is a property of the medium.

The second viscosity effect, volume viscosity, must be accounted for due to the "time lag" of molecules filling in holes in the molecular structure of the medium. A theoretical prediction for the volume viscosity affect is given by the equation:

$$\alpha_{volume} = \frac{16\pi^2}{3\rho c^3}\left(\frac{3}{4}\right)\mu_v f^2$$

where μ_v is the volume viscosity coefficient, again a property of the medium.

When both terms are combined and nominal values used for the density, speed of sound and the coefficients, the value for the attenuation coefficient becomes:

$$\alpha = 2.75\times10^{-4} f^2$$

where f is the sound wave frequency in kHz. This theoretical value is still 3 times less than actual measure values for higher frequencies. Another factor must account for further attenuation.

Ionic Relaxation

The most dominant disassociation-re-association process in the attenuation coefficient for seawater involves the finite time that it takes for magnesium sulfate ($MgSO_4$) ions to dissociate and reassociate as a sound wave passes through the medium.

$$\mathrm{MgSO_4 + H_2O \leftrightarrow Mg^{+3} + SO_4^{-2} + H_2O}$$

If the period of the wave is different than the time necessary for the molecule to recompose itself (relaxation time), the process is reproduced at every change in density as the wave moves by and permanently dissipates energy. An equation to describe how this process affects the attenuation coefficient is:

$$\alpha_{MgSO_4} = \frac{40f}{4100 + f^2}$$

where once again, the frequency, f, is in kHz.

It is particularly interesting that this process has such a large affect on the attenuation coefficient when $MgSO_4$ makes up less than 5% (by weight) of the total dissolved salts in seawater.

Another disassociation-re-association process that becomes a dominant factor in determining the attenuation coefficient below approximately 1 kHz, is the boron-borate relaxation process.

$$\mathrm{B(OH)_3 + (OH)^- \leftrightarrow B(OH)_4^-}$$

Though many factors affect this complex process, simply suffice it to say that an equation for this process' affect on α would be:

$$\alpha_{boron-borate} = \frac{0.1 f^2}{1+f^2}$$

A non-absorption factor, scattering

The last factor that we will discuss is the scattering of sound energy due to inhomogeneities in seawater. This factor we can approximate as a constant, not dependant on frequency and would only be a dominant factor below 100 Hz or so. This can be expressed as:

$$\alpha_{scattering} = 0.003 \ \mathrm{dB}/\mathrm{kyd}$$

Attenuation Summary

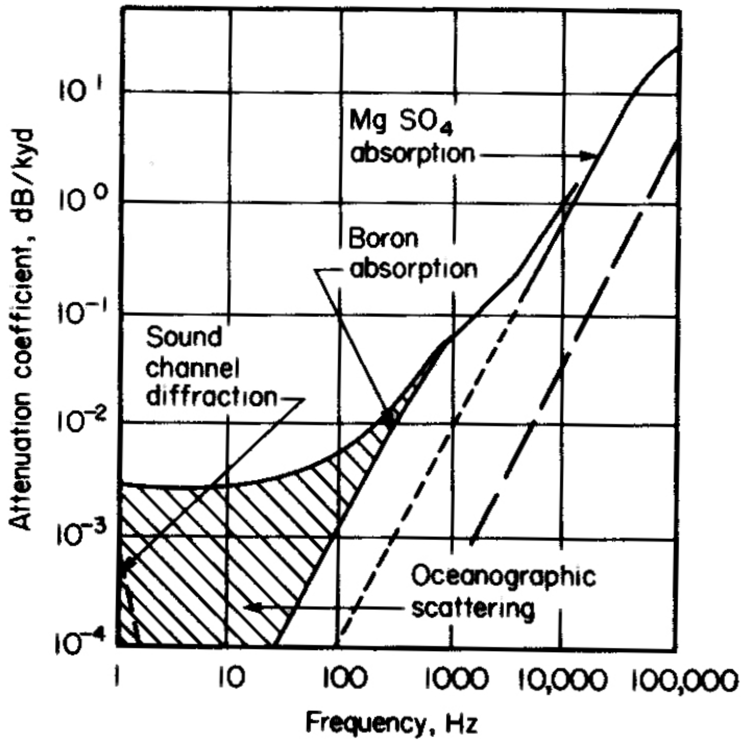
When all these factors are combined, the equation for transmission loss then becomes:

$$TL = \alpha\left(r \times 10^{-3}\right)\mathrm{dB}$$

where

$$\alpha = \left(0.003 + \frac{0.1 f^2}{1+f^2} + \frac{40 f^2}{4100+f^2} + 2.75\times10^{-4} f^2\right) \mathrm{dB}/\mathrm{kyd}$$

A plot of the transmission coefficient, α, as a function of frequency, is shown below.



(Source: **Principles of Underwater Sound,** Third Edition, Robert J. Urick, McGraw-Hill Book Company, 1983, p. 111) One should note from the graph that the attenuation coefficient is **very** small for any frequency below 10,000 Hz. Below 10 kHz, the attenuation coefficient is less than 1 dB **per thousand yards**. Generally speaking, one can usually neglect attenuation at any frequency below 10,000 Hz.

Francois-Garrison Attenuation Model

(Source: **An Introduction to Underwater Acoustics**, 1st Edition, Xavier Lurton, Springer-Verlag, 2002, p. 21)

$$\alpha = A_1 P_1 \frac{f_1 f^2}{f_1^2 + f^2} + A_2 P_2 \frac{f_2 f^2}{f_2^2 + f^2} + A_3 P_3 f^2$$

The first term is for Boric acid, the second for Magnesium Sulfate and the third for viscosity. For frequencies less then 1 kHz, this results in $\alpha < 0.01\,\text{dB/km}$, for f = 10 kHz, $\alpha \approx 1\,\text{dB/km}$, for f = 100 kHz, $\alpha \approx 30 - 40\,\text{dB/km}$, resulting in max ranges of @ 1km, and for f = 1MHz limits the max range to @ 100m.

Depth Dependence

Depth can have a large impact on sidescan sonar and underwater data transmission. If the frequency is high enough so that $MgSO_4$ is the major attenuation effect, we can multiply α by the pressure effect, P_2. As an example, α = 40 dB/km at the ocean surface, 30 dB/km at 2000 m, and 22 dB/km at 4000 m.

Active Transmission Loss

For the case of active sonar, the one way transmission loss is determined exactly the same as the passive case. The return echo experiences exactly the same transmission loss on its way back. Because of this, the active transmission loss is **twice** that of the passive case for the same range. Note that because spherical and cylindrical spreading are logarithmic, it would be incorrect to double the range.

Summary

What is the dominant factor in transmission loss? Does the frequency of the signal determine the amount of transmission loss? Can any of the factors in the transmission loss equation be neglected? These are some of the questions that the student should try to answer before using the transmission loss equations.

The most dominant factor in all of the transmission loss problems will be spherical spreading. If we have a source level 1 yard from our source, at what distance will the received signal level drop by 3 dB (½ original intensity)? In other words, due to spherical spreading, at what range, r, will the transmission loss be 3 dB? Solving for this using the spherical spreading equation:

$$TL = 3\text{ dB} = 20\log\left(r \big/ (1\text{ yd})\right)$$
$$\therefore r = 1.4\text{ yds}$$

In other words, the intensity drops quickly due to spherical spreading. If we do the same for a signal at 50 kHz and only consider attenuation, the range where the intensity drops to ½ its original intensity would be:

$$TL = 3\text{ dB} = \alpha\left(r - (1\text{ yd})\times 10^{-3}\right)$$
$$\alpha = 3.16\ \text{dB}\big/\text{kyd}\ \text{from graph above}$$
$$\therefore r = 950\text{ yds}$$

This is a significantly further range than calculated for the spherical spreading case. Notice also that the frequency was very high. If the frequency were lower, the range for a 3 dB attenuation transmission loss would increase significantly. **Thus spherical spreading is usually the only significant factor in figuring transmission loss.**

Problems:

1. Calculate the transmission loss for the following:
    a) spherical spreading only at 10 yds, 100 yds, 1000 yds.
    b) cylindrical spreading only at 10 yds, 100 yds, 1000 yds. (assume transition range, r_o=1yd)

2. What is the transmission loss over a range of 70 kyds if the transition range is 30 kyds? (Ignore attenuation.)

3. For only spherical spreading with absorptive losses, calculate the transmission loss for a 5 kHz sound at:
    a) 1 km
    b) 5 kyds
    c) At what range would the transmission loss be 70 dB?

The following information is to be used to do the next problem:

4. An acoustic signal is used to control a remote sub. The signal has a source level of 220 $dB_{re\ 1\ \mu Pa}$ at 1 yard from the source attached to the bottom of the research ship.
    a) If the absorption coefficient of seawater at a particular frequency is 0.095 dB/Kyd, at what distance would the acoustic signal lose ½ of its intensity due to absorption?
    b) Given the signal above, at what range from the source would the signal lose ½ its intensity due to spherical spreading?
    c) What is the absorption coefficient of seawater if the frequency of the transmitter is raised to 10 kHz?

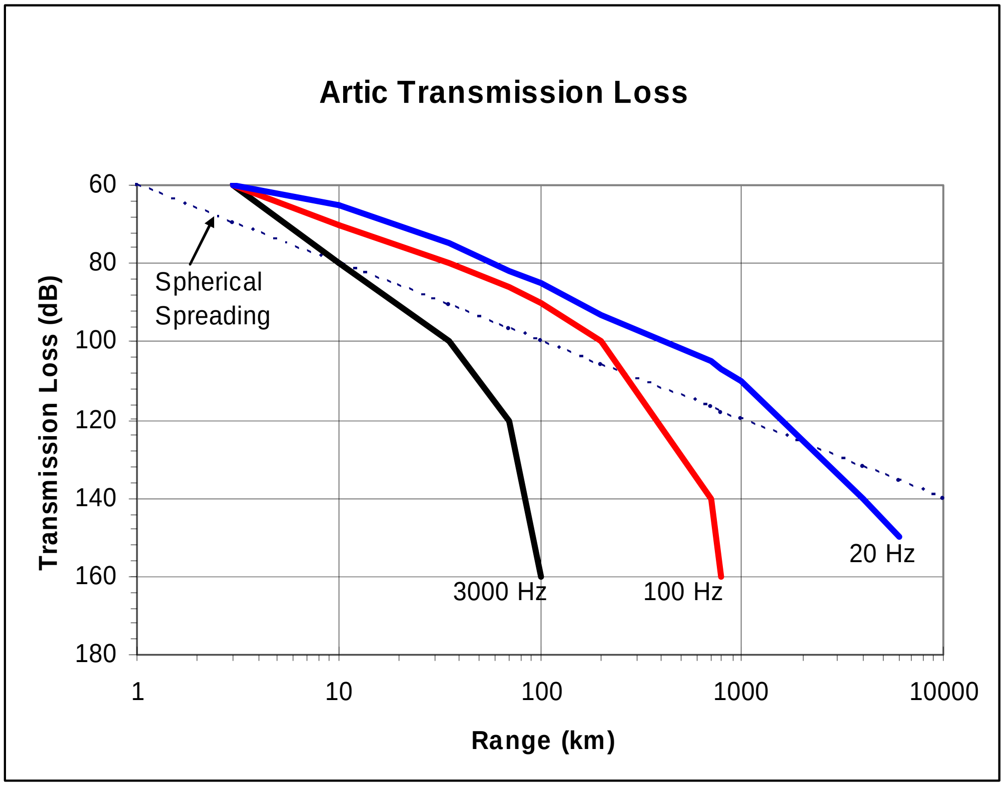
5. Where is cylindrical spreading more likely to occur?

6. A Sonarman on an Aegis is trying to calculate the expected two-way transmission loss of an active sonar ping directed at a target 20,000 yards away. If he assumes that absorption losses are negligible and IMAT tells him that the transition range is 6000 yards for the current environment, what is the **two-way** transmission loss?

7. A submarine is attempting to detect an aircraft carrier transiting the Straits of Malacca. The aircraft carrier has a source level of 90 dB. The submarine's passive sonar has a directivity index of 20 dB and a detection threshold of 15 dB. Biological noise in the Straits is 54 dB. The submarine's self noise is 50 dB. Assuming spherical spreading and neglecting attenuation, at what range can the carrier be detected.

8. Arctic transmission loss measurements are made at the three frequencies as a function of range in km and shown in the below graph. Theoretical spherical spreading is also shown for comparison by the dashed line. If 100 Hz frequency signals with a source level of 150 dB are propagating, find the detection range of the source using an omni directional hydrophone

(DI = 0 dB) operating in a total noise level of 60 dB. The signal to noise level of detection (DT) is 10 dB. Ignore attenuation at 100 Hz.



9. The intensity level 200 m from a sound source is measured as 100 dB re 1 μPa. What is the intensity level at 2000 m assuming:
   a) Spherical spreading
   b) Cylindrical spreading

10. For a given shallow water homogeneous bounded water column, the transition range between spherical and cylindrical spreading occurs at 1500 yards. Assuming only geometric spreading losses (no attenuation), plot transmission loss vs. range from 500 yds to 10,000 yds.

11. The absorption coefficient is 2 dB/kyd. Calculate the transmission loss due to spherical spreading and absorption:
   a) At a range of 5,000 yds.
   b) At a range of 10,000 yds.

12. The intensity level at 4,000 yards from a source is 150 dB and at 12,000 yards is 130 dB. Assuming that this loss is due to spherical spreading plus absorption, calculate the absorption coefficient in dB/kyd.

Transmission Loss

Review of Passive Sonar Equation

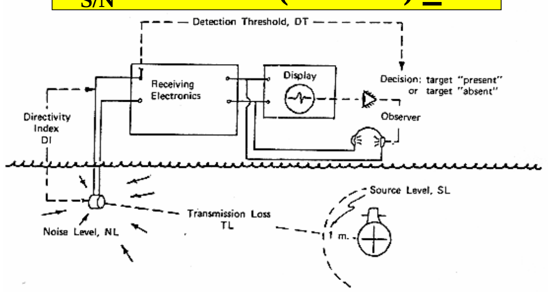
Terminology

- Signal to Noise
  The ratio of received echo from target
  to background noise produced by everything else.
- Detection Threshold (DT)
  The measure of return signal required for an operator using installed equipment to detect a target 50% of the time.

$$\mathbf{L_{S/N} = L_S - L_N \geq DT}$$

Passive Sonar Equation

$$\mathbf{L_{S/N} = SL - TL - (NL - DI) \geq DT}$$



The Passive Sonar Equation

$$L_{S/N} = SL - TL - (NL - DI)$$

$$SL = 10\log\left(\frac{I_S}{I_0}\right) \qquad NL = 10\log\left(\frac{I_N}{I_0}\right)$$

$$TL = 10\log\left(\frac{I_S}{I_R}\right) \qquad DI = 10\log(d)$$

Making the Sonar Equations Useful

Passive Example

Known

Can Measure

Function of Equipment

SL - TL - NL + DI > DT → Can Measure Experimentally

ONLY UNKNOWN

Figure of Merit

- Often a **detection threshold** is established such that a trained operator should be able to detect targets with that $L_{S/N}$ half of the time he hears them. Called "**Recognition Differential.**" **(RD)**
- Passive sonar equation is then solved for TL allowable at that threshold. Called "**Figure of Merit.**" **(FOM)**

$TL_{allowable}$ = Figure of Merit = SL- $L_{S/N\ Threshold}$ - (NL-DI)

- Since TL logically depends on range, this could provide an estimate of range at which a target is likely to be detected. Called "**Range of the Day.**" **(ROD)**
- Any $L_{S/N}$ above the Recognition Differential is termed "**Signal Excess.**" **(SE)** Signal Excess allows detection of targets beyond the Range of the Day.

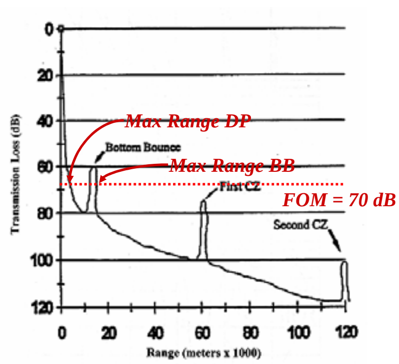
Range ???

- FOM helps to predict RANGE.
  - The ***higher*** the FOM, the ***higher*** the signal loss that can be suffered and, therefore, the ***greater*** the expected detection range.
- Probability of Detection
  - Passive
    - If FOM > TL then > 50% prob det
    - If FOM < TL then < 50% prob det
- Use Daily Transmission Loss (Prop Loss/FOM) curve provided by Sonar Technicians

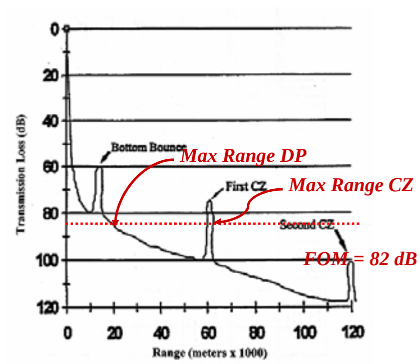
HW Example

- A submarine is conducting a passive barrier patrol against a transiting enemy submarine. The friendly sub has a directivity index of 15 dB and a detection threshold of 8 dB. The enemy sub has a source of 140 dB. Environmental conditions are such that the transmission loss is 60 dB and the equivalent isotropic noise level is 65 dB.
- What is the received signal level?
- What is the signal to noise ratio in dB?
- What is the figure of merit?
- Can the sub be detected? Why?

Prop Loss Curve



Prop Loss Curve



Transmission Loss

- Sound energy in water suffers two types of losses:
  - –Spreading
  - –Attenuation

Combination of these 2 losses:

TRANSMISSION LOSS (TL)

Spreading

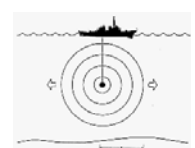
- Spreading
  - Due to divergence
  - No loss of energy
  - Sound spread over wide area
  - Two types:
    - Spherical
      - Short Range: r_o < 1000 m

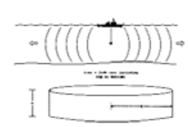
$$TL = 20 \log r$$

    - Cylindrical
      - Long Range: r_o> 1000 m

$$TL = 10 \log \frac{r}{r_o} + 20 \log \frac{r_o}{1}$$

Spherical component





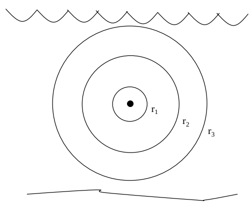
Spherical Spreading

$$TL = 10\log\left(\frac{I_S}{I_R}\right)$$

$$P_1 = P_2$$

$$I_1 4\pi r_1^2 = I_2 4\pi r_2^2$$

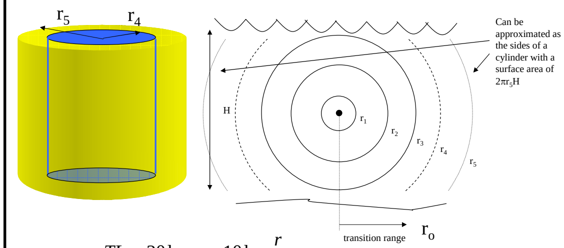
$$\frac{I_1}{I_2} = \frac{4\pi r_2^2}{4\pi r_1^2} = \left(\frac{r_2}{r_1}\right)^2$$



$$TL = 20\log\left(\frac{r_2}{r_1}\right) = 20\log\left(\frac{r}{1}\right) = 20\log r$$

Cylindrical Spreading

$$TL = 10\log\frac{I(1\ \text{yd})}{I(r)} = \underbrace{10\log\frac{I(1\ \text{yd})}{I(r_0)}}_{\text{spherical}} + \underbrace{10\log\frac{I(r_0)}{I(r)}}_{\text{cylindrical}}$$



$$TL = 20\log r_0 + 10\log\frac{r}{r_0}$$

Spherical to Cylindrical Transition Range in a Mixed Layer

$$r_0 = \sqrt{\frac{RH}{8}}\sqrt{\frac{H}{H-d}}$$

H = mixed layer thickness

d = depth of the source

$R = \dfrac{c_n}{g\cos\theta_n}$ = radius of curvature of sound ray

Attenuation

- **2 Types**
- Absorption
  - Process of converting acoustic energy into heat.
    - Viscosity
    - Change in Molecular Structure
    - Heat Conduction
  - Increases with higher frequency.
- Scattering and Reverberation
  - All components lumped into Transmission Loss Anomaly (A).
  - Components:
    - Volume: Marine life, bubbles, etc.
    - Surface: Function of wind speed.
    - Bottom Loss.
      - Not a problem in deep water.
      - Significant problem in shallow water; combined with refraction and absorption into bottom.

Absorption

- Decrease in intensity, proportional to:
  - Intensity
  - Distance the wave travels
- Constant of Proportionality, a

$$dI = -aIdr$$

$$\frac{I_2}{I_1} = e^{-a(r_2 - r_1)}$$

Absorption Coefficient

$$TL = 10\log\frac{I_1}{I_2} = 10\log e^{a(r_2 - r_1)}$$

$$TL = a(r_2 - r_1)10\log e = 4.343a(r_2 - r_1)$$

$$TL = \alpha(r_2 - r_1)$$

$\alpha = 4.343a$ Has units of dB/yard

$TL = \alpha(r_2 - r_1)x10^{-3}$ α Has units of dB/kiloyard

Example

- Spherical Spreading
- Absorption coefficient, α = 2.5 dB/kyd
- Find the TL from a source to 10,000 yards
- Find the TL from 10,000 yards to 20,000 yards

$$TL = 20\log\left(\frac{r_2}{r_1}\right) + \alpha\left(r_2 - r_1\right)x10^{-3}$$

General Form of the Absorption Coefficient

$$\alpha = \frac{Af_r f^2}{f_r^2 + f^2}$$

f_r = relaxation frequency. It is the reciprocal of the relaxation time. This is the time for a pressure shifted equilibrium to return to 1/e of the final position when pressure is released

f = frequency of the sound

When $f \ll f_r$,

$$\alpha = \frac{Af^2}{f_r}$$

Estimating Absorption Coefficient

- Viscosity – Classical Absorption - Stokes

$$\alpha = \frac{16\pi^2}{3\rho c^3}\mu f^2$$

$$\mu = \mu_s + \frac{3}{4}\mu_v$$ Shear and volume viscosity

$$\alpha = 2.75x10^{-4} f^2$$ For seawater, dB/m, f in kHz

Chemical Equilibrium

Magnesium Sulfate:

$$MgSO_4 + H_2O \leftrightarrow Mg^{+3} + SO_4^{-2} + H_2O$$

$$\alpha = \frac{40f^2}{4100 + f^2}$$ f in kHz

Boric Acid:

$$B(OH)_3 + (OH)^- \leftrightarrow B(OH)_4^-$$

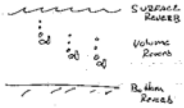
$$\alpha = \frac{.1f^2}{1 + f^2}$$ f in kHz

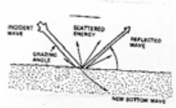
Scattering

- Scattering from inhomogeneities in seawater

$$\alpha = 0.003 dB / kyd$$

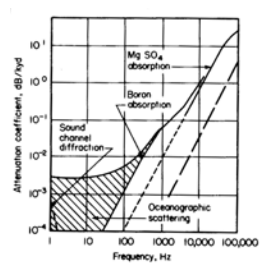
- Other scattering from other sources must be independently estimated





All lumped together as Transmission Loss Anomaly

Attenuation Summary



Note that below 10000Hz, attenuation coefficient is extremely small and can be neglected,

$$TL = \alpha\left(r \times 10^{-3}\right) dB$$

where

$$\alpha = \left(0.003 + \frac{0.1f^2}{1+f^2} + \frac{40f^2}{4100+f^2} + 2.75\times10^{-4} f^2\right) dB/kyd$$

Example

- Submarine sonar systems include a forward looking, high resolution active sonar which can be used to detect moored mines and ice keels. This system operates at 30 kHz and emits a source level of 130 dB re 1 μPa.
  - Calculate the attenuation coefficient
  - Assuming a transition range of 1000 yards, what is the transmission loss at a range of 5000 yards?
  - What is the Signal Intensity Level received by a second submarine 5000 yards away.
  - At this frequency, Noise Level is 40 dB and DI for the receiving sub is 20 dB. What is the signal to noise level?
  - If the Detection Threshold is 25 dB, will the second sub detect the first?

Transmission Loss Equations

$TL = 10 \log R + 30 + \alpha R + A$

Range ≥ 1000 meters

Transmission Loss Anomaly

Absorption

Cylindrical Spreading

$TL = 20 \log R + \alpha R + A$

Range < 1000 meters

Spherical Spreading

Absorption

TLA

Terminology

- Source Level (SL)
  - For ***ACTIVE*** sonar operations:
    - The SONAR's sonic transmission (transducer generated)
  - For ***PASSIVE*** sonar operations:
    - Noise generated by target
- Noise Level ($NL = NL_s \oplus NL_A$)
  - Self (NL_s)
    - Generated by own ship at the frequency of interest.
  - Ambient (NL_A)
    - Shipping (Ocean Traffic), Wind and Weather - Sea State (Hydrodynamic)
    - Biologic and Seismic obtained from other methods

Terminology

- Directivity Index (DI)
  - Receiver directional sensitivity.
  - $L_N = NL - DI$
- Transmission Loss (TL)
  - Amount the Source Level is reduced due to spreading and attenuation (absorption, scattering).

Passive SONAR Equation

(Signal Radiated by the Target)

- SNR required for detection = DT
- To achieve detection > 50% of the time…
  - SNR > DT
  - $L_S - L_N > DT$
    - $L_S = SL - TL$ (one way)
    - $L_N = NL - DI$
      - Remember $NL = NL_s \oplus NL_a$
- Therefore…

$$\mathbf{L_{S/N} = SL - TL - (NL - DI) \geq DT}$$

Ambient Noise

The background noise of the sea.

When trying to detect a target or contact out in the ocean, one of the most difficult parts is to "hear" the target through all the background noise. This is just like trying to hear a friend talk while standing in a crowd of people at a noisy rock concert. But out in the ocean, what are the sources of the background noise?

Major sources of background noise in deep water

Tides

A small contribution to ambient noise is the movement of water due to tides. This movement can create large changes in ambient pressure in the ocean. These changes will be most significant at very low frequencies (<100 Hz) but will decrease in strength with increasing depth. Overall though, tides contribute little to the ambient noise level.

Seismic

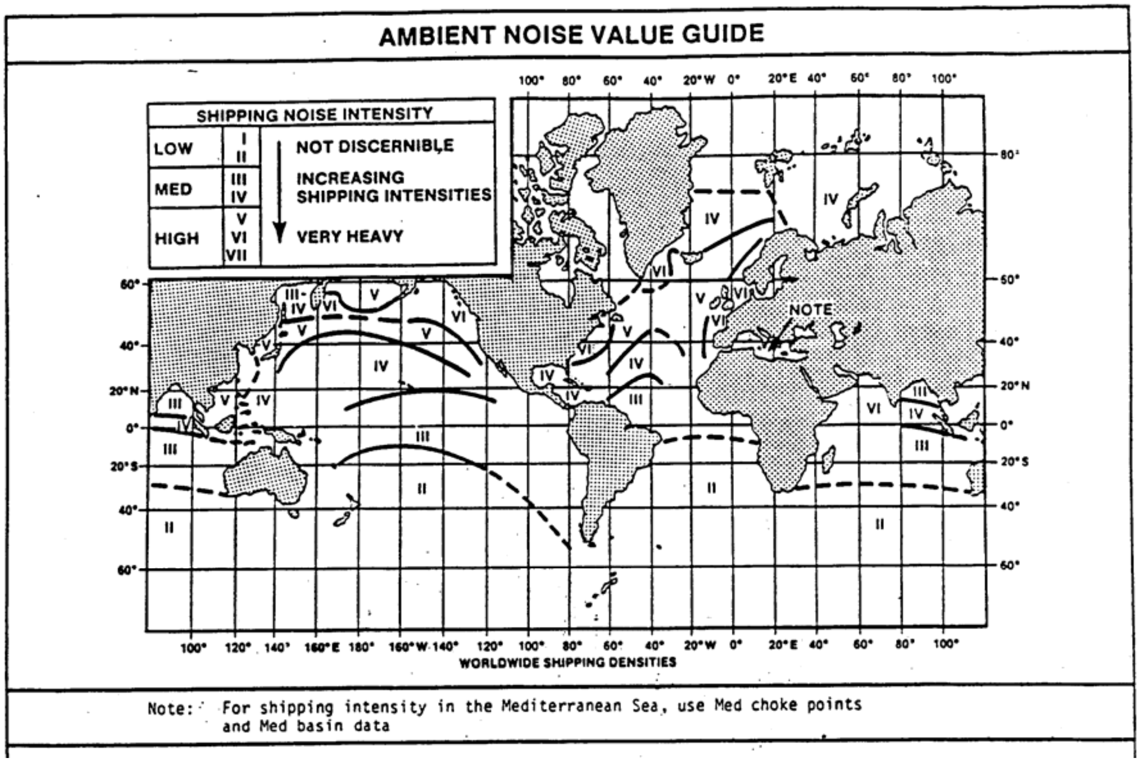
Another source of very low frequency noise is that due to the earth's seismic activity. The noise due to seismic activity is not significant though, above 10 Hz. As with tides, we will treat seismic sources as being insignificant in our calculations of ambient noise levels.

Turbulence

This can be a significant factor in ambient noise levels below 100 Hz but generally, we will not consider the affect of turbulence in our calculations.

Ship Traffic

In the North Atlantic, there can be more than 1000 ships underway at any one time. The noise from this shipping traffic can sometimes travel up to distances of 1000 miles or more. The frequency range where this man-made noise is most dominant is from 10 Hz to 1000 Hz. Noise levels depend on area operating in and "shipping density". Close proximity to shipping lanes and harbors increases noise levels. Shipping traffic is one of the two dominant factors we will use to determine ambient noise levels. The below chart shows how shipping density varies throughout the oceans of the world.

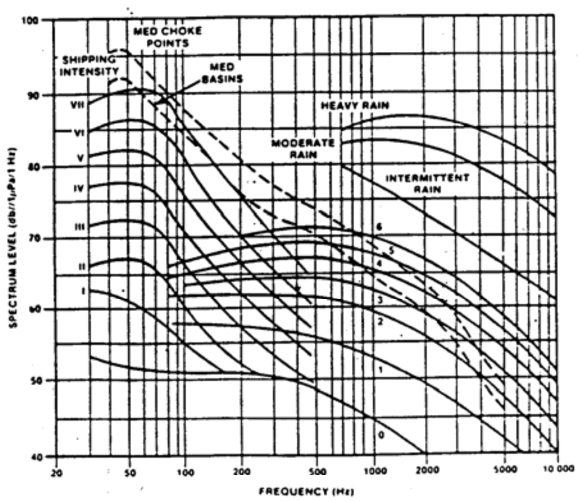


Sea State

Sea State (or more importantly wind speed) is the dominant factor in calculating ambient noise levels above 500 Hz. The noise levels depend on sea state and wind speed. Less the 10 Hz, wind-generated turbulence induces pressure variations similar to acoustic pressure variations. Greater than 100 Hz, wind generated microbubbles in the shallow water layers burst and cause pressure changes.

Wenz Curves

For ASW operators to predict the ambient noise levels for a given condition and frequency band, we have the Wenz Curves. Wenz Curves are plots of the average ambient noise spectra for different levels of shipping traffic, and sea state conditions (or wind speeds). Below is a complex example given in the Naval Warfare Publication. At the end of this handout are the simplified Wenz curves you will use for all homework, quizzes and exams.



SEA STATE	WIND SPEED (kts)
6	28-47
5	22-27
4	17-21
3	11-16
2	7-10
1	4-6
0	1-3

NOTE: ADD 2 dB TO SHIPPING NOISE LEVELS FROM JANUARY THROUGH MARCH AND SUBTRACT 2 dB FROM SHIPPING NOISE LEVELS FROM JULY THROUGH SEPTEMBER.

AMBIENT NOISE LEVEL VERSUS FREQUENCY

10-100 Hz – Noise levels depend heavily on shipping density and industrial activities. Levels are typically in range of 60-90 dB with very little frequency dependence.

100-1000 Hz – Noise in this band is dominated by shipping (decreasing intensity with frequency increases). A significant contribution is also from sea surface agitation. Urick (1986) developed a model for predicting this shipping noise:

$$NL_{SHIPPING} = NL_{100} - 20\log\left(\frac{f}{100}\right)$$

Where NL_{100} is 60-90 dB based on shipping density

1-100 kHz – Sea surface agitiation is now the dominant factor, unless marine mammals or rain is present. Knudsen (1948) presented a model to predict this contribution:

$$NL_{SURF} = \begin{Bmatrix} NL_{1K} & \text{IF f } < 1000\text{ Hz} \\ NL_{1K} - 17\log\left(\frac{f}{1000}\right) & \text{IF f } > 1000\text{ Hz} \end{Bmatrix}$$

NL_{1K} is in the below table, and is based on sea state.

A new model has been developed by APL (1994), it is more accurate but is more complex.

Beaufort Force	Sea State	Windspeed Knots	NL 1K dB	Description	Sea Condition
0	0	0	44.5	Calm	Sea like a mirror
1	0.5	1 - 3	50	Light Air	Ripples but without foam crests
2	1	4 - 6	55	Light Breeze	Small wavelets. Crests do not break
3	2	7 - 10	61.5	Gentle Breeze	Large wavelets. Perhaps scattered white horses
4	3	11 - 16	64.5	Moderate Breeze	Small waves. Fairly frequent white horses.
5	4	17 - 21	66.5	Fresh Breeze	Moderate waves, many white horses
6	5 - 6	22 - 27	68.5 - 70	Strong Breeze	Large waves begin to form; white foam crests, probably spray
7	7	28 - 33		Near Gale	Sea heaps up and white foam blown in streaks along the direction of the wind
8	8	34 - 40		Gale	Moderately high waves, crests begin to break into spindrift
9	9	41 - 47		Strong Gale	High waves. Dense foam along the direction of the wind. Crests of waves begin to roll over. Spray may affect visibility
10	9	48 - 55		Storm	Very high waves with long overhanging crests. The surface of the sea takes a white appearance. The tumbling of the sea becomes heavy and shock like. Visibility affected
11	9	56 - 63		Violent Storm	Exceptionally high waves. The sea is completely covered with long white patches of foam lying in the direction of the wind. Visibility affected
12	9	64+		Hurricane	The air is filled with foam and spray. Sea completely white with driving spray. Visibility very seriously affected.

***>100 kHz* –** Noise is dominated by electronic thermal noise (we will discuss causes later)
$NL_{TH} = -75 + 20\log f$

The total ambient noise level is derived by calculating the level sum of the contributing noise factors given by the following equation:

$$NL_{amient} = NL_{ship} \oplus NL_{SS}$$

The appropriate level of shipping is selected based on location. The "heavy shipping" curves should be used when in or near the shipping lanes in the North Atlantic. The "light-shipping" curves should be used for more southerly, remote areas of the ocean.

The regions below 10 Hz and above 200 kHz are dominated by other factors that are quantified by the solid lines.

General Rules –

1. NL generally decreases with frequency increasing
2. NL decreases at great depths since most noise sources are at the surface.
3. Ambient noise is greater in shallow water (noise is trapped between sea floor and the ocean surface).

Example

For a sonar receiver set with a band width of 100 Hz, centered around 200 Hz, what is the ambient noise level? (Shipping is heavy, sea state is 3.)

To calculate the upper and lower frequency of the band:

$$f_c = \sqrt{f_1 f_2}$$
$$200Hz = \sqrt{f_1\left(f_1 + 100Hz\right)}$$
$$\Rightarrow f_1 = 156Hz$$
$$f_2 = 256Hz$$

From the Wenz Curves (end of handout):

$ISL_{ave\ shipping} = 69$ dB $\quad\quad$ $ISL_{ave\ sea\ state} = 67$ dB

Thus:

$$NL_{tot} = NL_{ship} \oplus NL_{SS}$$
$$NL_{tot} = \left(ISL_{aveship} + 10\log\Delta f\right) \oplus \left(ISL_{aveSS} + 10\log\Delta f\right)$$
$$NL_{tot} = 89dB \oplus 87dB$$
$$NL_{tot} = 91dB$$

Transient Noise

Just for passing interest, there are numerous other sources of noise in the oceans. Many of these sources are transitory in nature though which makes them hard to quantify. They may only affect detectability of contacts for short periods of time. These sources may include but are certainly not limited to:

- Human industrial sources ashore – particularly in coastal areas
- Biological factors including
  - snapping shrimp – mostly in warm, shallow coastal areas
    - generate intense broadband noise, f = 1-10 kHz, SL =60-90 dB
  - whales, dolphins, etc – echolocation and communication
    - f = 12 Hz - @2-5 kHz for "whale songs", SL up to 188 dB
    - Echolocation – 50-200 kHz – similar to active sonar, SL up to 180-200 dB

- Weather – rain
    - Rain drops impacting sea surface and implosion of air bubbles caused by rain, f = 1-100 kHz, max SL @ 20 kHz, SL can be up to 30 dB above sea surface noise

Problems

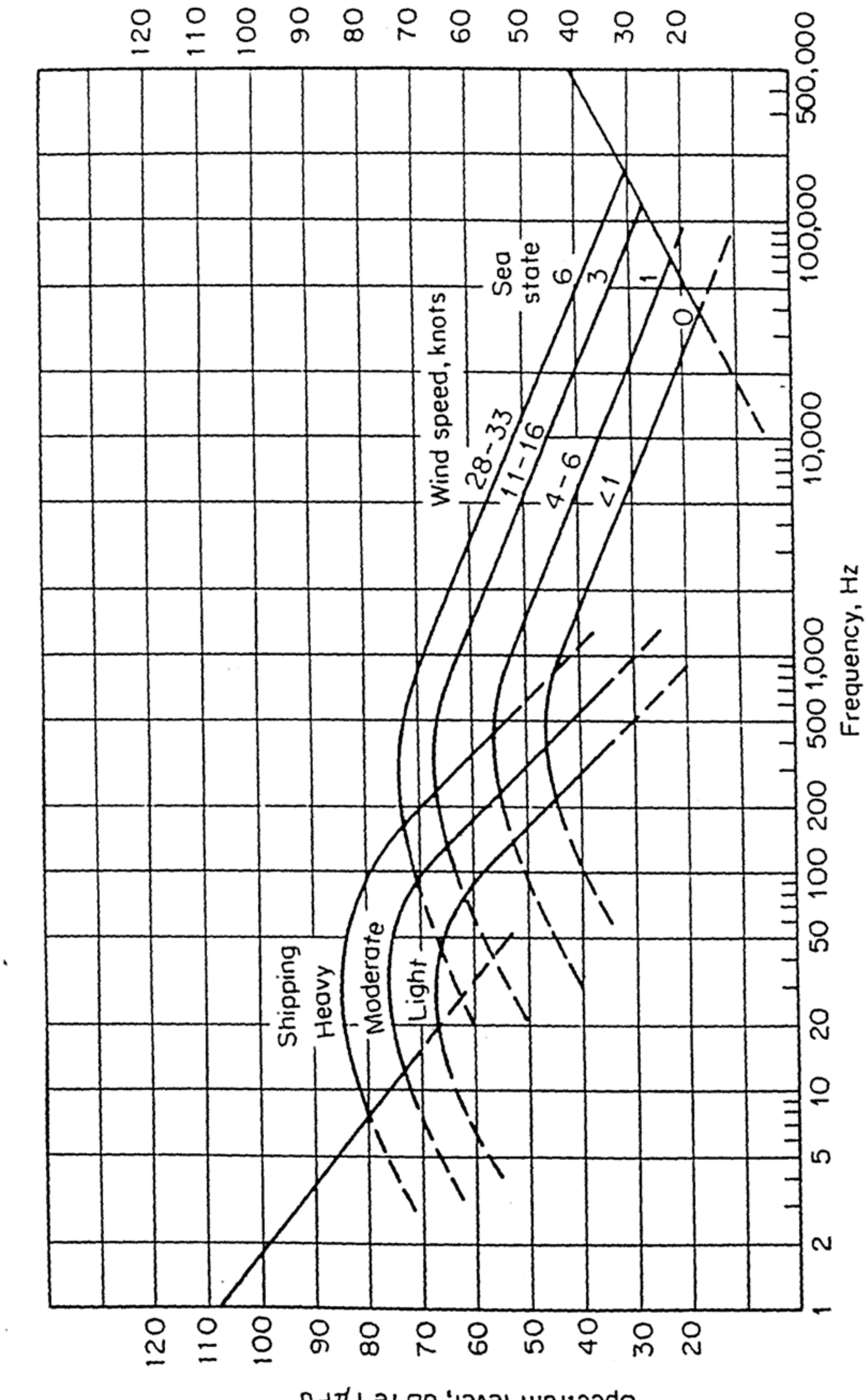
1. What is the principal cause of ambient noise at frequencies
    a. 1 to 20 Hz
    b. 20 to 500 Hz
    c. 500 to 50,000 Hz
    d. above 50,000 Hz

2. Using the Wenz curves, determine the isotropic ambient noise level for an area with heavy shipping. Assume that wind speeds are 14 knots and we are interested in the noise level at exactly 200 Hz (use a 1 Hz bandwidth).

3. The SONAR receiver onboard ship operates in a frequency range from 50 Hz < f < 1000 Hz. Using the Wenz curves, determine the isotropic ambient noise level in the operating band of the receiver. Assume that winds are light as 4-6 knots and shipping traffic is moderate. (Note: You will have to determine an average ISL from the Wenz curves and calculate the appropriate band levels.)

4. List three intermittent sources of ambient noise.

5. Using the wenz curves for average deep water ambient noise, estimate the band level noise for heavy shipping and sea state 6 for the following conditions:
    a. Noise received in a band between 20 and 50 Hz.
    b. Noise received in a band between 2000 and 5000 Hz.

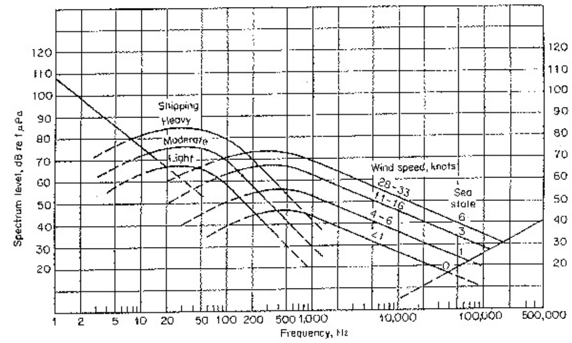


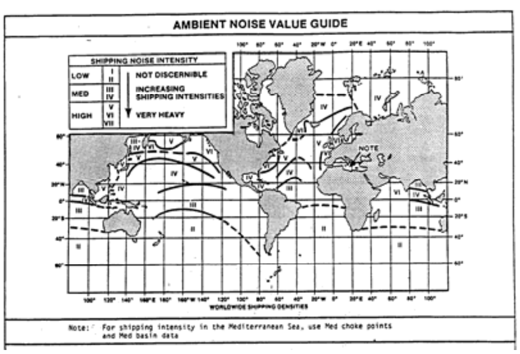
Spectrum level, dB re 1 μPa
Frequency, Hz
1
2
5
10
20
50
100
200
500
1,000
10,000
100,000
500,000
120
110
100
90
80
70
60
50
40
30
20
Shipping
Heavy
Moderate
Light
Wind speed, knots
28–33
11–16
4–6
<1
Sea state
6
3
1
0

Major Sources of Noise

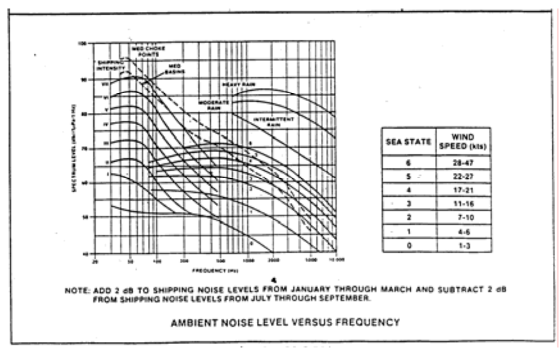
- Sea State – Dominant factor above 500 Hz
- Ship Traffic – Dominant factor 10 to 1000 Hz
- Minor Sources
  - Tides
  - Turbulance
  - Seismic
- Transients

Wentz Curves





NWP Wenz Curves

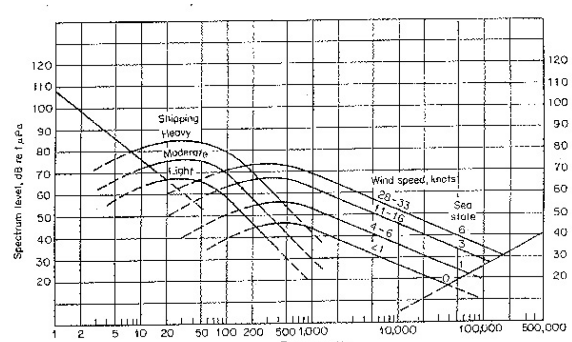


100-1000 Hz

- NL_{100} is 60-90 dB depending on shipping. Below 100 Hz NL is the same.
- Above 100 Hz, the noise decreases with frequency

$$NL_{SHIPPING} = NL_{100} - 20\log\left(\frac{f}{100}\right)$$

Wentz Curves



1-100 kHz

$$NL_{SURF} = \begin{cases} NL_{1K} & \text{IF f } < 1000 \text{ Hz} \\ NL_{1K} - 17\log\left(\frac{f}{1000}\right) & \text{IF f } > 1000 \text{ Hz} \end{cases}$$

NL falls at 17 dB per decade above 1000 Hz

Beaufort Force	Sea State	Windspeed Knots	NL 1K dB	Description	Sea Condition
0	0	0	44.5	Calm	Sea like a mirror
1	0.5	1 - 3	50	Light Air	Ripples but without foam crests
2	1	4 - 6	55	Light Breeze	Small wavelets. Crests do not break
3	2	7 - 10	61.5	Gentle Breeze	Large wavelets. Perhaps scattered white horses
4	3	11 - 16	64.5	Moderate Breeze	Small waves. Fairly frequent white horses.
5	4	17 - 21	66.5	Fresh Breeze	Moderate waves, many white horses
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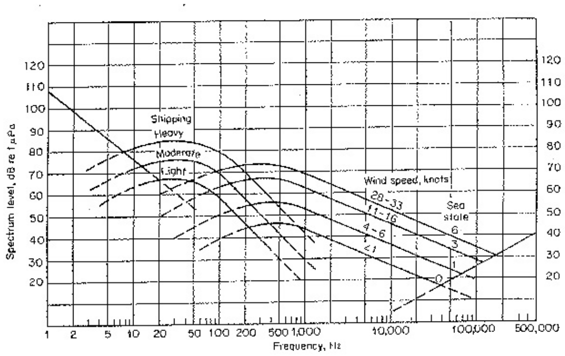
Wentz Curves



Above 50 kHz

- Thermal Agitation of water molecules
- Thermal noise in electronics
- 6 dB per octave increase in noise

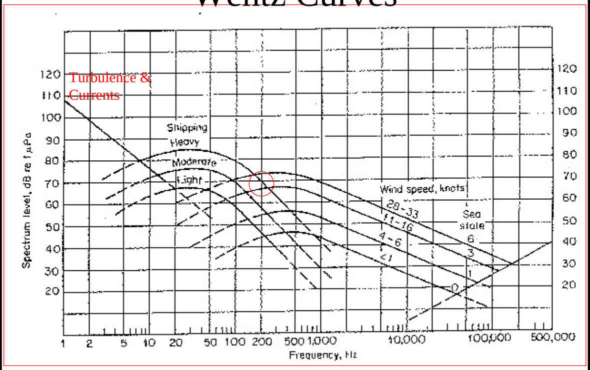
Total Noise

$$\text{NL}_{\text{ambient}} = \text{NL}_{\text{ship}} \oplus \text{NL}_{\text{SS}}$$

Example

- For a sonar receiver set with a width of 100 Hz, centered around 200 Hz,
- Shipping is heavy,
- Sea state is 3,
- What is the ambient noise level?

Wentz Curves



Transients

- Human industrial activity
- Biological Activity
  - Snapping Shrimp 1-10 kHz, SL = 60-90 dB
  - Whales, dolphins
    - Whale songs 2 – 5 kHz, SL = 188 dB
    - Echolocation 50 – 200 kHz, SL = 180 – 200 dB
- Weather – Rain
  - 1 – 100 kHz, SL(20 kHz) is 30 dB above sea state noise

General Noise Rules

- NL generally decreases with frequency increasing
- NL decreases at great depths since most noise sources are at the surface.
- Ambient noise is greater in shallow water (noise is trapped between sea floor and the ocean surface).

$$f_c = \sqrt{f_1 f_2}$$
$$200Hz = \sqrt{f_1(f_1 + 100Hz)}$$
$$\Rightarrow f_1 = 156Hz$$
$$f_2 = 256Hz$$

- From the Wenz Curves:

  $ISL_{\text{ave shipping}}$ = 69 dB
  $ISL_{\text{ave sea state}}$ = 67 dB

$$NL_{tot} = NL_{ship} \oplus NL_{SS}$$
$$NL_{tot} = \left(ISL_{aveship} + 10\log\Delta f\right) \oplus \left(ISL_{aveSS} + 10\log\Delta f\right)$$
$$NL_{tot} = 89dB \oplus 87dB$$
$$NL_{tot} = 91dB$$

Self-Noise

Self-noise is the noise that own ship produces and is detected by own ships sonar system, contributing to the overall noise interfering with the detection of other vessels. Some of the sources of self-noise are:

Propeller Noise - Rotating propellers generate spectral lines in VLF band, f = 0.1-10 Hz. These frequencies depend on rotation speed of propeller and its geometry. Cavitation induced creates broadband noise at higher frequencies. Cavitation depends on:

1. Rotation speed of propeller
2. Type of propeller (fixed or variable pitch)
3. Depth of propeller (deep depths inhibit cavitation)

Flow Noise – Turbulence generated by flow of water over ship's hull. Depends on ship speed, shape and placement of transducer.

Machinery Noise – Ships have numerous noisy machines - engines, reduction gears, generators and hydraulic machinery. This machinery causes vibrations in the hull by solid transmission through internal structures or through the air. These vibrations are then passed on to the water. Machinery noise is independent of ship's speed, therefore it is the major component of self noise at low speeds and is masked at high speeds by flow noise. Both primary frequencies and harmonics are transmitted.

Submarine Transient Noise - Acoustic stealth can be degraded by transmission of short transient noises (from few milliseconds to several seconds). These transients are caused by opening torpedo tube doors, control surface movement, and starting/stopping machinery. These are very characteristic of submarines and can help classify them as such.

Activity Noise – Civilian ship activities can be very noisy (drilling, trawling, seismic surveys, etc).

Ship Radiated Noise Model – Self noise represents continuous broadband noise spectrum whose level increases with ship speed. The maximum level is typically around 100 Hz and generally decreases by 6 dB per octave above a few hundred hertz. In VLF band, narrowband components (spectral lines) exceed broadband noise. The combination of spectral lines and broadband forms a ship's "acoustic signature" and can be used to classify a ship passively.

The ship's self noise can be modeled by two components:

1. RNL_{1K} at f = 1 kHz, from which we can derive the level at other frequencies.

$$RNL(f) = RNL_{1K} - 20\log\left(\frac{f}{1000}\right)$$

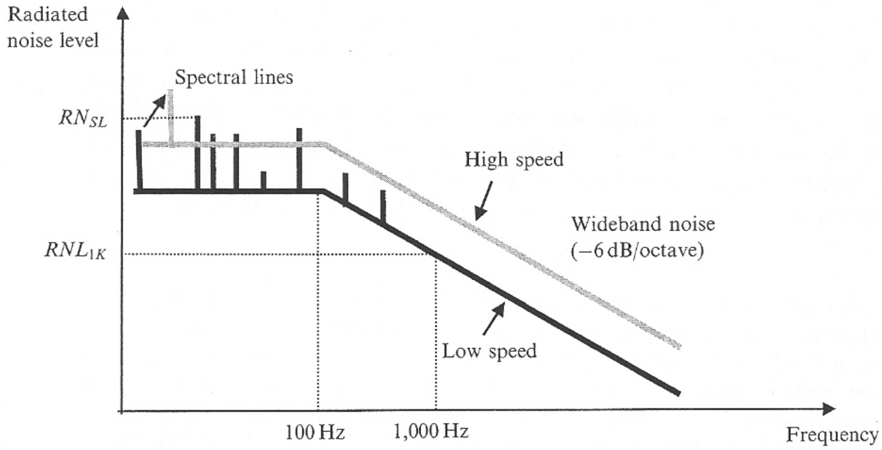


Figure 4.4. Generic spectrum of ship-radiated noise.

2. Radiated noise of the spectral lines RNL_{SL} - these lines are generally described by their maximum level.

Considerable time and money has been spent to improve acoustic performance on submarines and other naval vessels since WWII. In the below table from Lurton, we see how the RNL has decreased for French submarines since WWII (modern US submarines lower RNLs than those presented here.)

Radiated Noise at 4 kts

	RNL_{SL}	RNL_{1K}
WWII Deisel Sub (Electric)	140	120
Modern Diesel Sub (Electric)	100	80
Modern Diesel Sub	140	120
Modern SSN	110	90
Modern SSBN	120	100

Above 10-20 kts, flow noise becomes the dominant factor and significantly increases with speed (@1.5-2 dB/KT)

<u>Thermal Noise</u> – In electronic circuits, resistors create some electric noise due to electronic agitation. Nyquist formula describes the voltage created:

$$u = \sqrt{4KRT\Delta f}$$
$$K - \text{Boltzmann constant}$$
$$\text{T - °K}$$

For ideal hydrophone, we need only consider resistivity, therefore:

$$R_r = \frac{\pi\rho c}{\lambda^2} S_H^2 \qquad\qquad \text{S}_\text{H} \text{ - hydrophone sensitivity}$$

This creates electronic noise voltage:

$$u = \sqrt{4\pi k\rho cT}\,\frac{S_H}{\lambda} \qquad \text{AND} \qquad p_a = \frac{\sqrt{4\pi k\rho cT}}{\lambda}$$
$$\Rightarrow NIS_{therm} \approx -75 + 20\log f - 10\log\eta$$
$$\eta \text{ - transducer efficiency}$$

Noise Measurement

There is no way to calculate the self-noise of a platform however it can be measured for a given platform. Very difficult to assess the self-noise level of transducer, since it is a combination of several components:

1. Acoustic noise radiated in the water by the platform and received by the transducer through the water.
2. Mechanical vibrations passed on to the transducer.
3. Electronic noise radiated by other high-power electrical devices if inadequately shielded.

Most modern sonars have self-monitoring capabilities to measure real-time self noise levels. For this course, it will normally be given for students in a problem.

How self-noise is taken into account in our evaluation of the sonar equations is that self-noise is a second component of the overall noise level. The total noise level of sources that interferes with detection of other vessels is the combination of the ambient noise (that we determined in the previous section) and the self-noise. In other words:

$$NL_{tot} = NL_{ambient} \oplus NL_{self}$$

Additionally, this self-noise level is the combination of the broadband and tonal noise. This must be added to the self-noise as shown below:

$$NL_{self} = NL_{BB} \oplus NL_{tonal} \left(\oplus NL_{tonal2} \oplus NL_{tonal3} \oplus \ldots\right)$$

$$\text{where } \text{NL}_\text{BB} = \text{ISL}_\text{BB} + 10\log\Delta\text{f}$$

Problems:

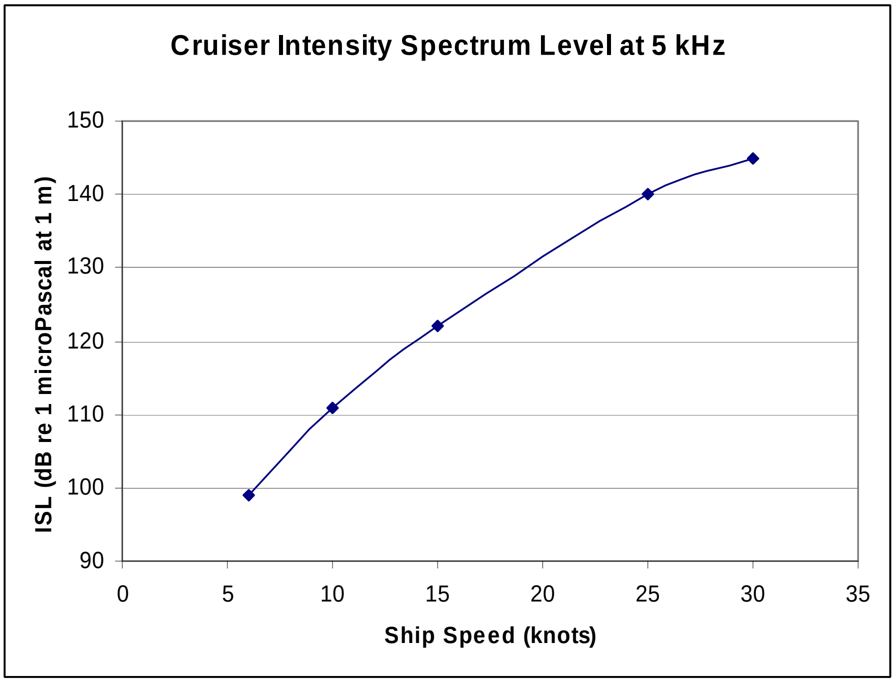
1. Determine the isotropic ambient noise level for a lightly traveled shipping lane. Assume that winds are moderate at 12 knots, the sea state is equal to 3 and the receiver is set to look in the 100 Hz ≤ f ≤ 500 Hz range.

2. Using the conditions from problem 1 above, and given the following table of data:

    a. Plot the total ISL as a function of frequency from 100 Hz to 500 Hz.
    b. Compute the total noise level.

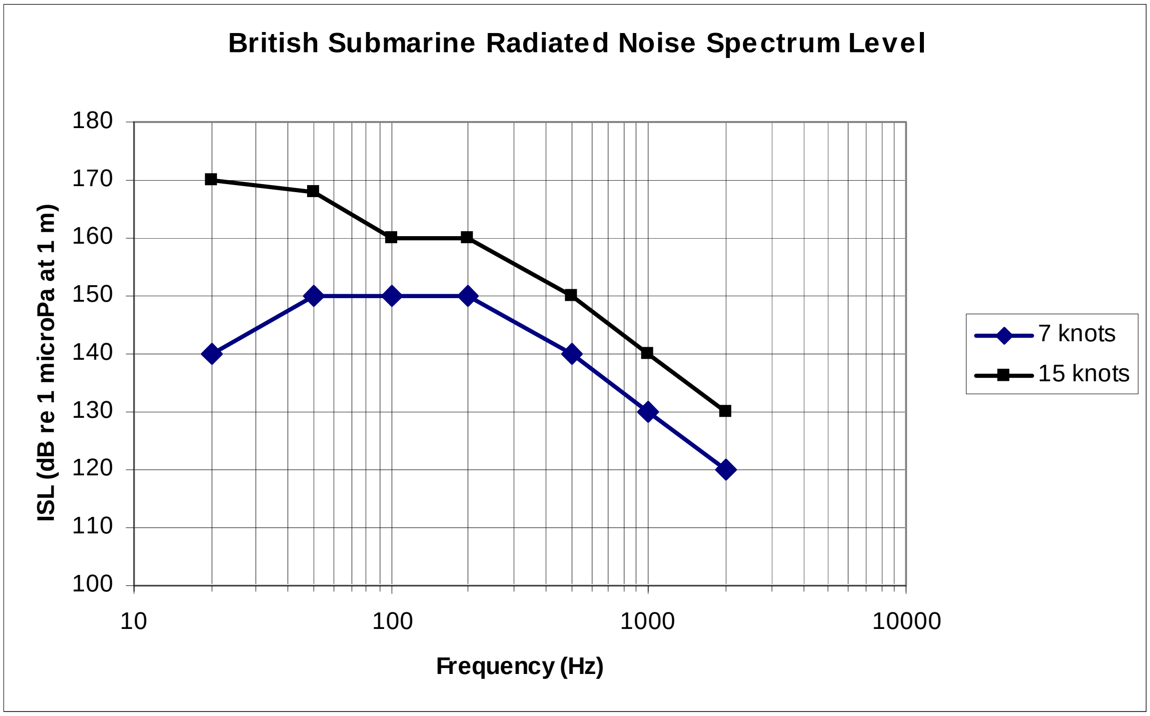
BL_{shrimp}	76 dB
$ISL_{engine\ at\ 400\ Hz}$	124 dB
$BL_{generators}$	99 dB
$BL_{propellers}$	89 dB
BL_{pumps}	92 dB
BL_{whales}	44 dB
$BL_{hull\ flow\ noise}$	80 dB
BL_{rain}	78 dB
$BL_{crew\ noise}$	82 dB

3. a) Using the figure below, compute the increase in radiated noise spectrum level when the cruiser's speed is doubled from 6 knots. Also compute the increase when the speed is increased by a factor of 5 from 6 knots.



b) If $ISL = 10\log k^n + A$, where k is the speed in knots, and n and A are constants. Use the figure above to solve for n and A.

4. a) Using the figure below, estimate the radiated noise level of the British sub traveling on the surface a 7 knots and in a band ranging from 50 to 200 Hz.



b) Estimate the radiated noise level of the sub traveling at 15 knots in a band from 50 to 200 Hz.

Self Noise

- Acoustic noise radiated in the water by the platform and received by the transducer through the water.
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- Electronic noise radiated by other high-power electrical devices if inadequately shielded.

Propeller Noise

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- Rotation speed of propeller
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Ship Radiated Noise Model

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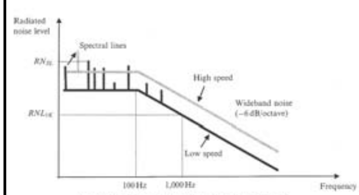

Figure 4.4. Generic spectrum of ship-radiated noise.

	RNL_{SL}	RNL_{1K}
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Total Noise

$$NL_{tot} = NL_{ambient} \oplus NL_{self}$$

$$NL_{ambient} = NL_{ship} \oplus NL_{SS}$$

$$NL_{self} = NL_{BB} \oplus NL_{tonal} \left(\oplus NL_{tonal2} \oplus NL_{tonal3} \oplus \ldots \right)$$

$$NL_{BB} = ISL_{BB} + 10 \log \Delta f$$

Beam Pattern Function for Two Element Array

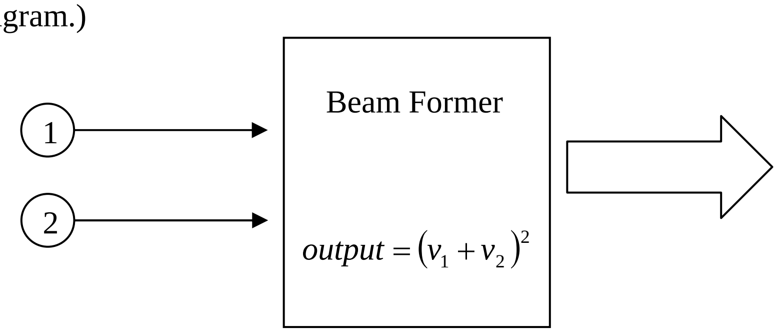
If we had a single hydrophone, with an omni-directional response, sounds would appear to come from all directions. In other words, we could not determine what direction a sound came from. If we could somehow limit the direction our system would listen, we could possibly determine the bearing a sound came from and maybe increase the ratio of the signal power received to the noise power received. (Increase SNR which is a good thing.)

One way to do this is to use more than one hydrophone. What if we use two hydrophones connected at a distance d apart from each other. Recall from our previous studies that the hydrophone converts the mechanical sound signal to an electrical signal or voltage. We can mathematically describe this process by introducing a quantity M, the transducer sensitivity constant. M is used to convert the mechanical pressure quantity to an electrical signal, where:

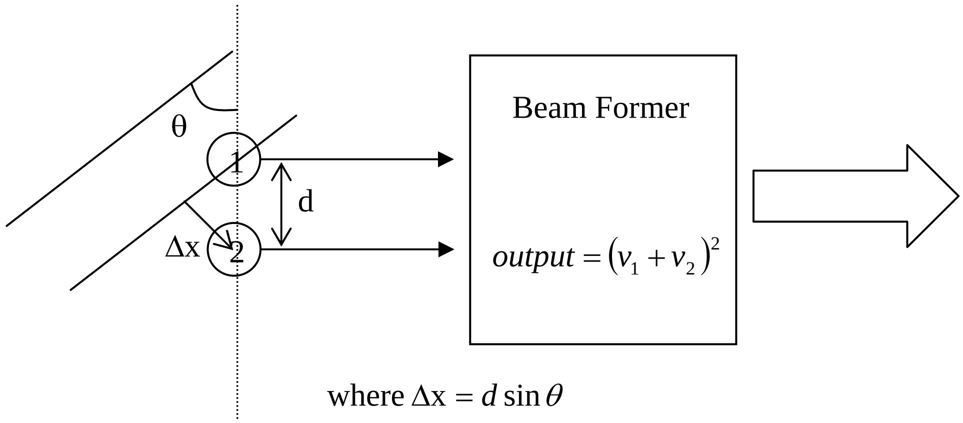
$$v(t) = M * p(t)$$

Now let's look at the arrangement of the two hydrophones and how their output is used.

First examine the diagram for a basic two-hydrophone array, sonar system. The outputs of each hydrophone are combined in a beam former (they are added together), then the quantity squared to find the amount of power in the signal and noise incident on the hydrophones. (See the following diagram.)



If a sound wave is incident upon the two hydrophones at some angle other than perpendicular to the axis of the two hydrophones, the sound wave will have to travel some distance Δx further to reach the second hydrophone. (See diagram below.)



where $\Delta x = d \sin\theta$

A phase factor, δ, can be inserted in the one-dimensional wave equation to describe the pressure of the wave as it is incident upon each hydrophone where:

$$
\begin{aligned}
&v_1 = Mp_1(t) = Mp_{max}\cos\left(k(0)+\omega t\right)\\
&v_2 = Mp_2(t) = Mp_{max}\cos\left(k(-\Delta x)+\omega t\right)\\
&v_1 + v_2 = Mp_{max}\left[\cos(\omega t)+\cos(-\delta+\omega t)\right]\\
&\text{where } \ \delta = k\Delta x = kd\sin\theta
\end{aligned}
$$

When the output is then squared it is actually measuring is the power of the incoming signal (or a signal proportional to the rate of sound energy incident on the hydrophones.)

$$Power = \frac{V^2}{R} = \frac{output^2}{R}$$

$$P = \frac{(Mp_{\max})^2}{R}\left[\cos(\omega t)+\cos(-\delta+\omega t)\right]^2$$

If we then display the time-averaged power derived from the equation above, we get:

$$
\begin{aligned}
\langle P\rangle &= \frac{\langle Mp_{\max}\rangle^2}{R}\left\langle \cos^2\omega t + \cos^2(\omega t-\delta) + 2\cos\omega t\cos(\omega t-\delta)\right\rangle\\
\langle P\rangle &= \frac{(Mp_{\max})^2}{R}\left\langle \cos^2\omega t + \cos^2(\omega t-\delta) + 2\left[\frac{1}{2}\left\{\cos(2\omega t-\delta)+\cos\delta\right\}\right]\right\rangle\\
\langle P\rangle &= \frac{(Mp_{\max})^2}{R}\left[\frac{1}{2}+\frac{1}{2}+\langle\cos(2\omega t-\delta)\rangle+\langle\cos\delta\rangle\right]\\
\langle P\rangle &= \frac{(Mp_{\max})^2}{R}\left[1+\cos\delta\right]
\end{aligned}
$$

So depending on the value of δ (which is equal to kdsin(θ)), the time averaged power will be:

$$0 \le \langle P \rangle \le \frac{2(Mp_{\max})^2}{R}$$

Two-dimensional Beam Pattern

Why have we calculated the time-averaged power? Since the value for δ depends on the angle of the incoming sound wave from the array axis, the power received depends on the angle at which the sound ray is incident on the array. We can describe this angular dependence with one equation to relate the actual power received to the time averaged power on the axis (where θ=0° and the power is a maximum.) This ratio is the two-dimensional beam pattern function of the array, b(θ) where:

$$b(\theta) = \frac{\langle P(\theta) \rangle}{\langle P(\theta = 0°) \rangle} = \frac{\frac{(Mp_{\max})^2}{R}(1 + \cos\delta)}{\frac{(Mp_{\max})^2}{R}(1 + \cos 0°)}$$

$$b(\theta) = \frac{\frac{(Mp_{\max})^2}{R}(1 + \cos(kd \sin\theta))}{2\frac{(Mp_{\max})^2}{R}} = \frac{(1 + \cos(kd \sin\theta))}{2}$$

using a trigonometric identity that $1 + \cos\theta = 2\left(\cos^2\left(\frac{\theta}{2}\right)\right)$:

$$b(\theta) = \left[\cos^2\left(\frac{kd \sin\theta}{2}\right)\right]$$

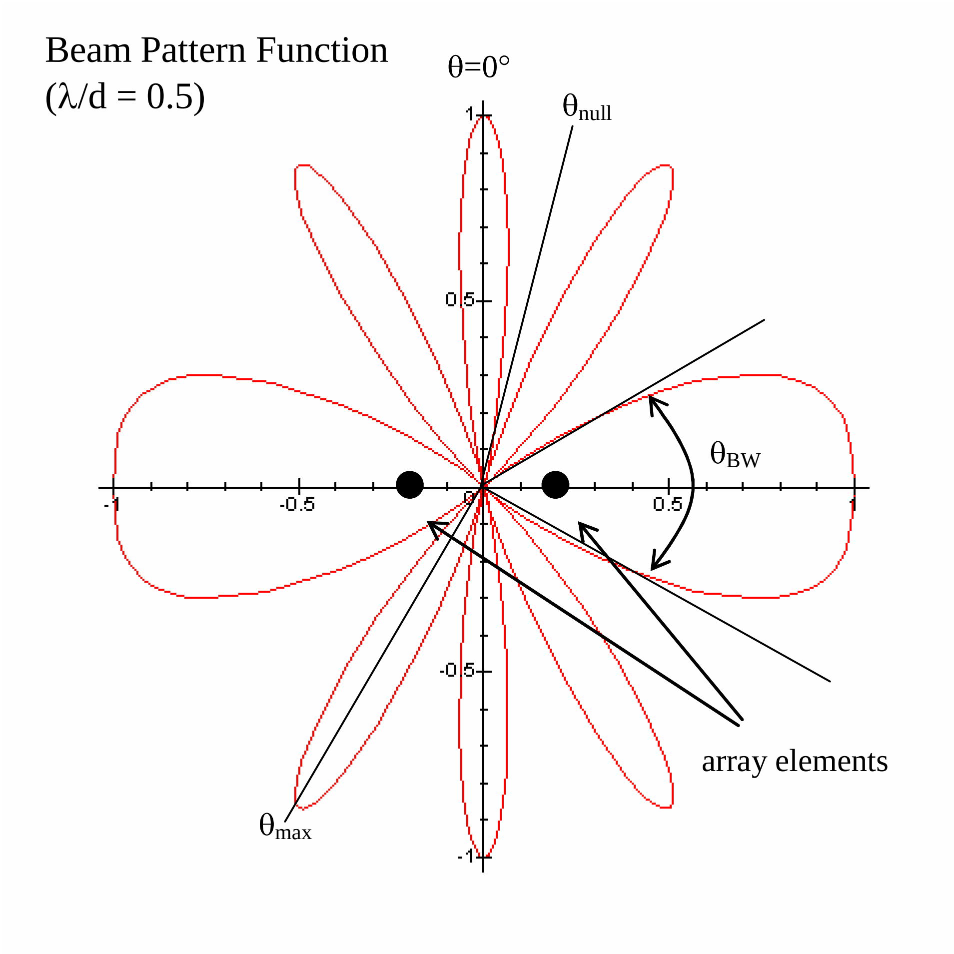
or

$$b(\theta) = \left[\cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)\right]$$

The beam pattern function determines the magnitude of the received power at every angle, to the maximum received power, thus the beam pattern function will vary (as a function of angle) between 0 and 1.

$$0 \le b(\theta) \le 1$$

The key now is to determine what important parameters we can determine from the beam pattern function. Below is a polar plot of the beam pattern function for a two element array where the separation in elements is equal to twice the wavelength.


Beam Pattern Function
(λ/d = 0.5)
θ=0°
θnull
θBW
θmax
array elements
1
0.5
-0.5
-1
0.5
1

Maximum Power Angles (θ_{max})

Any angle where b(θ) = 1. Using our previously derived formula for b(θ), there can be many angles where this occurs. From b(θ):

$$b(\theta_{\max}) = 1 = \cos^2\left[\frac{\pi d \sin\theta_{\max}}{\lambda}\right]$$

$$\cos\left[\frac{\pi d \sin\theta_{\max}}{\lambda}\right] = \pm 1$$

$$\frac{\pi d \sin\theta_{\max}}{\lambda} = n\pi \quad \text{where n} = 0,1,2,3,...$$

$$\sin\theta_{\max} = \frac{n\lambda}{d}$$

$$\theta_{\max} = \sin^{-1}\left[\frac{n\lambda}{d}\right]$$

Below are listed the max power angles for various ratios of λ/d:

λ/d	θ_{max} (between 0° and 90°)
2.0	0°
1.0	0°,90°
0.5	0°, 30°, 90°
0.333	0°, 19.5°, 41.8°, 90°
0.25	0°, 14.5°, 30°, 48.6°, 90°

Notice that the lower the ratio of λ/d, the higher the number of maximum power angles.

Null Angles (θ_{null})

The angles where the beam pattern function is equal to zero. If any sound ray arrives at any of the null angles, little or no power from the incoming sound ray is received because of destructive interference between the signals received by each of the separate elements in the array. We calculate the null angle by setting the beam pattern function equal to zero as shown below.

$$b(\theta_{null}) = 0 = \cos^2\left[\frac{\pi d \sin\theta_{null}}{\lambda}\right]$$

$$\cos\left[\frac{\pi d \sin\theta_{null}}{\lambda}\right] = 0$$

$$\frac{\pi d \sin\theta_{null}}{\lambda} = n\frac{\pi}{2} \quad \text{where n} = 1,3,5,7,...$$

$$\sin\theta_{\text{null}} = \frac{n\lambda}{2d}$$

$$\theta_{null} = \sin^{-1}\left[\frac{n\lambda}{2d}\right]$$

Below are listed the null angles for various ratios of λ/d:

λ/d	θ_{null} (between 0° and 90°)
2.0	90°
1.0	30°
0.5	14.5°, 48.6°
0.333	9.6°, 30°, 56.4°
0.25	7.2°, 22.0°, 38.7°, 61.0°

Beamwidth (θ_{BW})

The beamwidth of a beam is the angular displacement between the angles where the beam pattern function, b(θ), is greater than 0.5. If any sound ray arrives at any angle within the beamwidth, the sound ray may be detectable. We assume that if a ray arrives at an angle outside the beamwidth that the signal will not be detectable. Within each beam, at least half of the power of the original wave will be received (not cancelled due to destructive interference between the elements of the array.)

The beamwidth is important because it is proportional to the bearing accuracy of the specific beam.

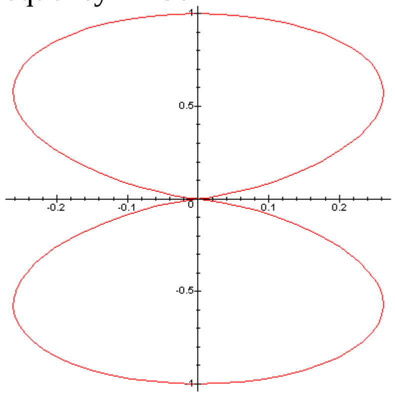
When we detect a sound, we can electronically determine which beam that the sound arrived in but not specifically at what exact bearing in that beam. Thus, the smaller the beam width, the greater the bearing accuracy. It is important to not then that beam width is not only a function of the frequency of the sound but what beam the sound arrives in.

Referring to the diagram on page 13-4, the beams on the "beam" of the array (perpendicular to the array axis) are much narrower than the beams on the array axis (also called the "end-fire" beams.)

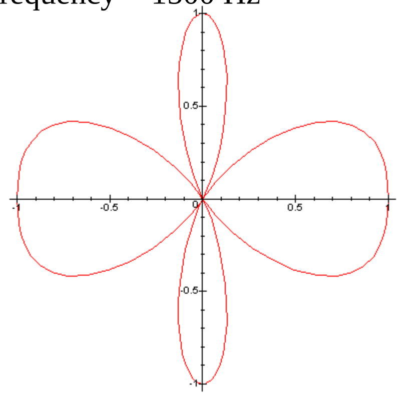
Dependence Of Beam Pattern On Frequency

For most physical arrays, the separation distance between the elements, d, is a fixed distance. Since all of the previous parameters depended on the ratio of λ/d, every one of the parameters will depend on the frequency (and thus the wavelength) of the sound incident on the array. To show the dependence of the beam pattern of a fixed array on frequency, several beam patterns are shown below:

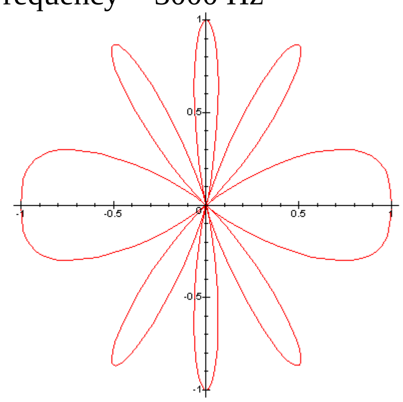
Frequency = 750 Hz



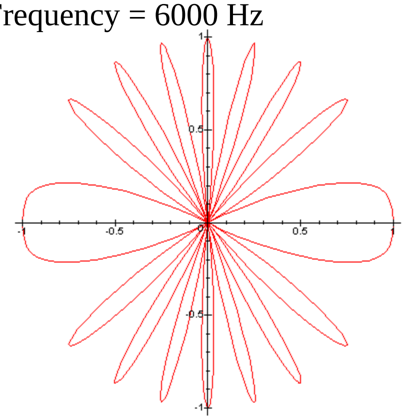
Frequency = 1500 Hz



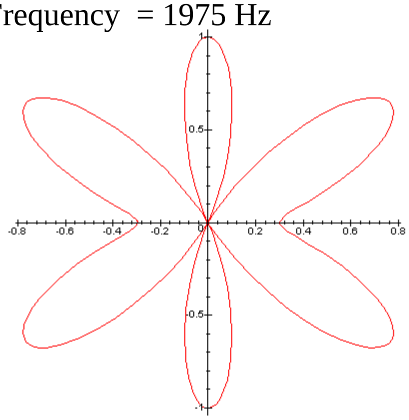
Frequency = 3000 Hz



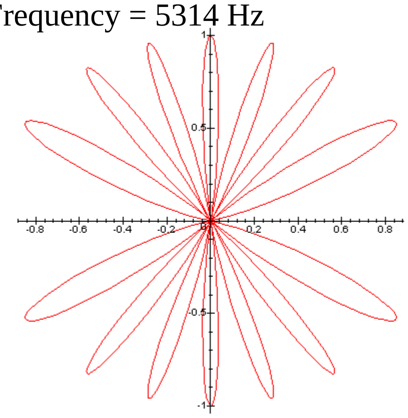
Frequency = 6000 Hz



Frequency = 1975 Hz

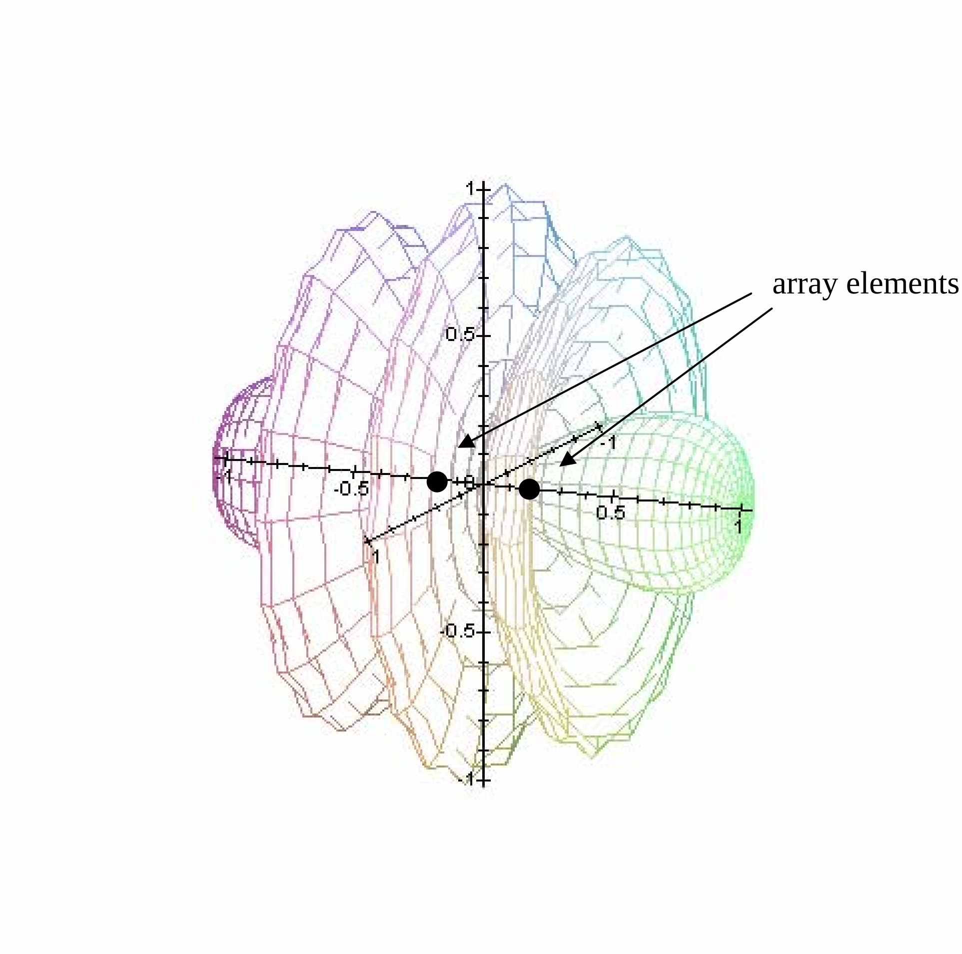


Frequency = 5314 Hz



Three-Dimensional Beam Pattern

Lastly, we must remember that we live in a three-dimensional world. So why did we spend so much time exploring the two-dimensional beam pattern? The beam pattern is independent of the angle ϕ in a three-dimensional environment. An example of a three-dimensional beam pattern is shown below.

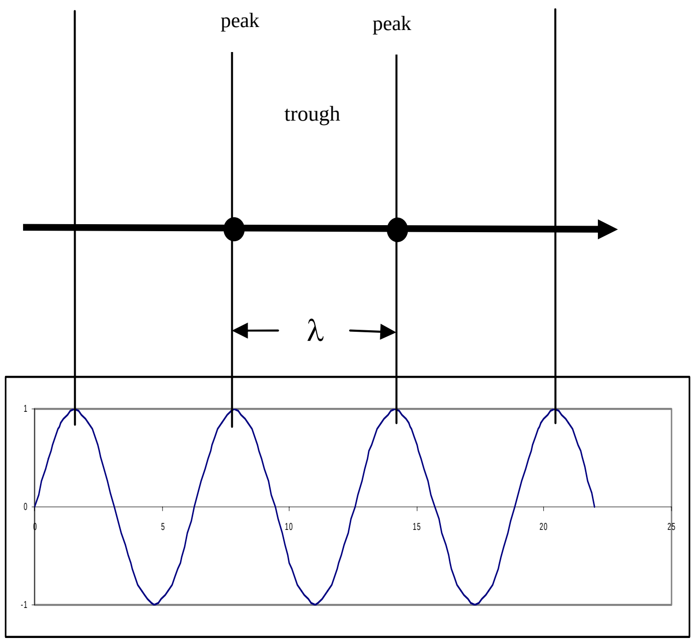


The only difference between the two-dimensional beam patterns we previously derived and the three-dimensional beam pattern shown above is that the three-dimensional beam pattern is the two-dimensional pattern rotated about the array axis. In the example above, the elements lie on the x-axis as shown.

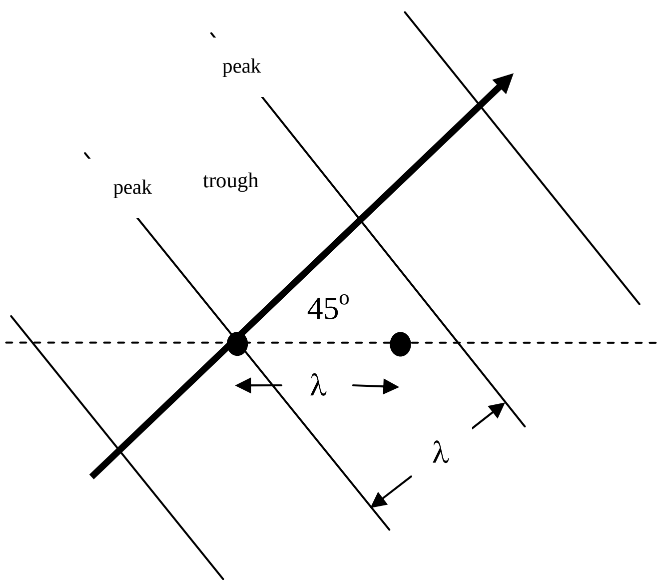
Problems:

1. You have a two element array as shown in the sketches below. The separation between the elements is as indicated. Each point element is omni-directional and calibrated to give 0.001 volt per Pascal. Find the total voltage generated from the array for a traveling wave $p(x) = p_o \cos\left(\frac{2\pi}{\lambda}x - \omega t\right)$ (with maximum amplitude p_o = 1 Pa) in each of the following situations. The time is at the instant shown in the sketch

a)



b)



c) Repeat b) for an angle of 30°. (Draw your own sketch)
d) Repeat a) for the case where one of the elements is moved to the trough. (draw your own sketch)

2. Given a 2 element array with a 1.0 m spacing between elements, determine the following assuming the frequency is 3000 Hz and c = 1500 m/s.

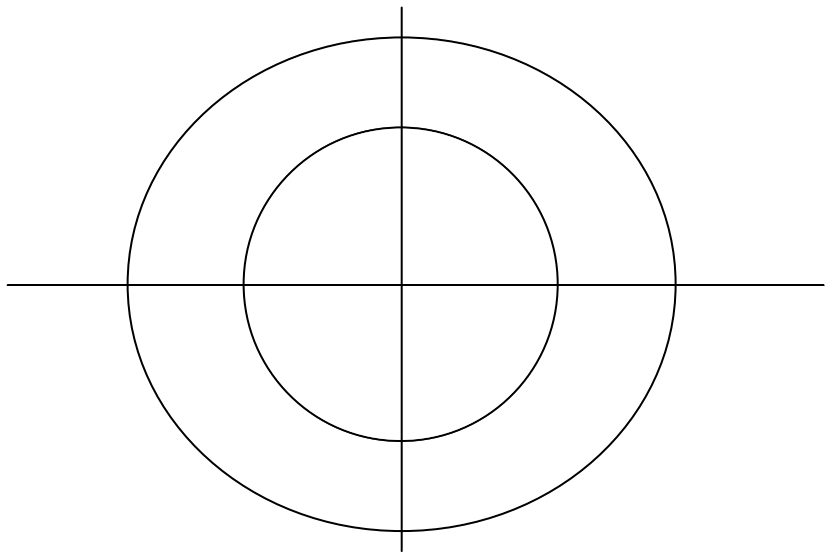
   a) The wavelength of the sound.

   b) The maximum power angles from $0° \le \theta \le 90°$.

   c) The null angles from $0° \le \theta \le 90°$.

   d) The beam width about 0°
.
   e) The beam width about 30°.

   f) Complete a polar plot of $b(\theta)$.



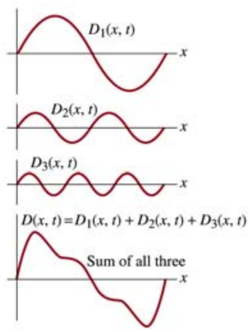
3. The half power beamwidth is defined as:
   a) The angular separation between the first two null angles of an array.
   b) The angular separation between the two "3dB down" angles of the main beam of the array.
   c) The directivity index of the array divided by 2.
   d) The area of the beam pattern of an array where there is no chance of detection.

4. You are given a two element array with identical omni directional hydrophones. Let the spacing between the hydrophones be $\lambda/2$. Calculate the beam width of the main lobe (beam width is the angular separation of the half power points)

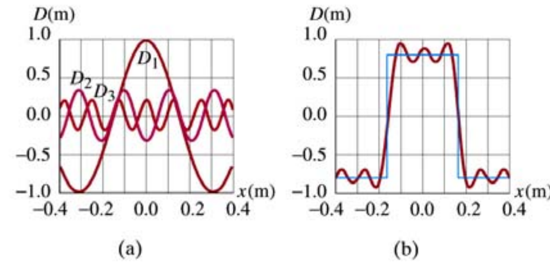
5. An array consisting of two identical elements placed 40 cm apart is receiving sound of a wavelength of 12 cm.
   a) Locate the angles where there are nulls in the beam pattern function.
   b) Locate the angles where there are maxima (or side lobes).
   c) Calculate the value of $b(\theta)$ for a sufficient number of additional angles such that you can plot $b(\theta)$ for $0<\theta<90$. Plot $b(\theta)$ vs θ on polar graph paper.

6. Design a 2 element array with a half –power full beam width of 25 degrees at 15 kHz. The spacing between the two elements is:______________________

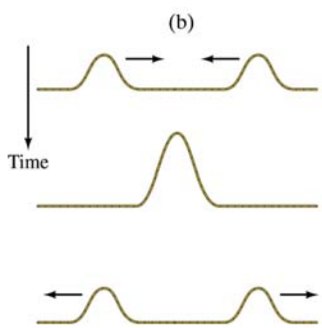
Superposition



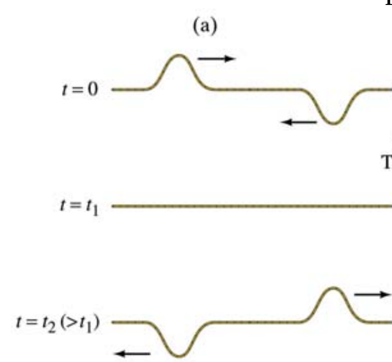
Fourier Series



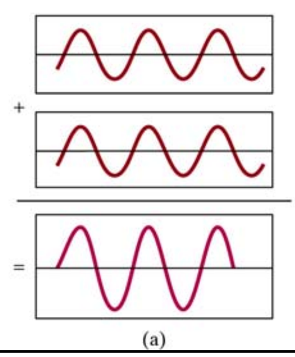
Constructive Interference of a pulse



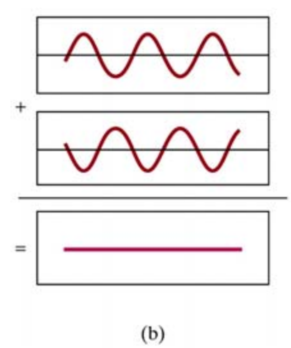
Destructive Interference of a pulse



Constructive Interference of Harmonic Waves



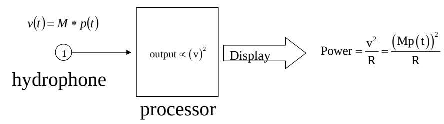
Destructive Interference of Harmonic Waves



2 Dimensional Example

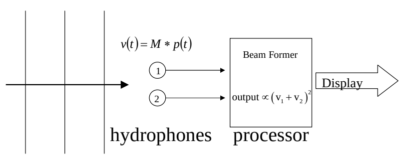
(a) (b)

Single Hydrophone



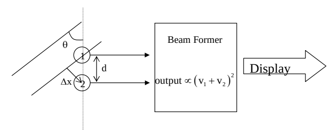
"omni-directional"

Two Hydrophones



Why not output $\propto v_1^2 + v_2^2$???

Incident Wave



$$v_1 = Mp_1(t) = Mp_{max}\cos(k(0) + \omega t)$$
$$v_2 = Mp_2(t) = Mp_{max}\cos(k(-\Delta x) + \omega t)$$
$$\text{output} \propto (v_1 + v_2)^2 = \left\{Mp_{max}\left[\cos(\omega t) + \cos(-\delta + \omega t)\right]\right\}^2$$
$$\text{where } \delta = k\Delta x = kd\sin\theta$$

Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$
$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$
$$\cos\theta + \cos\phi = 2\cos\frac{\theta + \phi}{2}\sin\frac{\theta - \phi}{2}$$
$$\cos\theta\cos\phi = \frac{1}{2}\left[\cos(\theta + \phi) + \cos(\theta - \phi)\right]$$
$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

Power Output from the Processor

$$P = \frac{(Mp_{max})^2}{R}\left[\cos(\omega t) + \cos(-\delta + \omega t)\right]^2$$
$$\langle P\rangle = \frac{(Mp_{max})^2}{R}\left\langle \cos^2\omega t + \cos^2(\omega t - \delta) + 2\cos\omega t\cos(\omega t - \delta)\right\rangle$$
$$\langle P\rangle = \frac{(Mp_{max})^2}{R}\left\langle \cos^2\omega t + \cos^2(\omega t - \delta) + 2\left[\frac{1}{2}\{\cos(2\omega t - \delta) + \cos\delta\}\right]\right\rangle$$
$$\langle P\rangle = \frac{(Mp_{max})^2}{R}\left[\frac{1}{2} + \frac{1}{2} + \langle\cos(2\omega t - \delta)\rangle + \langle\cos\delta\rangle\right]$$
$$\langle P\rangle = \frac{(Mp_{max})^2}{R}\left[1 + \cos\delta\right] \qquad \delta = k\Delta x = kd\sin\theta$$
$$0 \le \langle P\rangle \le \frac{2(Mp_{max})^2}{R}$$

Beam Pattern Function

$$b(\theta)=\frac{\langle P(\theta)\rangle}{\langle P(\theta=0°)\rangle}=\frac{\frac{(Mp_{max})^2}{R}(1+\cos\delta)}{\frac{(Mp_{max})^2}{R}(1+\cos 0°)}$$

$$\delta = k\Delta x = kd\sin\theta$$

$$b(\theta)=\frac{(1+\cos(kd\sin\theta))}{2}$$

Trig identity

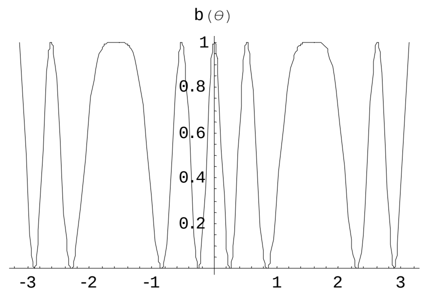
$$1+\cos\theta = 2\left(\cos^2\left(\frac{\theta}{2}\right)\right)$$

$$b(\theta)=\left[\cos^2\left(\frac{kd\sin\theta}{2}\right)\right]$$

or

$$b(\theta)=\left[\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)\right]$$

Example λ = .5 d

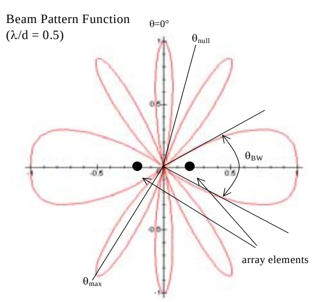


$$b(\theta)=\left[\cos^2\left(\frac{kd\sin\theta}{2}\right)\right]$$

or

$$b(\theta)=\left[\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)\right]$$

Example λ = .5 d



Maximum Power Directions

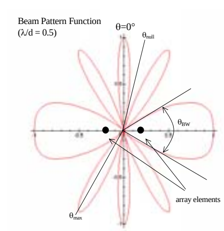
$$b(\theta_{max})=1=\cos^2\left[\frac{\pi d\sin\theta_{max}}{\lambda}\right]$$

$$\cos\left[\frac{\pi d\sin\theta_{max}}{\lambda}\right]=\pm 1$$

$$\frac{\pi d\sin\theta_{max}}{\lambda}=n\pi \quad \text{where } n = 0,1,2,3,...$$

$$\sin\theta_{max}=\frac{n\lambda}{d}$$

$$\theta_{max}=\sin^{-1}\left[\frac{n\lambda}{d}\right]$$



Null Angles

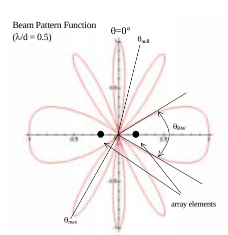
$$b(\theta_{null})=0=\cos^2\left[\frac{\pi d\sin\theta_{null}}{\lambda}\right]$$

$$\cos\left[\frac{\pi d\sin\theta_{null}}{\lambda}\right]=0$$

$$\frac{\pi d\sin\theta_{null}}{\lambda}=n\frac{\pi}{2} \quad \text{where } n = 1,3,5,7,...$$

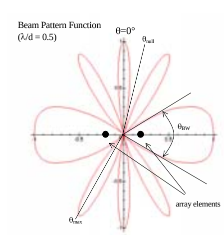
$$\sin\theta_{null}=\frac{n\lambda}{2d}$$

$$\theta_{null}=\sin^{-1}\left[\frac{n\lambda}{2d}\right]$$

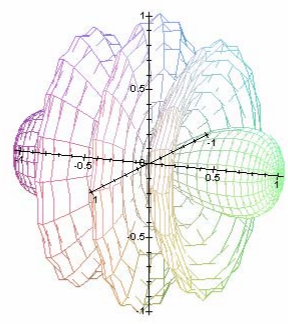


Beam Width

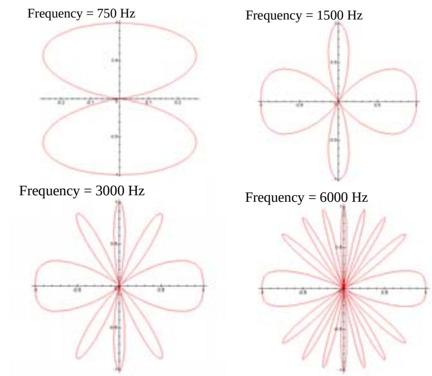
- The beamwidth of a beam is the angular displacement between the angles where the beam pattern function, b(θ), is greater than 0.5.
- 3 dB down points
- The beamwidth is important because it is proportional to the bearing accuracy of the specific beam.



3-d Beam Pattern



Effect of Increasing Frequency



Directivity Index and Multi-element Arrays

At the beginning of the last section, we began discussing how it would be possible to increase the response and the signal-to-noise ratio by increasing the number of elements that we used to receive sound. This also led to the formulation of the beam pattern function and drawing the response patterns for a simple two-element array.

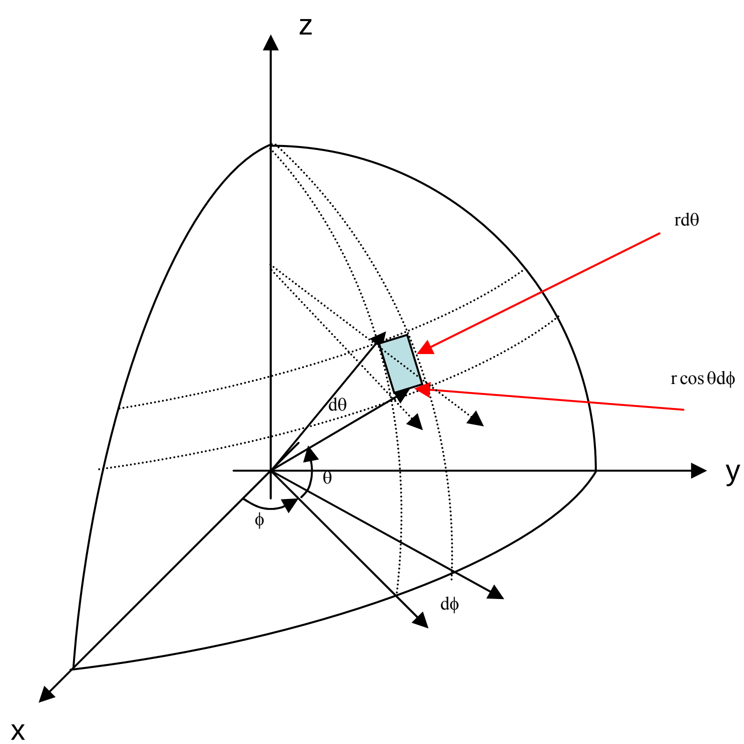
We will quantify the affect of increasing the number of elements in our array by deriving an expression called the Directivity Index. The Directivity Index is the ratio of the total noise power in an isotropic noise filled environment, incident on an array, compared to the power actually received by the system.

$$DI = 10\log\frac{N_{\text{omni-directional noise}}}{N_{\text{directional noise}}}$$

where $N_{\text{omni-directional noise}}$ (N_{ND}) is the power of the isotropic noise incident on the array and $N_{\text{directional noise}}$ (N_D) is the power of the isotropic noise received by the array.

To calculate the Directivity Index of an array,

$$N_{ND} = 4\pi r^2 I_i$$
$$N_D = \iint I_i b(\theta,\phi)\, dA$$



As shown in the above sketch, θ is the latitude angle measured up from the plane of the equator (x-y plane) and ϕ is the longitude angle measured from the x-z axis. The area of a small elemental area on this surface can be found from the following equation, obtained by multiplying the dimensions of the element.

$$dA = r^2 \cos\theta d\theta d\phi$$

The integrations over θ must be from 0 to 2π and the integration over ϕ is from –π/2 to +π/2. When calculating the omni or non-directional power, b =1 and it is easy to show that the integration over θ and ϕ result in a factor of 4π. Similarly, to calculate the directional noise level:

$$N_D = \iint I_i b(\theta,\phi) r^2 \cos\theta d\theta d\phi$$

$$N_D = I_i r^2 \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b(\theta,\phi) \cos\theta d\theta d\phi$$

Since the beam pattern function is independent of θ such that b(θ,ϕ)=b(θ) and because the beam pattern function is symmetrical about the x-axis, the double integrals can be evaluated as below.

$$N_D = I_i r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b(\theta) \cos\theta d\theta \int_0^{2\pi} d\phi$$

$$N_D = 2\pi I_i r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b(\theta) \cos\theta d\theta$$

$$N_D = 4\pi I_i r^2 \int_0^{\pi/2} b(\theta) \cos\theta d\theta$$

When this is all combined to calculate the Directivity Index:

$$DI = 10\log\frac{N_{ND}}{N_D}$$

$$DI = 10\log\frac{4\pi r^2 I_i}{4\pi I_i r^2 \int_0^{\pi/2} b(\theta)\cos\theta d\theta}$$

$$DI = 10\log\frac{1}{\int_0^{\pi/2} b(\theta)\cos\theta d\theta}$$

If we can solve the integral of the beam pattern function in the formula above, we can determine the Directivity Index of a given array. The key will be to determine the beam pattern function for the specific array and to evaluate the integral.

Directivity Index for a 2-element Array

If we evaluate the integral in the equation above for a 2-element array, we get the following:

$$DI = 10\log\frac{2}{\left[1+\frac{\sin\left(2\pi d/\lambda\right)}{2\pi d/\lambda}\right]}$$

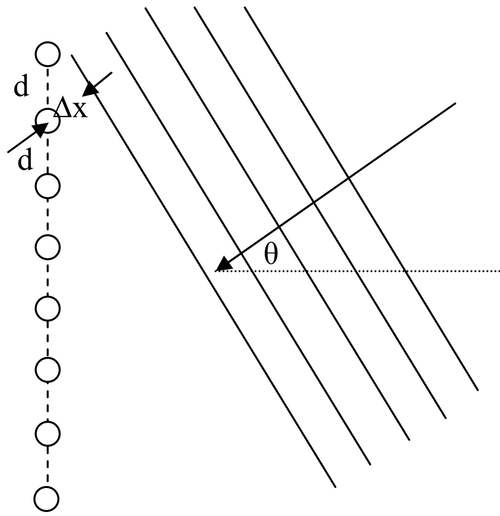
Since the denominator inside the logarithm is simply:

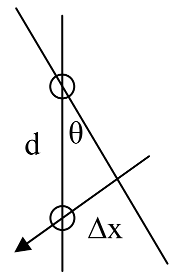
$$\int_0^{\pi/2} b(\theta)\cos\theta d\theta = \int_0^{\pi/2}\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)\cos\theta d\theta = 1+\frac{\sin\left(2\pi d/\lambda\right)}{2\pi d/\lambda}$$

The student should note then that **the Directivity Index of an array varies as a function of frequency** (or wavelength) of the incident sound. When we are evaluating the Directivity Index for an array, normally we will calculate the DI using the center frequency of the frequency band of the processor.

n-Element Array

Beam Pattern Function



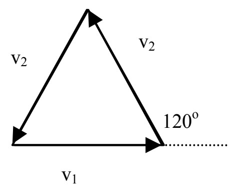


We will study an n-element array with separation, d, between elements and an acoustic wave incident at an angle θ just as we did for the two element array. To find the total voltage from all n-elements we have to add up the voltage from each element and then square the result. For the two element case we were able to accomplish this mathematical task using trigonometric identities. The task is more complicated with 3 or more elements so we will use a technique borrowed from electrical engineering called phasor addition.

Recall from our electrical engineering that we often used phasor addition to add up AC sin waves in three phase systems. In this technique, the voltage from each array element is represented by a vector-like arrow whose direction is defined by the difference in phase that the element has from the voltage of the adjacent array elements. This "phase angle" representation is where the technique gets its name. The so called "phasor" diagram if formed by connecting the individual "phasors" head to tail analogous to vector addition. If the output from a hypothetical array with three elements each differed by 120^o or $2\pi/3$ radians, the below expressions would represent the output from each:

$$v_1 = V_o \cos(\omega t)$$
$$v_2 = V_o \cos\left(\omega t + \frac{2\pi}{3}\right)$$
$$v_3 = V_o \cos\left(\omega t + \frac{4\pi}{3}\right)$$

If we added up these three voltages, the phasor diagram would appear as below at the time, t=0 sec. If somehow we had an output equal to the sum of these three voltages, the output must be zero volts.

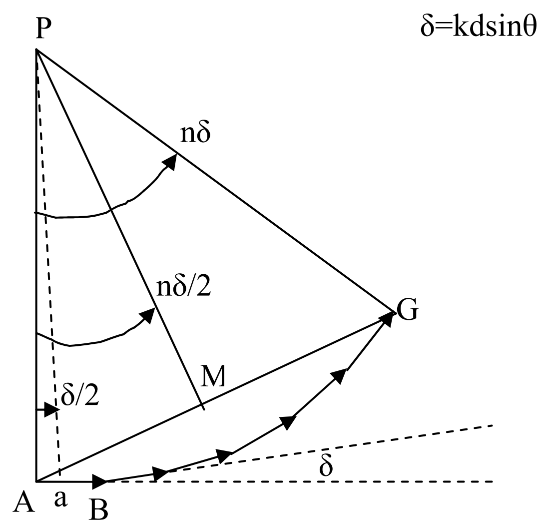


More often in our EE class, we were interested in the difference between 2 phases of a system. We employed phasor subtraction to find the real and reactive parts of this difference. Hopefully it is obvious that the magnitude of the difference between v_1 and v_2 in our example is V_o.

For our multi-element array, the difference in phase between adjacent elements is $\delta = k\,\Delta x$. In the above diagram, we see that each element of the array sees the same wavefront after it has traveled an additional distance $\Delta x = d\sin\theta$ from the element next to it. The phase difference between elements is then $\delta = kd\sin\theta$. The total voltage of beamformer obtained by summing the individual elements is therefore:

$$v_{TOT} = Mp_0\left\{\cos(k[0]+\omega t)+\cos(k[-d\sin\theta]+\omega t)+\cos(k[-2d\sin\theta]+\omega t)+\ldots\cos(k[-(n-1)d\sin\theta]+\omega t)\right\}$$
$$v_{TOT} = v_0\left\{\cos(\omega t)+\cos(\omega t-\delta)+\cos(\omega t-2\delta)+\ldots+\cos(\omega t-(n-1)\delta)\right\}$$
$$v_{TOT} = A\cos(\omega t+\phi)$$

Using a phasor representation, we want to find the resulting amplitude of the sum, A, and sometimes even the resulting phase angle, ϕ. A geometric construction of each of the phasor elements in the sum is drawn as in the diagram below. In this case a 6 element array is shown.



Segment AG is the resulting amplitude of the sum, A. We see that the phasors are approximating the arc of a circular path of radius, R, such that

$$\sin\frac{\delta}{2} \approx \frac{V_0/2}{R} \text{ where R is distance AP}$$

since $\angle$ APa is $\frac{\delta}{2}$. Similarly, since $\angle$ APM = $\frac{n\delta}{2}$ and the midpoint of the chord is $\frac{A}{2} = \frac{V(\theta)}{2}$

$$\sin\left(\frac{n\delta}{2}\right) \approx \frac{V(\theta)/2}{R} \quad \text{Where R is distance AP}$$

Combining these two results and solving for $V(\theta)$,

$$V(\theta) = nV_0\left[\frac{\sin n\frac{\delta}{2}}{n\sin\frac{\delta}{2}}\right] = A$$

It is customary to write nV_o in the numerator since this would be the voltage if the wave arrived at each element of the array at the same time. In this case we would call nV_o the maximum voltage, V_m.

The overall phase of the resulting sum is simply,

$$\phi = \frac{n\delta}{2}$$

Since $\delta = k d\sin\theta = 2\pi d\sin\theta/\lambda$, the total voltage can be written as a function of the angle, θ,

$$V(\theta) = V_m\left[\frac{\sin\left(\frac{n\pi d\sin\theta}{\lambda}\right)}{n\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right]$$

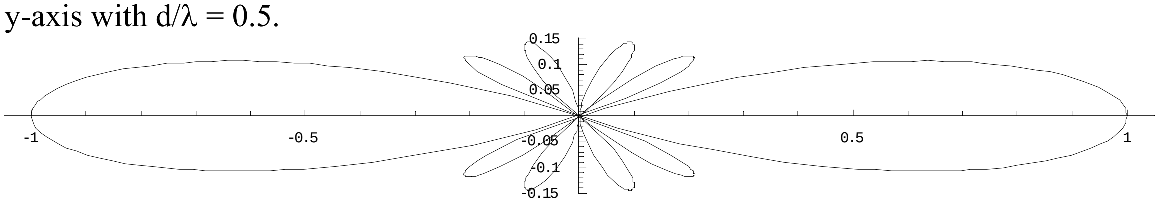
The power seen by the beamformer is then,

$$P(\theta) = \frac{V_m^2}{R}\left[\frac{\sin\left(\frac{n\pi d\sin\theta}{\lambda}\right)}{n\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right]^2$$

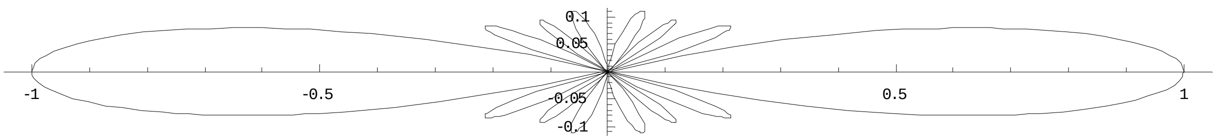
Finally, the beam pattern function is defined,

$$b(\theta) = \frac{\langle P(\theta)\rangle}{\langle P(\theta=0)\rangle} = \left[\frac{\sin\left(\frac{n\pi d}{\lambda}\sin\theta\right)}{n\sin\left(\frac{\pi d}{\lambda}\sin\theta\right)}\right]^2$$

Side lobes and maximums are dependent on the number of elements in the array. For six elements, a null can be created from a hexagon of the 6 representative phasors. This corresponds to a phase angle, δ, of 60 degrees between phasors. Additional nulls can be found when δ is 120° (triangle), 180°, 240°, and 300°. Below is the beam pattern ($\sqrt{b(\theta)}$)for a six element array along the y-axis with $d/\lambda = 0.5$.



In general, the greater the greater the number of elements, the more nulls and therefore more side lobes are created. Each lobe is narrower resulting in increased bearing resolution. Below is the beam pattern for an eight element array along the y-axis with d/λ= 0.5. Can you describe the phasor diagram that creates each of the nulls?



Directivity Index

Calculating the Directivity index for an n-element array is fairly difficult. Using the definition of Directivity Index,

$$DI = 10\log\frac{1}{\int_0^{\pi/2} b(\theta)\cos\theta d\theta}$$

we state without proof that if the beam pattern function for an n-element array is evaluated, the result is:

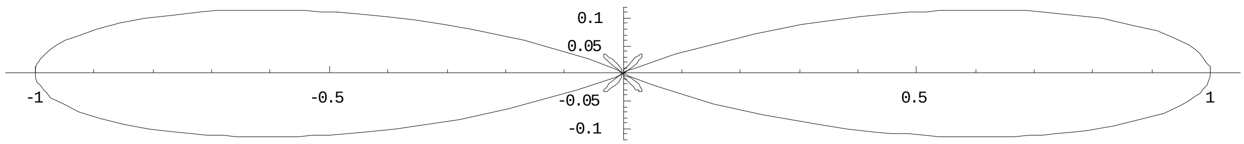
$$DI = 10\log\left[\frac{n}{1+\frac{2}{n}\sum_{\rho=1}^{n-1}\frac{(n-\rho)\sin\left(\frac{2\pi\rho d}{\lambda}\right)}{\frac{2\pi\rho d}{\lambda}}}\right]$$

Linear Arrays

A linear array is a continuous collection of many very small elements. The phasor diagram is similar to the one above with n a very large number and each individual element having a very small length. Because of this, the same beam pattern function can be used as the n-element array with the substitution that array length L = nd. Additionally, with many small elements, the denominator is the sine of a very small angle allowing us to use the small angle approximation, $\sin\alpha = \alpha$.

$$b(\theta)=\left[\frac{\sin\left(\frac{n\pi d}{\lambda}\sin\theta\right)}{n\sin\left(\frac{\pi d}{\lambda}\sin\theta\right)}\right]^2=\left[\frac{\sin\left(\frac{\pi L}{\lambda}\sin\theta\right)}{n\frac{\pi d}{\lambda}\sin\theta}\right]^2=\left[\frac{\sin\left(\frac{\pi L}{\lambda}\sin\theta\right)}{\frac{\pi L}{\lambda}\sin\theta}\right]^2$$

Below is the beam pattern function for a linear array along the y axis with L/λ = 2.



Nulls and Side Lobes

Nulls occur when $\sin\left(\frac{\pi L\sin\theta}{\lambda}\right)=0$. The sine function has zeros at integer multiples of 180 degrees or π radians.

$$\frac{\pi L\sin\theta}{\lambda}=n\pi,\quad n=1,2,3,...$$

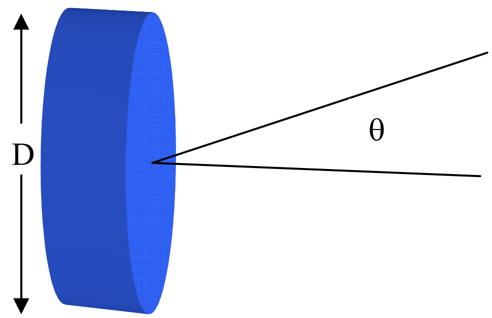
Between these nulls are secondary maxima or side-lobes that occur when the function $\frac{\sin\alpha}{\alpha}$ is a maxima. ($\alpha=\frac{\pi L\sin\theta}{\lambda}$). We can find cases where this occurs with a computer and observe that smallest value is $\alpha=1.43\pi$. For this value, $b(\theta)=0.04719$ and $10\log b(\theta)=-13.3\,\text{dB}$. This means that the first side lobe next to the main lobe at θ = 0 degrees is reduced in amplitude by 13.3 dB.

Directivity Index

Again without proof, the directivity index of a linear array reduces to the following simple result so long as the array length is much greater than the wavelength.

$$DI=10\log\frac{1}{\int_0^{\pi/2}b(\theta)\cos\theta d\theta}=10\log\left(\frac{2L}{\lambda}\right)$$

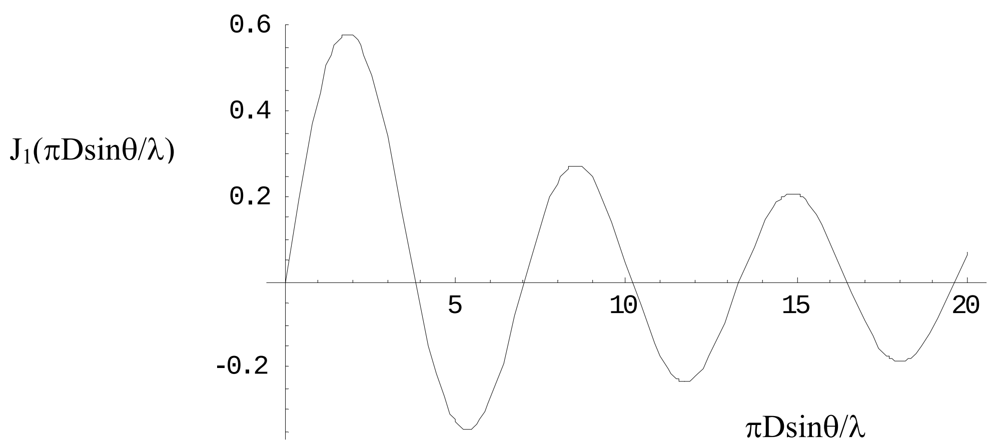
Piston Arrays



A plane piston array as shown above is thought of as composed of a very large number of elements arranged in 2 dimensions on it's surface. Since there is no fixed phase relationship between these elements, phasor addition will not work. Instead, it is necessary to integrate over the elements making up the surface. Experience has shown this is best done in polar coordinates and the results will not be repeated here. The resulting beam pattern function is

$$b(\theta) = \left[\frac{2J_1\left(\frac{\pi D \sin\theta}{\lambda}\right)}{\frac{\pi D \sin\theta}{\lambda}} \right]^2$$

where J_1 is the Bessel Function of the first order and first kind. It's values are well tabulated in mathematical handbooks much like the trigonometric functions. As seen below, maximum values and zero crossings for this Bessel function are not as orderly as the trigonometric functions.



$J_1(\pi D\sin\theta/\lambda)$ has zero crossings (nulls) at $\pi D\sin\theta/\lambda = 3.83, 7.02, 10.17, 13.32, 16.47, \ldots\ldots$
$J_1(\pi D\sin\theta/\lambda)$ has extremes (near the side lobes) at $\pi D\sin\theta/\lambda = 1.84, 5.33, 8.54, 11.71, 14.86, \ldots.$

From this we see that the first zero crossing corresponding to a null in the beam pattern function occurs when

$$\sin\theta = \frac{3.83\lambda}{\pi D} = 1.22\frac{\lambda}{D}$$

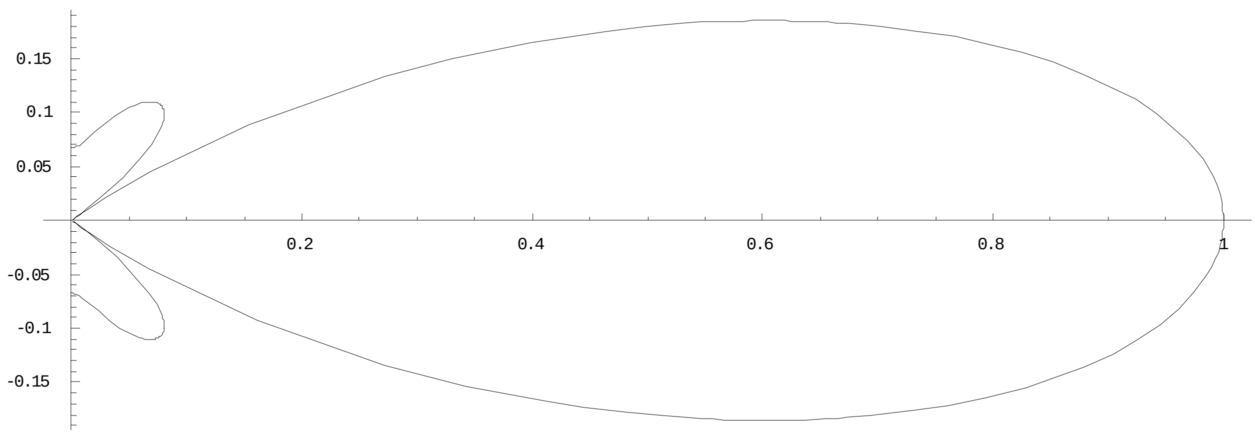
The first side lobe occurs when

$$\frac{2J_1\left(\dfrac{\pi D\sin\theta}{\lambda}\right)}{\dfrac{\pi D\sin\theta}{\lambda}} = \max$$

The actual value of the maximum corresponding to the first side lobe is found by iterating with a computer. It is near the place where $\dfrac{\pi D\sin\theta}{\lambda} \approx 5.33$, and the exact value is $\sin\theta = \dfrac{1.66\lambda}{D}$.

Note that the center beam occurred at θ = 0 where both the numerator and denominator are approaching zero.

Below is the beam pattern ($\sqrt{b(\theta)}$)for a piston array along the y-axis with D/λ = 2.0.



A Table showing the piston array results for lobes, nulls, and beam widths as well as those for linear and two element arrays appears on the following page.

	2-element array	**continuous line array**	**circular piston**
defining parameters	element separation distance – d	array length – L	array diameter - D
beam pattern function **$b(\theta) =$**	$\cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right)$	$\left(\frac{\sin\left[\frac{\pi L}{\lambda}\sin\theta\right]}{\frac{\pi L}{\lambda}\sin\theta}\right)^2$	$\left[\frac{2J_1\left(\frac{\pi D}{\lambda}\sin\theta\right)}{\frac{\pi D}{\lambda}\sin\theta}\right]^2$
directivity index DI	$10\log\left[\frac{2}{1+\left(\frac{\sin\left(2\pi d/\lambda\right)}{2\pi d/\lambda}\right)}\right]$	$10\log\frac{2L}{\lambda}$ for $L >> \lambda$	$10\log\left(\frac{\pi D}{\lambda}\right)^2$ for $D >> \lambda$
null angles **$b(\theta) = 0$** **θ_{null}**	$\sin\theta = (m)\frac{\lambda}{2d}$ m = 1, 3, 5, ...	$\sin\theta = (m)\frac{\lambda}{L}$ m = 1, 2, 3, ...	$\sin\theta = (z)\frac{\lambda}{D}$ $z = 1.22, 2.23, 3.24, 4.24,$. roots of $J_1\left(\frac{\pi D}{\lambda}\sin\theta\right) = 0$
side lobes **$b(\theta)=1$** **θ_{max}**	$\sin\theta = m\frac{\lambda}{d}$ $m = 0, 1, 2, 3$	$\tan\left(\frac{\pi L\sin\theta}{\lambda}\right) = \left(\frac{\pi L\sin\theta}{\lambda}\right)$ $\sin\theta = y\left(\frac{\lambda}{L}\right)$ where y = 1.43, 2.46, 3.47, 4.4	$\sin\theta = w\frac{\lambda}{D}$ where w = 1.64, 2.68, 3.70,
half power angles **$b(\theta)=0.5$** **θ_{hp}** **$\theta_{BW}=2\theta_{hp}$** (only for beam about array axis)	$\sin\theta_{hp} = \frac{n\lambda}{4d}$ $n = 1, 3, 5, 7, \ldots$	$\sin\theta_{hp} = 0.442\frac{\lambda}{L}$	$\sin\theta_{hp} = 0.51\frac{\lambda}{D}$

Problems:

1. Given a 2 element array with a 1.0 m spacing between elements, determine the Directivity Index assuming the frequency is 3000 Hz and c = 1500 Hz.

2. Find the directivity Index of a line of 6 elements spaced 10 cm apart when receiving sound of wavelength 30 cm.

3. The Directivity Index of a sonar array depends on all of the following except:

   a) the physical dimensions of the array.
   b) the speed of sound in the water.
   c) the layout of the hydrophones in the array.
   d) the efficiency of the array.

4. Determine the null angles from 0 to 90° of a 0.25 m active linear array operating at 25 kHz.

5. A 200m linear array is used for receiving a 300 Hz signal. What is the directivity index.

6. A continuous line array of length 150 cm is receiving sound of 5 kHz. The sound speed is 1500 m/s.
   a) Find the angles at which there is a null in the directivity pattern.
   b) Find the angles to the maximum points of all side lobes.
   c) Calculate the half power beam width.
   d) Calculate $b(\theta)$ for $\theta = 10^o, 20^o, 30^o, 40^o, 50^o, 60^o, 70^o, 80^o, 90^o$
   e) Calculate the Directivity Index.

7. Find the directivity index for a linear array of length 125 cm, when operating at 15 kHz in water where c = 1500 m/s.

8. Find the directivity index for a circular piston array of diameter 125 cm, when operating at 15 kHz in water where c = 1500 m/s.

9. A plane circular piston array of diameter 100 cm is receiving sound of frequency 7 kHz. The sound speed is 1500 m/s.
   a) Find the angles at which there are nulls in the directivity pattern
   b) Find the angles to the maximum points of all side lobes.
   c) Calculate the half-power beam width.

10. a) Design a plane circular array with a half-power beam width of 25^o at 15 kHz. The diameter of the array is _________.
    b). Design a continuous line array with a half-power beam width of 25^o at 15 kHz. The length of the array is _________.

11. What is the spacing, d, required for a 4-element line array (detecting frequencies of 10 kHz in water) so that:

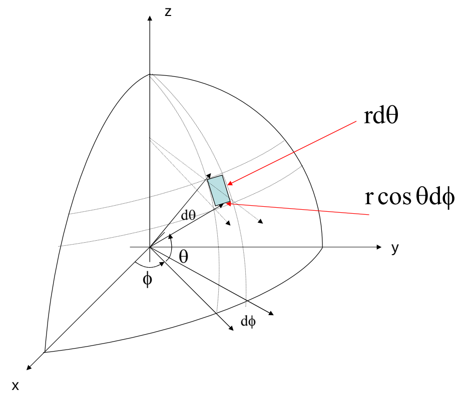
a) The first null in the beam pattern occurs at 90^o.
b) The second major lobe (of magnitude unity) in the beam pattern occurs at 90^o.
c) Compute DI for a)
d) Compute DI for b)

Directivity Index

$$DI = 10\log\frac{N_{\text{omni-directional noise}}}{N_{\text{directional noise}}}$$

$$N = \iint I_i b(\theta,\phi)\,dA$$

$$\text{but } dA = r^2\cos\theta\, d\theta\, d\phi$$



Directional Case

$$N_{ND} = 4\pi r^2 I_i$$

$$N_D = \iint I_i b(\theta,\phi)\,dA$$

$$N_D = \iint I_i b(\theta,\phi)\, r^2\cos\theta\, d\theta\, d\phi$$

$$N_D = I_i r^2 \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b(\theta,\phi)\cos\theta\, d\theta\, d\phi$$

With Rotational Symmetry

$$N_D = I_i r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b(\theta)\cos\theta\, d\theta \int_0^{2\pi} d\phi$$

$$N_D = 2\pi I_i r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b(\theta)\cos\theta\, d\theta$$

$$N_D = 4\pi I_i r^2 \int_0^{\pi/2} b(\theta)\cos\theta\, d\theta$$

DI with Rotational Symmetry

$$DI = 10\log\frac{N_{ND}}{N_D}$$

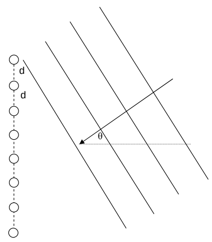
$$DI = 10\log\frac{4\pi r^2 I_i}{4\pi I_i r^2\int_0^{\pi/2} b(\theta)\cos\theta\, d\theta}$$

$$DI = 10\log\frac{1}{\int_0^{\pi/2} b(\theta)\cos\theta\, d\theta}$$

DI for the Two-element Array

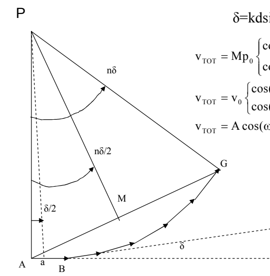
$$DI = 10\log\frac{2}{\left[1+\dfrac{\sin\left(2\pi d/\lambda\right)}{2\pi d/\lambda}\right]}$$

N Element Array



$$v_1 = Mp_1(t) = Mp_{max}\cos\left(k(0) + \omega t\right)$$
$$v_2 = Mp_2(t) = Mp_{max}\cos\left(k(-\Delta x) + \omega t\right)$$
$$v_3 = Mp_3(t) = Mp_{max}\cos\left(k(-2\Delta x) + \omega t\right)$$
$$v_4 = Mp_4(t) = Mp_{max}\cos\left(k(-3\Delta x) + \omega t\right)$$
$$\vdots$$
$$v_N = Mp_N(t) = Mp_{max}\cos\left(k(-(N-1)\Delta x) + \omega t\right)$$
$$\text{output} \propto (v_1 + v_2)^2 = \left\{Mp_{max}\left[\cos(\omega t) + \cos(-\delta + \omega t) + \cos(-2\delta + \omega t) + \cos(-3\delta + \omega t) + \ldots + \cos(-(N-1)\delta + \omega t)\right]\right\}^2$$
$$\text{where } \delta = k\Delta x = kd\sin\theta$$

Phasor Addition

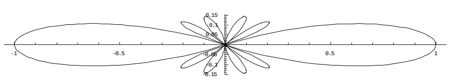


$$\delta = kd\sin\theta$$
$$v_{TOT} = Mp_0\begin{Bmatrix}\cos(k[0] + \omega t) + \cos(k[-d\sin\theta] + \omega t) + \\ \cos(k[-2d\sin\theta] + \omega t) + \ldots\cos(k[-(n-1)d\sin\theta] + \omega t)\end{Bmatrix}$$
$$v_{TOT} = v_0\begin{Bmatrix}\cos(\omega t) + \cos(\omega t - \delta) + \cos(\omega t - 2\delta) + \ldots \\ \cos(\omega t - (n-1)\delta)\end{Bmatrix}$$
$$v_{TOT} = A\cos(\omega t + \phi)$$

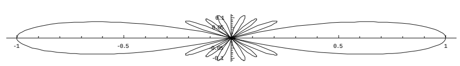
$$\frac{V(\theta)}{2} = R\sin\left(\frac{n\delta}{2}\right) \quad \text{Where R is distance AP}$$
$$\Rightarrow V(\theta) = nV_0\left[\frac{\sin n\frac{\delta}{2}}{n\sin\frac{\delta}{2}}\right]$$
$$\Rightarrow b(\theta) = \left[\frac{\sin\left(\frac{n\pi d}{\lambda}\sin\theta\right)}{n\sin\left(\frac{\pi d}{\lambda}\sin\theta\right)}\right]^2$$

Beam Patterns for 6 and 8 Element Arrays (λ/d = 0.5)

$$\sqrt{b(\theta)}$$



6 element array oriented vertically



8 element array oriented vertically

Directivity Index for an n-Element Array

$$DI = 10\log\left[\frac{n}{1 + \frac{2}{n}\sum_{\rho=1}^{n-1}\frac{(n-\rho)\sin\left(\frac{2\pi\rho d}{\lambda}\right)}{\frac{2\pi\rho d}{\lambda}}}\right]$$

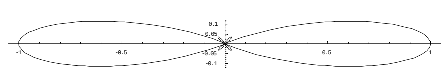
Linear Array

$$b(\theta) = \left[\frac{\sin\left(\frac{n\pi d}{\lambda}\sin\theta\right)}{n\sin\left(\frac{\pi d}{\lambda}\sin\theta\right)}\right]^2 = \left[\frac{\sin\left(\frac{\pi L}{\lambda}\sin\theta\right)}{n\frac{\pi d}{\lambda}\sin\theta}\right]^2 = \left[\frac{\sin\left(\frac{\pi L}{\lambda}\sin\theta\right)}{\frac{\pi L}{\lambda}\sin\theta}\right]^2$$

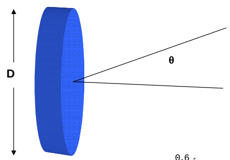
Nulls: $$\frac{\pi L\sin\theta}{\lambda} = n\pi, \quad n = 1, 2, 3, \ldots$$

$$DI = 10\log\frac{1}{\int_0^{\pi/2} b(\theta)\cos\theta\, d\theta} = 10\log\left(\frac{2L}{\lambda}\right)$$

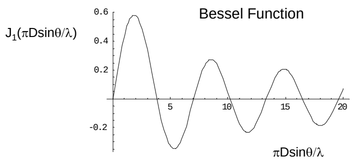
Beam Pattern for a Vertical Linear Array L/λ = 2.0



Piston Array



$$b(\theta) = \left[\frac{2J_1\left(\frac{\pi D\sin\theta}{\lambda}\right)}{\frac{\pi D\sin\theta}{\lambda}}\right]^2$$

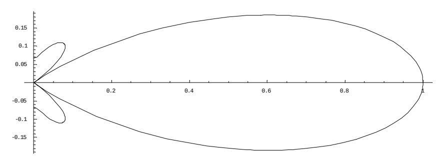


$J_1(\pi D\sin\theta/\lambda)$ has zero crossings (nulls) at $\pi D\sin\theta/\lambda$ = 3.83, 7.02, 10.17, 13.32, 16.47, ...
$J_1(\pi D\sin\theta/\lambda)$ has extremes (side lobes) at $\pi D\sin\theta/\lambda$ = 1.84, 5.33, 8.54, 11.71, 14.86, ...

Beam Pattern for a Piston Array

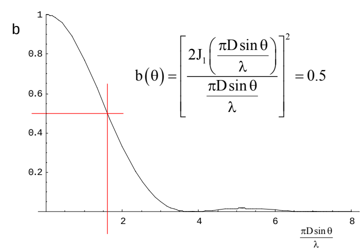
$D/\lambda = 2.0$

$\sqrt{b(\theta)}$



	2-element array	continuous line array	circular piston
defining parameters	element separation distance – d	array length – L	array diameter - D
beam pattern function b(θ) =	$\cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right)$	$\left(\frac{\sin\left[\frac{\pi L}{\lambda}\sin\theta\right]}{\frac{\pi L}{\lambda}\sin\theta}\right)^2$	$\left[\frac{2J_1\left(\frac{\pi D}{\lambda}\sin\theta\right)}{\frac{\pi D}{\lambda}\sin\theta}\right]^2$
directivity index DI	$10\log\left[\frac{2}{1+\left(\frac{\sin(2\pi d/\lambda)}{2\pi d/\lambda}\right)}\right]$	$10\log\frac{2L}{\lambda}$ for $L >> \lambda$	$10\log\left(\frac{\pi D}{\lambda}\right)^2$ for $D >> \lambda$
null angles b(θ) = 0 θ_{null}	$\sin\theta = (m)\frac{\lambda}{2d}$ $m = 1, 3, 5, \ldots$	$\sin\theta = (m)\frac{\lambda}{L}$ $m = 1, 2, 3, \ldots$	$\sin\theta = (z)\frac{\lambda}{D}$ $z = 1.22,\ 2.23,\ 3.24, 4.24, \ldots$ roots of $J_1\left(\frac{\pi D}{\lambda}\sin\theta\right) = 0$
side lobes b(θ)=1 θ_{max}	$\sin\theta = m\frac{\lambda}{d}$ $m = 0, 1, 2, 3 \ldots.$	$\tan\left(\frac{\pi L\sin\theta}{\lambda}\right) = \left(\frac{\pi L\sin\theta}{\lambda}\right)$ $\sin\theta = y\left(\frac{\lambda}{L}\right)$ where $y = 1.43, 2.46, 3.47, 4.48, \ldots$	$\sin\theta = w\frac{\lambda}{D}$ where $w = 1.64, 2.68, 3.70, \ldots.$
half power angles b(θ)=0.5 θ_{hp} $\theta_{BW}=2\theta_{hp}$ (only for beam about array axis)	$\sin\theta_{hp} = \frac{n\lambda}{4d}$ $n = 1, 3, 5, 7, \ldots$	$\sin\theta_{hp} = 0.442\frac{\lambda}{L}$	$\sin\theta_{hp} = 0.51\frac{\lambda}{D}$

Beam width of a piston array



$$b(\theta) = \left[\frac{2J_1\left(\frac{\pi D\sin\theta}{\lambda}\right)}{\frac{\pi D\sin\theta}{\lambda}}\right]^2 = 0.5$$

$$\frac{\pi D\sin\theta}{\lambda} = 1.6 \qquad \sin\theta = \frac{1.6}{\pi}\frac{\lambda}{D} = .51\frac{\lambda}{D}$$

Detection Theory

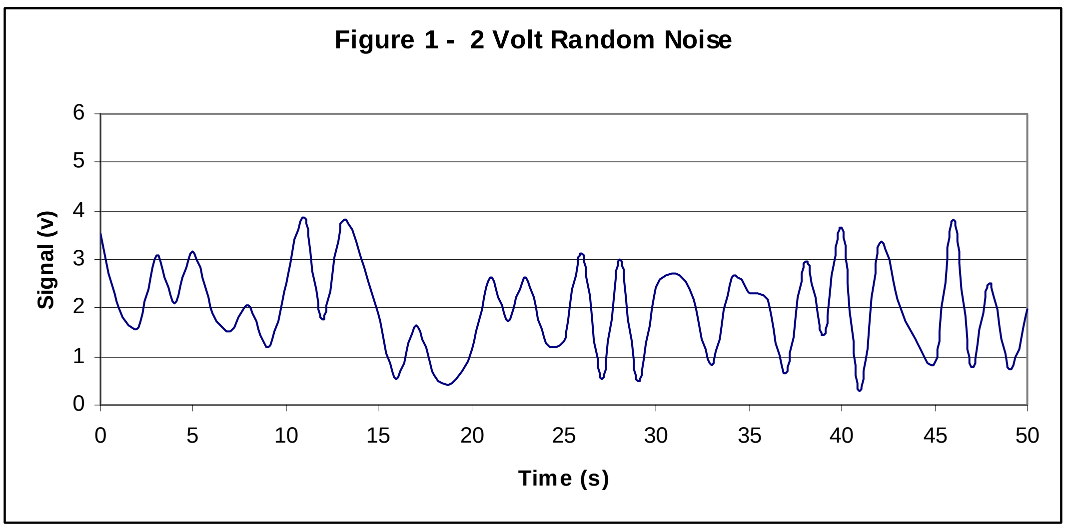
The criterion for **detection** requires that the amount of sound energy collected by the receiver must exceed a threshold level to register a detection. The most common way to do this is first to express the ratio of signal to noise in decibels where:

$$SNR = 10 \log\left(\frac{\text{Signal}}{\text{Noise}}\right)$$

The minimum SNR that is required to determine that there is a signal present in the environment a pre-established percentage of the time, is called the detection threshold, DT. The goal of this lesson is to be able to determine by calculation, a DT for our sonar system. It should be apparent that detection threshold is a statistical concept since the background noise that masks our signal fluctuates in randomly in time. Because of this, we will have to discuss some statistics ideas before we can calculate our Detection Threshold.

Threshold setting

Let's assume that in our environment there is random noise. Let us also assume we have a sonar system with a hydrophone that converts incident acoustic pressure into a voltage sent to the sonar processor. A plot of the voltage output from a hydrophone in an environment with noise might look something like that in figure 1.



Now let's assume that there is a signal also present in the same environment as shown in figure 2 where the hypothetical signal is plotted without noise.

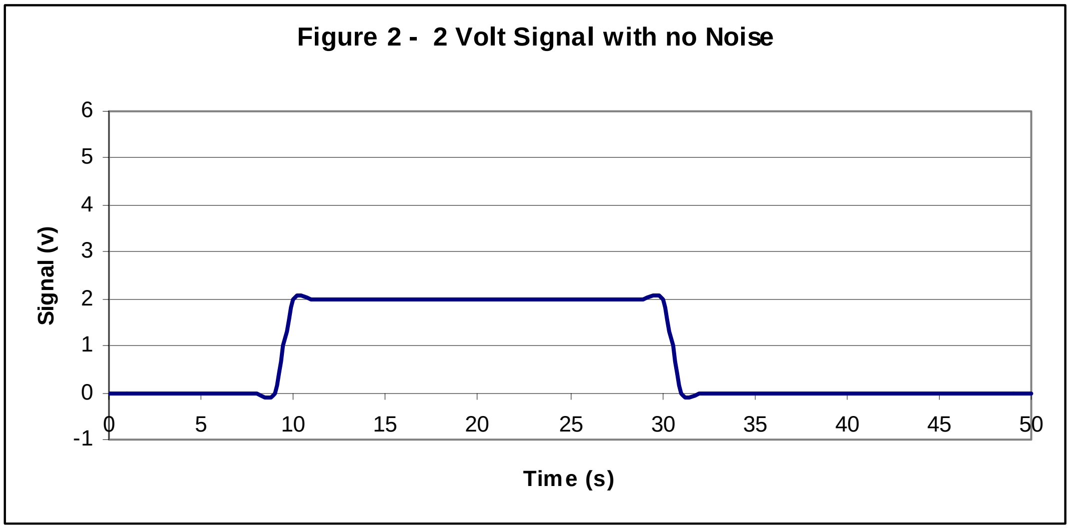
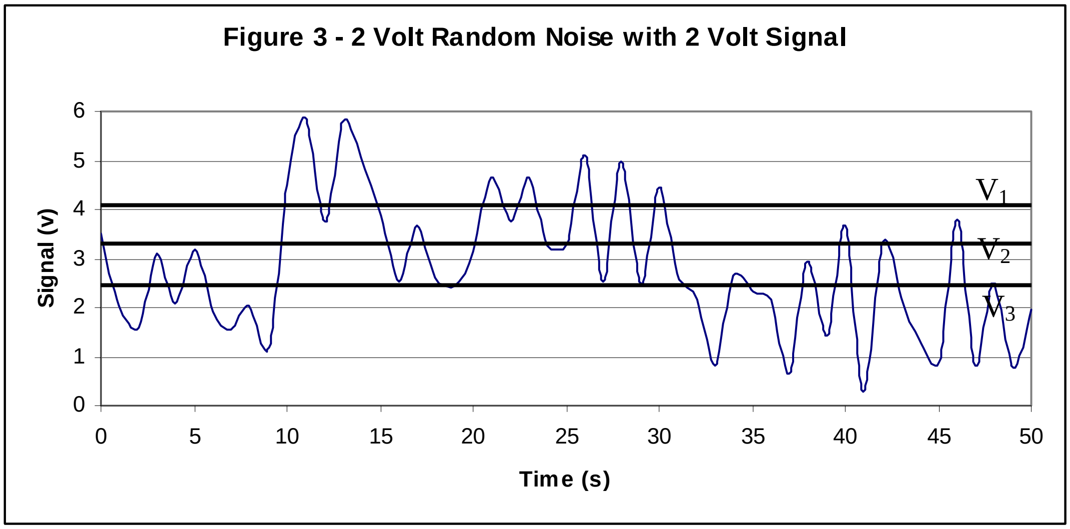


Figure 3 is a depiction of the sum of the signal plus noise. The question then becomes, what detection threshold should be set in the sonar system so that the signal can be detected through the background noise.



Should the threshold voltage be set at voltage V_1 where only the signal that is well above the noise will cause a detection? Or should it be V_2 where not only will some of the signals be detected but also some of the noise will cause a false detection? Or should it be V_3 where a good portion of the noise as well as most of the signals cause detections? What threshold voltage to set is a very difficult question to answer. The more important question though is, what percentage of the time can we tolerate a false alarm and tolerate missing a detection. Both circumstances are directly related to one another.

Binary Decision Table

Though there are only two possible answers at any moment for the conclusion that there is a signal present or not, there are two possible outcomes for both answers, the conclusion was either correct or incorrect. This is best summed up by the following two tables:

		Decision/Hypothesis	
		Signal present	**Signal not present**
Actual input	**Signal present**	Correct detection p(D)	Missed detection p(miss)=1-p(D)
	Signal not present	False Alarm p(FA)	Correct no detection p(null)=1-p(FA)

In this Matrix presentation, statisticians call the "decision" the "hypothesis." It should be clear that for a given situation, the two hypothesis are mutually exclusive. If a signal is actually present, either hypothesis "signal present" or hypothesis "signal not present" must be selected. We are not allowing for an unknown hypothesis. Because of this, the sum of the probability that the "signal is present" and the probability that the "signal is not present" must add up to one.

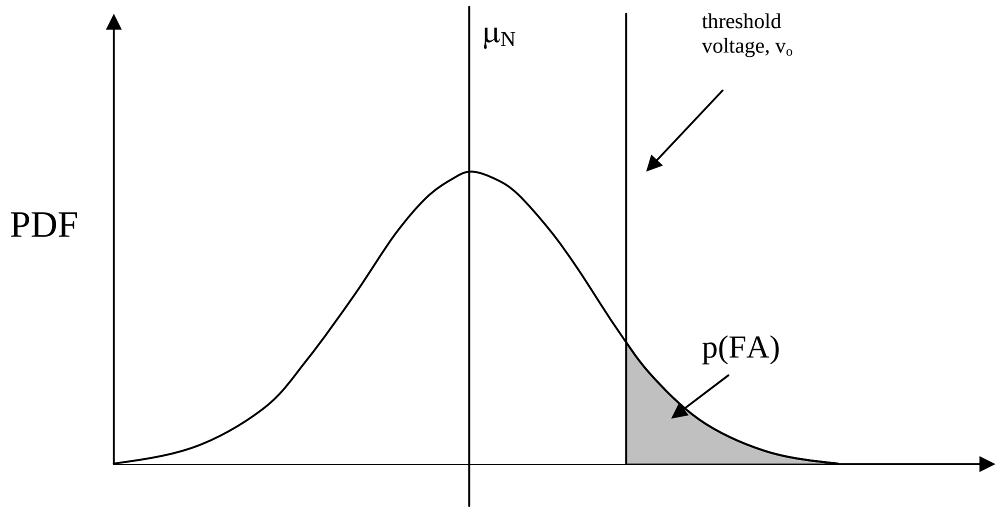
There are two desired outcomes. We hope that anytime a signal actually exists, we chose the "signal present" hypothesis. Otherwise we have selected a "false negative" and have missed a valid target. On the other hand, if there is no signal present, we hope to always select the "signal not present" hypothesis. In this case, selecting "signal present" would be a "false positive" and would represent a false alarm. The below chart summarizes this idea.

	When there is noise only		***When there is signal and noise***	
Decision	noise only	signal + noise	noise only	signal + noise
	correct	wrong	wrong	correct
	p(null)	p(FA)	P(miss)	p(D)
comments	*you are correct, continue searching*	*wasted torpedoes*	*you missed the hostile*	*hostile sunk*

Probability Density Function

A better representation of the voltage output of a hydrophone to be used in determining the threshold setting is to plot the probability density function of the voltage. The probability density function represents the number of times the voltage was at a certain voltage (represented on the x-axis) per unit time. For good empirical reasons, we mathematically model the noise as a

Figure 4 - Noise Only



normal or "Gaussian" distribution of voltages about a mean value, μ. The mathematical description of a Gaussian probability distribution function (PDF) is:

$$p(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(v-\mu)^2}{2\sigma^2}}$$

This equation tells us the probability of a particular value of voltage occurring in an interval of time. For each distribution we define the variance, σ^2. Variance tells us how much the distribution of the voltage "varies" about the mean value.

$$\sigma^2 = \frac{\int (v-\mu)^2\, dv}{\int dv}$$

Standard deviation, σ, is the square root of the variance. We say that the probability that the voltage will lie within one standard deviation of the mean is about 67%. More exactly,

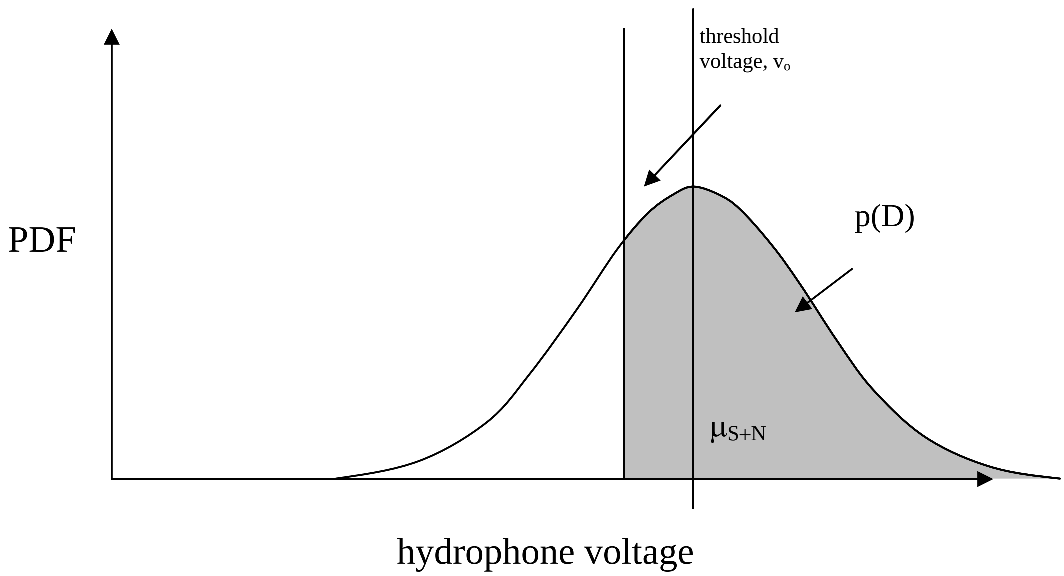
$$0.67 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\sigma}^{+\sigma} e^{-\frac{(v-\mu)^2}{2\sigma^2}}\, dv$$

Figure 4 above represents the probability density function of the receiver voltage for gaussian background noise only. The x-axis represents the voltage output of the hydrophone and the y-axis represents the probability that the voltage was at the level on the x-axis. This is roughly the same as the percentage of samples the sonar system will get at a particulate value of voltage in a particular time interval. From the curve, depending on where the threshold level, v_o, is set, the shaded area under the curve and to the right of the threshold represents the probability of getting a false alarm. Since the total area under the curve represents 100% of the time, the remaining area represents, p(null), the probability that there is no signal.

For simplicity, we will often shift the distribution of noise such that it has a mean value of zero. With this shift, we could calculate P(FA),

$$p(FA) = \frac{1}{\sigma\sqrt{2\pi}} \int_{v_0}^{\infty} e^{-\frac{(v-0)^2}{2\sigma^2}} dv$$

Figure 5 -Signal + Noise



If we look at the probability density function of a signal plus gaussian background noise, we get a distribution like Figure 5. This curve is shifted to the for exactly the same reason that the curve in figure 3 is shifted up while the signal is present. From this curve, the shaded area under the curve represents the probability that a detection p(D) will occur. This probability can be calculated as follows:

$$p(D) = \frac{1}{\sigma\sqrt{2\pi}} \int_{v_0}^{\infty} e^{-\frac{(v-\mu_{S+N})^2}{2\sigma^2}} dv$$

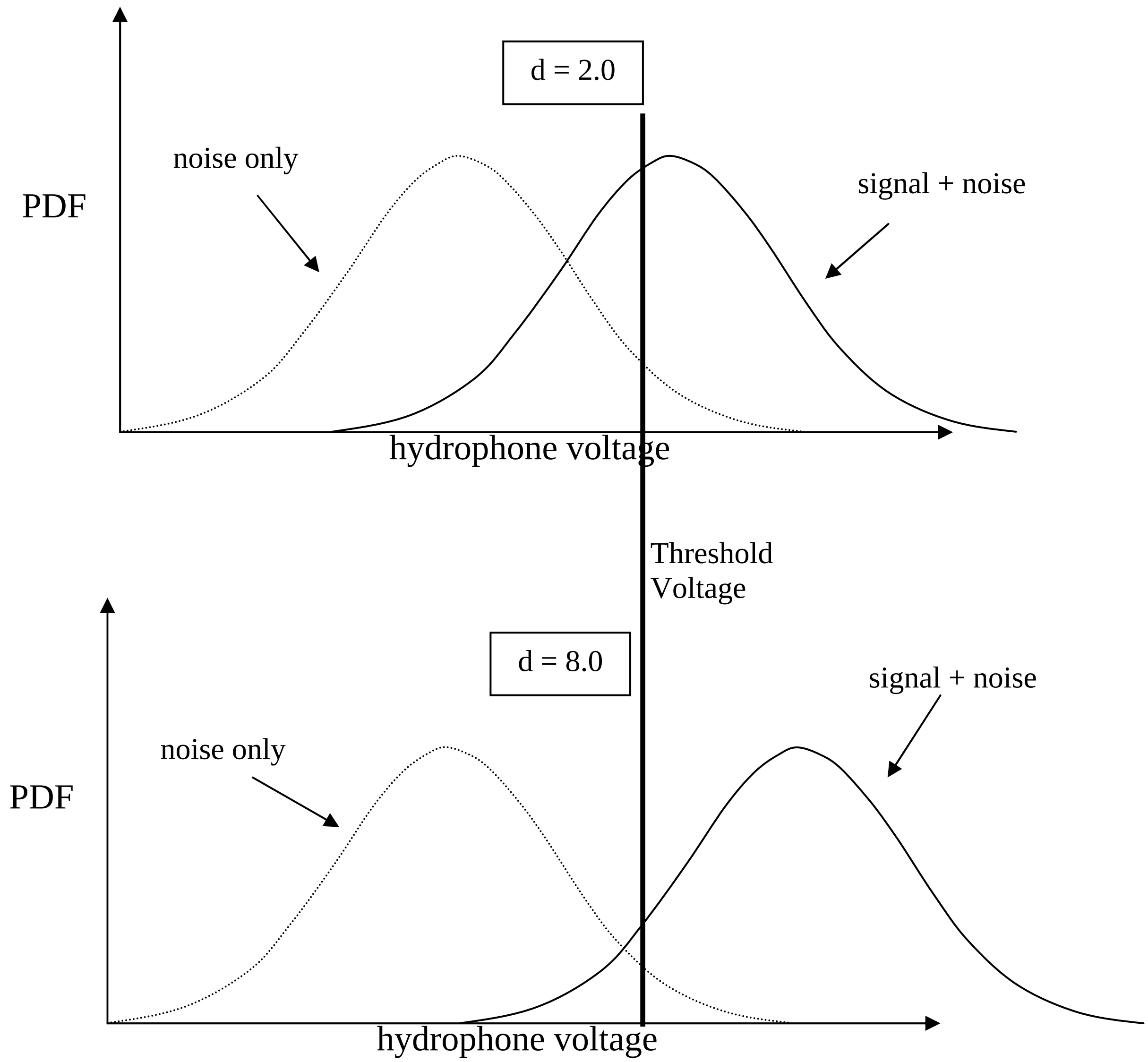
The area to the left of the threshold voltage represents the probability of a missed detection, p(Miss).

Detection Index

Now to relate the time varying magnitude of the hydrophone voltage due to noise, to the time varying magnitude of the signal plus the noise, we define the quantity, d, the detection index. The detection index can be thought of as the "processed signal to noise ratio". That is the ratio of the signal to the noise after the sound energy has been converted to a voltage level and processed electronically. The formula for the detection index is:

$$d = \frac{\left(\mu_{s+n} - \mu_n\right)^2}{\frac{1}{2}\left(\sigma_{s+n}^2 + \sigma_n^2\right)}$$

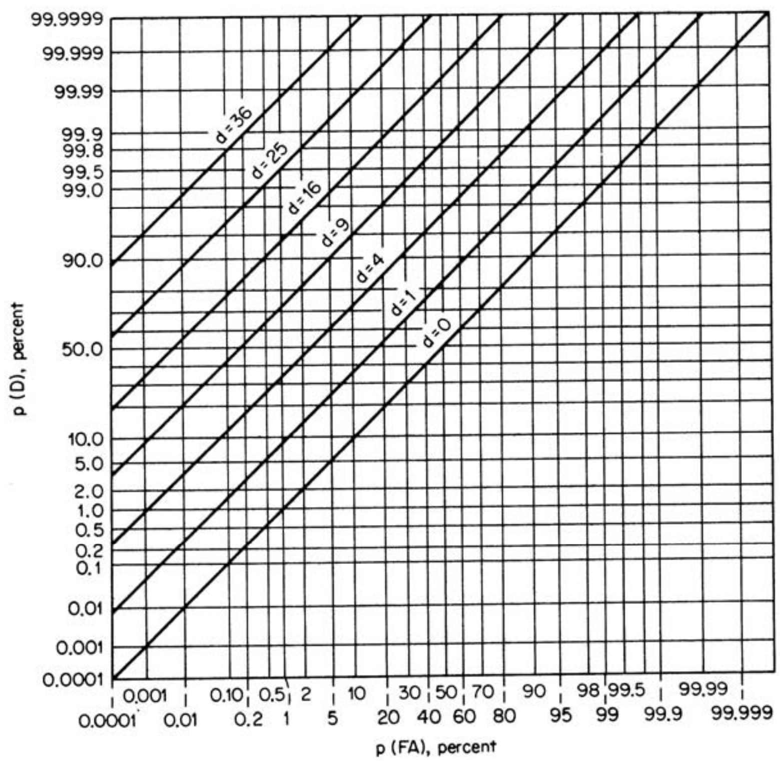
where μ is the mean voltage of the signal plus noise (s+n) or of the noise (n) (denoted by the subscript), and σ^2 is the variance. An example of the detection index for two PDF curves is shown below.



Compare the detection indices for the two PDF's shown above. Notice that the higher the ratio of signal to noise, the higher the detection index or in other words, the more likely a detection will occur at a particular threshold. Since the noise is the same in both cases, the probability of false alarm is the same for both detection indices. But since the d = 8 case is shifted to the right, more area is under the signal + noise curve to the right of the threshold. This tells us the probability of detection is greater in this case.

Receiver Operating Characteristic (ROC) Curves

To put all the preceding information together we can plot the probability for detection as a function of the probability of false alarm for various detection indices. The ROC curves are a set of curves that make our lives simpler by allowing us to be able to determine the probabilities for a sonar system for various signal to noise ratios. An example set of ROC curves for an ideal receiver system is shown below.



From Urick, R. J. Principles of Underwater Sound, 3rd ed. McGraw-Hill Book Company. 1983. p 383

This plot shows that for a given detection index, d, (which is the "processed signal to noise ratio"), that choosing a probability of detection determines the probability of false alarm or vice versa. Understand also that these ROC curves are dependent on the sonar system being analyzed and will look different for a real sonar system compared to the idealized curves shown above.

One characteristic common to all ROC curves is the detection index line labeled d = 0, called the chance diagonal. With improved signal to noise ratio, the series of curves moves up and to the left of the chance diagonal corresponding to improved probability of detection, p(D), and fewer false alarms, p(FA).

Detection Threshold

The ROC curves discussed above are important in that from these curves, we can determine a good detection threshold, DT in dB, for our sonar system. As a first step, you must decide (or be provided) the necessary detection probability you desire. This must be balanced with a reasonable probability of false alarms. It does you no good to insist on perfect probability of detection if you sonar system is constantly crying wolf with false alarms. Often the probability of detection specified is as low as 50%.

As an example, consider a required p(D) = 50% and a p(FA) = 0.2%. The necessary detection index is then 9. Conversely, if the relationship between signal and noise is such that d = 4, then a probability of detection of 70% can not be obtained without accepting a value of 10% for p(FA).

Imagine yourself in a noisy stadium at the concert of the year by your favorite artist. Can you hear what your friend is trying to tell you? Well that depends on many things including how loud the concert is as well as how loud your friend is talking. One other thing that can help you though is whether you see their lips moving or not. If you can "correlate" their lip movement to what little that you do hear from them, it is easier to tell what they are saying. The same holds true for sonar systems.

Active Sonar System or Correlator Detector

If we can compare the received signal and noise to a known signal, as in the example above, it will be easier to determine if there is an actual signal present or not. This is exactly what an active sonar system does. The active system sends out a signal with a known frequency, and pulse shape, and looks for a return signal with the same frequency and pulse shape through the background noise. Knowing this, we can better relate the detection threshold to the detection index. To find the equation for an ideal correlator detector, we must first review the meaning of detection index.

Previously we defined detection index,

$$d = \frac{\left(\mu_{s+n} - \mu_n\right)^2}{\frac{1}{2}\left(\sigma_{s+n}^2 + \sigma_n^2\right)}$$

For the case of correlation, we might expect signal and noise to have the same variance. Detection index then is proportional to a ratio whose numerator is related to the average signal intensity and whose denominator is related to the average noise intensity.

$$d = \frac{\left(\mu_{s+n} - \mu_n\right)^2}{\sigma_n^2} \propto \frac{\langle S \rangle}{\langle N \rangle}$$

Further we state without rigorous proof that the constant of proportionality is the number of samples, m, obtained by the sonar system in a time period T, called the integration time.

$$d = m\frac{\langle S \rangle}{\langle N \rangle}$$

A well known sampling theorem by Nyquist states that "the sampling rate must be at least twice the bandwidth, ω, of the received power so that no signal information is lost." Nyquist's theorem requires that the number of samples be at least, $m = 2(\Delta f)T$. The average signal to noise ratio is then,

$$\frac{\langle S \rangle}{\langle N \rangle} = \frac{d}{m} = \frac{d}{2\left(\Delta f\right)T}$$

Detection threshold for a correlation detector is then defined the expected way band levels in dB are calculated.

$$DT = 10\log\frac{\langle S \rangle}{\langle N \rangle} = 10\log\frac{d}{2\left(\Delta f\right)T}$$

Passive Sonar System or Energy Detector

Imagine yourself at the same concert that we discussed above but now, your friend is facing away from you towards the stage. It would be much harder to determine what they were saying or even if they were talking to you, without the visual clue of seeing their lips moving. The same holds true for a passive sonar system. With a passive system, the operator is looking for a signal even though he does not know what type or frequency signal or even if there is one present. For this case, the equation for how the detection index relates to the signal and noise is different. For the passive sonar, we can show with some difficulty that d is given by the equation:

$$d = \left(\Delta f\right)T\left(\frac{\langle S \rangle}{\langle N \rangle}\right)^2$$

Again solving for the average signal to noise ratio,

$$\frac{\langle S \rangle}{\langle N \rangle} = \left(\frac{d}{\left(\Delta f\right)T}\right)^{\frac{1}{2}}$$

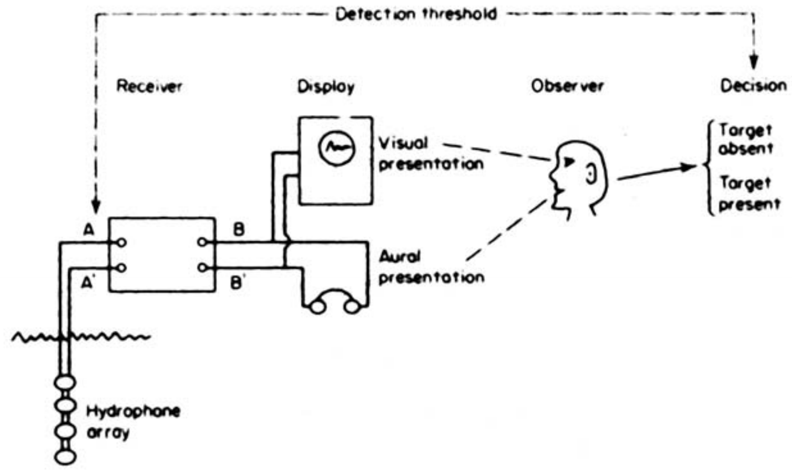
Detection threshold for a passive detector is then defined the expected way band levels in dB are calculated

$$DT = 10\log\frac{\langle S \rangle}{\langle N \rangle} = 10\log\left(\frac{d}{\left(\Delta f\right)T}\right)^{\frac{1}{2}} = 5\log\left(\frac{d}{\left(\Delta f\right)T}\right)$$

** Note that this only holds true for small signal to noise ratios (S/N<<1) and large sample sizes ($\left(\Delta f\right)T >> 1$).

The Sonar System Detection Threshold

Now let's put this together starting with a very basic illustration of the components of a sonar system. This system is composed of an array of hydrophones, a receiver, a display and an operator. Each one of these components including the observer or operator contributes to the detection threshold of the system.



From Urick, R. J. Principles of Underwater Sound, 3rd ed. McGraw-Hill Book Company. 1983. p 378

We have only discussed an idealized prediction of the detection threshold of the above system. Many other things will reduce the detectability of the system but we can not increase the detectability above the idealized case. Some of the items that can affect the systems detection threshold.

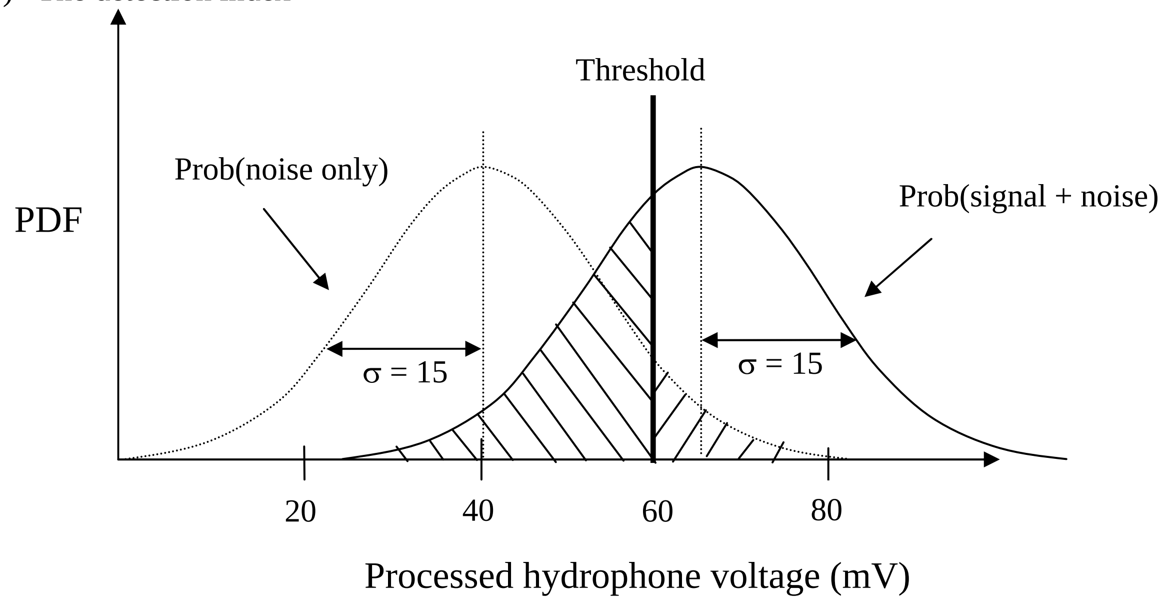
- Fluctuating signal from the target will degrade system performance. P(D) will be a function of amplitude density probability of signal. If signal follows a Rayleigh distribution it can be shown that p(D) can be approximated with threshold Y_0 and detection index d.
  - $$p(D) = \frac{e-\left(\dfrac{Y_0^2}{2+d}\right)}{1+\dfrac{2}{d}}$$
    - valid if $p_d > 0.1$ and $p_{fa} < 0.01$
- If there are more than one signal present.
- If there is multipath propagation.
- If bandwidth- time product (ωT) is not greater than 1.

- If post-detector averager or smoothing filter is used to remove noise from processor output.

We will leave the study of these factors to a more in depth study of sonar systems.

Problems

1. The curves below represent the probability density functions for the “voltage” distribution of signal plus noise and noise alone. The detection threshold is set at 60 mV. If the area under the curve shaded ////=0.10 and the area under the curve shaded \\\\=0.30, calculate:
   a) p(D)
   b) p(miss)
   c) p(FA)
   d) p(null)
   e) The detection index



2. A series of 5 processed voltage readings, for the case of noise alone , is tabulated below:

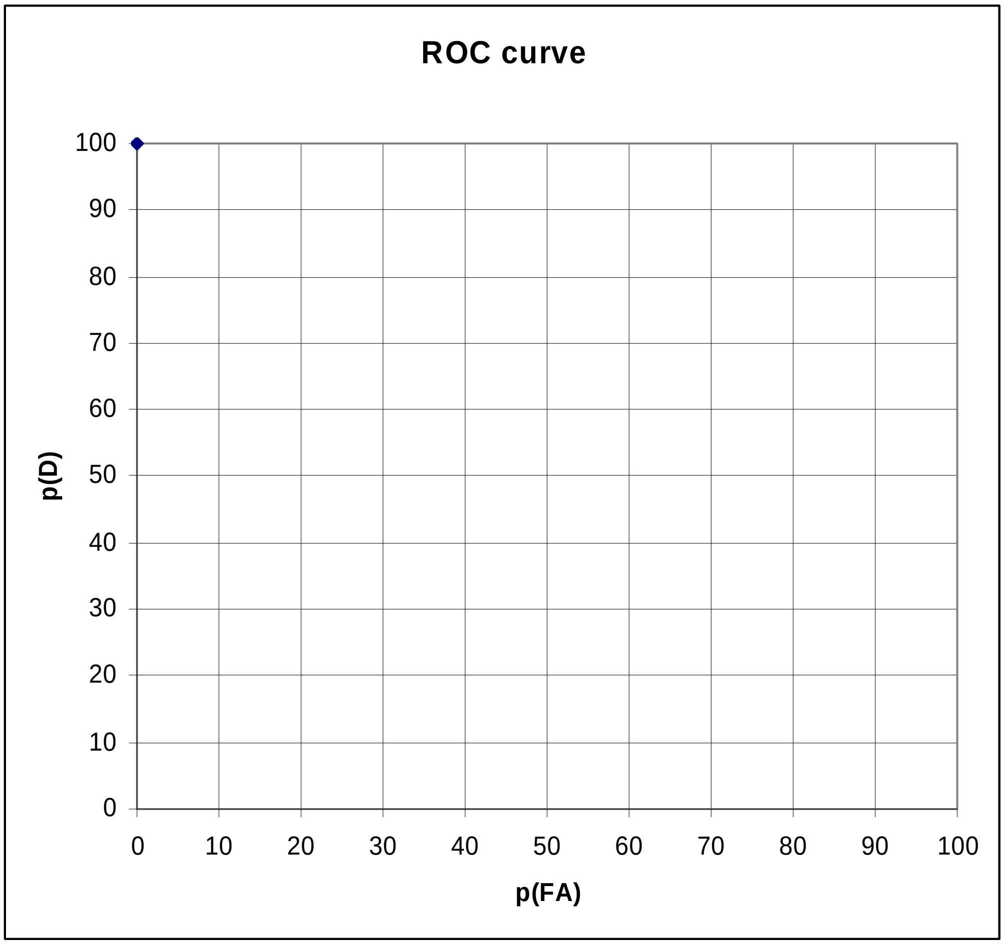
Trial #	Processed “noise” voltage
1	2
2	1
3	3
4	2
5	1

A series of processed voltage readings for the case of signal plus noise is tabulated below

Trial #	Processed “noise” voltage
1	2
2	3
3	4
4	2
5	3

a) Complete the following table and draw the receiver operating curve corresponding to the coordinate (p(FA), P(D)).

Threshold Voltage	P(FA)	P(D)
0.5		
1.5	3/5 = 0.6	5/5 = 1.0
2.5		
3.5		
4.5		



b) Compute the mean, μ_{s+n} and the standard deviation, σ_{s+n} for the processed "signal +noise" case.

Note that sample variance is found from $\sigma^2 = \dfrac{\sum_{i=1}^{n}(x_i - \mu)^2}{n-1}$

c) Compute the mean, μ_n and the standard deviation, σ_n for the processed "noise alone" case.

d) Compute the processed signal to noise ratio parameter, d

3. The human ear can be modeled as an energy detector (passive system) of bandwidth 50 Hz and integration time of 0.50 seconds. What will be the detection threshold for the ear given a 60% probability of detection and a 5% probability of false alarm?

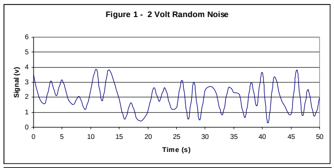
4. A scuba diver must be able to hear a 1000 Hz tone in a backround of ocean noise that is isotropic with a constant intensity spectrum level of 80 dB. If the ear is modeled as an energy detector with a 50 Hz bandwidth and an integration time of 0.5 sec, what is the minimum rms pressure in μPa necessary for him to hear the tone with a probability of detection of 50% and a probability of false alarm of 0.05%? Transmission loss is neglected here. Let the directivity index equal 3 dB.

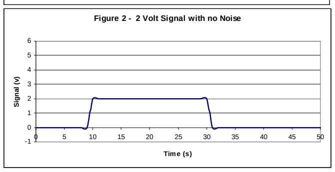
5. A cross-correlator receiving (active) system is used to detect a known signal in a background of Gaussian noise. The predetermined criterion for detection is such that p(D) = 50% and p(FA) = 0.2%. Calculate the system's detection threshold given that the signal duration is 200 milliseconds. The bandwidth is 100 Hz.

6. A surface ship is trying to prosecute an enemy submarine. If the surface ships sonar system has P(D) = 75% and P(FA) = 0.1%, what is the probability that a torpedo will be wasted on a false target?

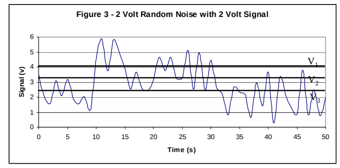
7. A passive continuous line array sonar 30 m long receives signals in a one half octave bandwidth centered on a frequency of 400 Hz. The sonar's receiver may be modeled as a passive energy detector with an integration time of 2.0 seconds. The line array is towed in an environment where the ambient noise spectrum level due to distant shipping is 51 dB, the ambient noise spectrum level due to wind driven waves is 54 dB, the self noise spectrum level is 52 dB, and the local sound speed is 1500 m/s. All spectrum levels are constant in the range of the frequencies in the sonar's receiver bandwidth. What is the sonar's figure of merit (FOM) against a target radiating white noise (with a spectrum level of 120 dB at the sonar's center frequency) given a requirement for p(D) = 50.0% and p(FA) = 0.10%. The directivity index is given by DI = 10 log(2L/λ) where L is the array length and λ is the wavelength. Considering only spherical spreading and no attenuation (TL = 20 log r), solve for the detection range.

Signal and Noise





Combined Signal and Noise



SNR = 1

Binary Decision Table

Actual input

	Decision/Hypothesis: **Signal present**	**Signal not present**
Signal present	Correct detection p(D)	Missed detection p(miss)=1-p(D)
Signal not present	False Alarm p(FA)	Correct no detection p(null)=1-p(FA)

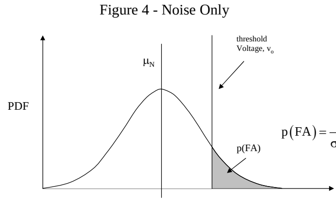
Binary Decision Table

	When there is noise only		*When there is signal and noise*	
Decision	noise only	signal + noise	noise only	signal + noise
	correct	wrong	wrong	correct
	p(null)	p(FA)	P(miss)	p(D)
comments	*you are correct, continue searching*	*wasted torpedoes*	*you missed the hostile*	*hostile sunk*

Probability Density Function - Noise

Normal or Gaussian Distribution

Figure 4 - Noise Only



$$p(FA) = \frac{1}{\sigma\sqrt{2\pi}} \int_{v_o}^{\infty} e^{-\frac{(v-0)^2}{2\sigma^2}} dv$$

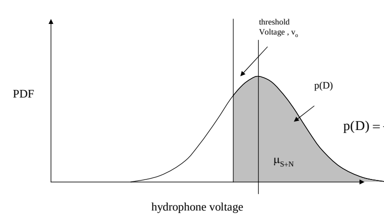
$$p(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(v-\mu)^2}{2\sigma^2}}$$

$$\sigma^2 = \frac{\sum_{i=1}^{n} (v_i - \mu)^2}{n-1}$$

$$0.68 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\sigma}^{+\sigma} e^{-\frac{(v-\mu)^2}{2\sigma^2}} dv$$

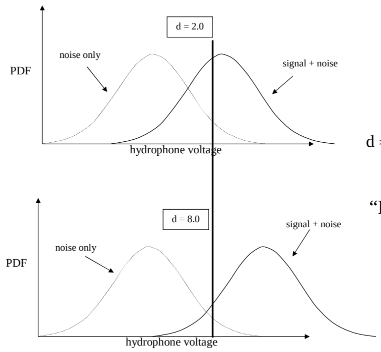
PDF – Signal and Noise

Figure 5 -Signal + Noise



$$p(D) = \frac{1}{\sigma\sqrt{2\pi}} \int_{v_o}^{\infty} e^{-\frac{(v-\mu_{S+N})^2}{2\sigma^2}} dv$$

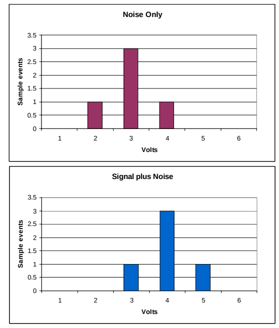
Detectivity (Detection) Index



$$d = \frac{(\mu_{s+n} - \mu_n)^2}{\frac{1}{2}(\sigma^2_{s+n} + \sigma^2_n)}$$

"Processed SNR"

Example ROC Curve



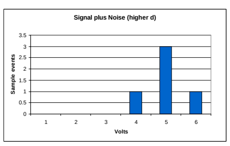
Threshold	P(FA)	P(D)
0.5	1.00	1.00
1.5	1.00	1.00
2.5	0.80	1.00
3.5	0.20	0.80
4.5	0.00	0.20
5.5	0.00	0.00

$$d = \frac{(\mu_{s+n} - \mu_n)^2}{\frac{1}{2}(\sigma^2_{s+n} + \sigma^2_n)}$$

$$\sigma^2 = \frac{(2-3)^2 + 3(3-3)^2 + (4-3)^2}{(5-1)} = .1$$

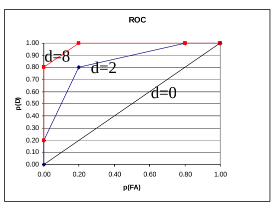
$d = 2$

Receiver Operating Curves

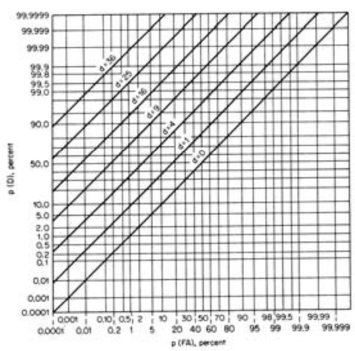


$d = 8$

Threshold	P(FA)	P(D)
0.5	1.00	1.00
1.5	1.00	1.00
2.5	0.80	1.00
3.5	0.20	1.00
4.5	0.00	0.80
5.5	0.00	0.20



ROC Curves



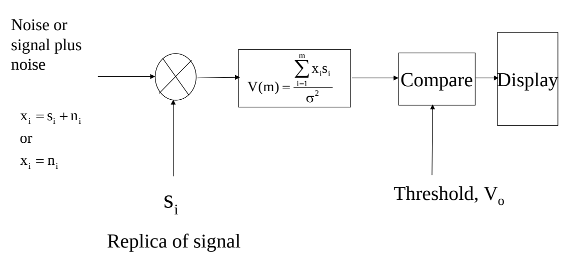
Chance diagonal

P(D) = 50%
P(FA) = 0.2%
d = ?

Nyquist Theorem

- The sampling rate must be at least twice the bandwidth, Δf, of the received power so that no signal information is lost.

Correlation Process



Correlation Detector (active)

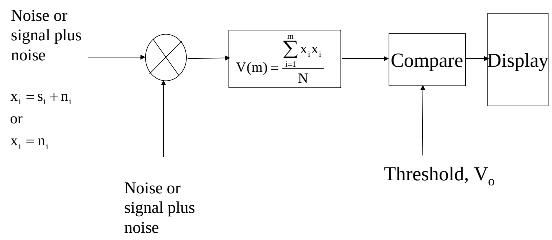
$$d = \frac{\left(\mu_{s+n} - \mu_n\right)^2}{\sigma_n^2} \propto \frac{\langle S \rangle}{\langle N \rangle}$$

$$d = m\frac{\langle S \rangle}{\langle N \rangle}$$

$$\frac{\langle S \rangle}{\langle N \rangle} = \frac{d}{m} = \frac{d}{2(\Delta f)T}$$

$$DT = 10\log\frac{\langle S \rangle}{\langle N \rangle} = 10\log\frac{d}{2(\Delta f)T}$$

Square Law (Energy Detector) Process



Energy Detector (passive)

$$d = (\Delta f)T\left(\frac{\langle S \rangle}{\langle N \rangle}\right)^2$$

$$\frac{\langle S \rangle}{\langle N \rangle} = \left(\frac{d}{(\Delta f)T}\right)^{\frac{1}{2}}$$

$$DT = 10\log\frac{\langle S \rangle}{\langle N \rangle} = 10\log\left(\frac{d}{(\Delta f)T}\right)^{\frac{1}{2}} = 5\log\left(\frac{d}{(\Delta f)T}\right)$$

NAME______________________

Passive Sonar Wrap-up Exercise

The two submarines described below are to engage in a sonar detection exercise off the coast of Kauai. Use the following data for all questions.
(Note: all numbers are made up and do not reflect reality.)

USS Memphis (Target)
SL of sub 180 dB in band from 100 Hz to 500 Hz
Main tonal in band = 400 Hz due to sound short of 400 Hz generator

USS Seawolf (Attacker)
Use hull array consisting of 50 hydrophones in groups of 4 spaced 30 m down the side of the sub (consider as continuous line array)
Bandwidth of sonar suite = 400 Hz from 100 to 500 Hz
Integration time of sonar suite = 20 ms
Want P(D) = 90%, P(FA) = 0.2% assume ideal sonar processor
NL_{self} = 81 dB in band from 100 Hz to 500 Hz

Environment
Sea State = 1
Shipping = light
transition range = 14 Kyds

1. What is the detection index required for detection of the Memphis by the Seawolf?

2. What would be the detection threshold for detecting the Memphis using passive sonar?

3. What are the angles for the nulls of the Seawolf's 30 m long hull array (only give from 0° to 90°)?

4. What is the Directivity Index for this 30 m long array at the frequency of the principle tonal?

5. What is the attenuation coefficient at the frequency of the principle tonal?

6. What is the Ambient Noise Level (Sea State and Shipping)?

7. What is the Total Noise Level?

8. If the Memphis were at 16,000 yds from the Seawolf, what would be the Transmission Loss (include attenuation)?

9. If the Memphis were at 16,000 yds from the Seawolf, what would be the signal-to-noise level?

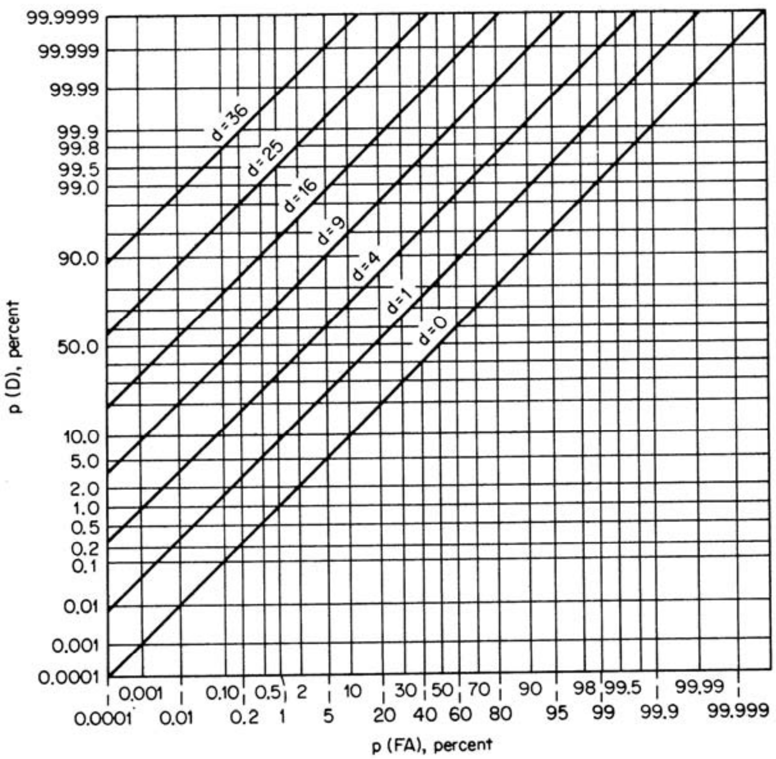
10. Is Memphis detectable? If so, what is the Signal Excess?

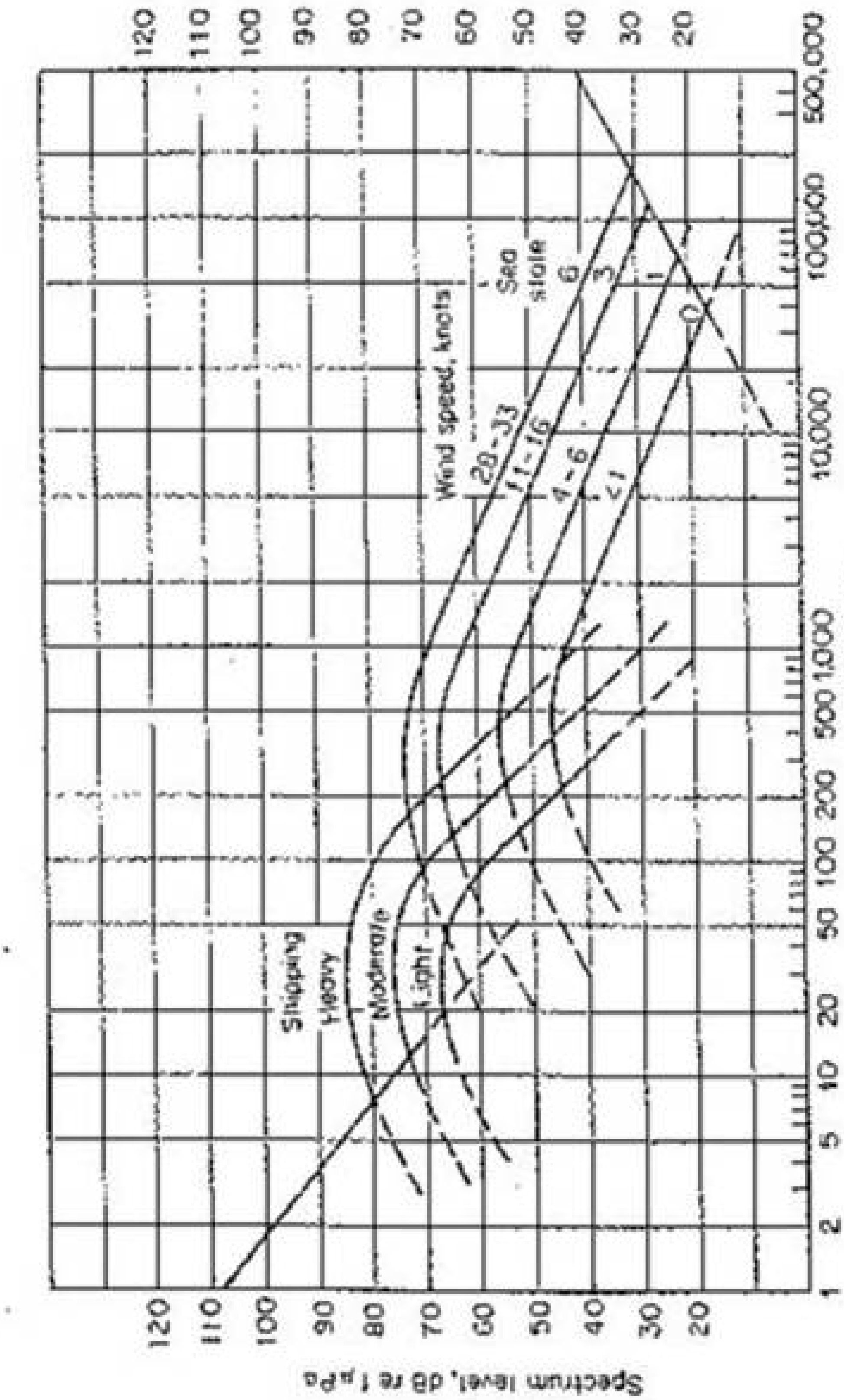
11. What is the max detection range of the Memphis by the Seawolf (this time you can ignore attenuation)?

Miscellaneous Questions

12. If P(D) = 75% and P(FA) = 1%, what is the probability that you will miss a detection of an actual contact?

13. Is a passive sonar system a correlator detector or an energy detector?





Spectrum level, dB re 1 μPa
120
110
100
90
80
70
60
50
40
30
20
Frequency, Hz
1
2
5
10
20
50
100
200
500
1,000
10,000
100,000
500,000
Shipping
Heavy
Moderate
Light
Wind speed, knots
28-33
11-16
4-6
<1
Sea state
6
3
1
0

Passive Sonar Wrap Up Exercise

And Exam Review

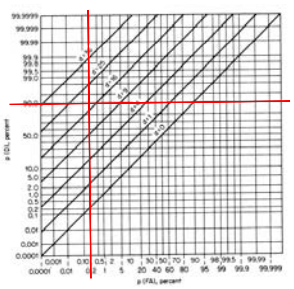
Data

- The two submarines described below are to engage in a sonar detection exercise off the coast of Kauai. Use the following data for all questions.
- (Note: all numbers are made up and do not reflect reality.)
- **USS Memphis (Target)**
  - SL of sub 180 dB in band from 100 Hz to 500 Hz
  - Main tonal in band = 400 Hz due to sound short of 400 Hz generator
- **USS Seawolf (Attacker)**
  - Use hull array consisting of 50 hydrophones in groups of 4 spaced **30 m** down the side of the sub (consider as continuous line array)
  - Bandwidth of sonar suite = 400 Hz from 100 to 500 Hz
  - Integration time of sonar suite = 20 ms
  - Want P(D) = 90%, P(FA) = 0.2% assume ideal sonar processor
  - NLself = 81 dB in band from 100 Hz to 500 Hz
- **Environment**
  - Sea State = 1
  - Shipping = light
  - transition range = 14 Kyds

Part 1

- What is the detection index required for detection of the Memphis by the Seawolf?

p(D)=90%
p(FA)=0.2%



$$d \approx 18$$

Part 2

- What would be the detection threshold for detecting the Memphis using passive sonar?

–Bandwidth of sonar suite = 400 Hz from 100 to 500 Hz
–Integration time of sonar suite = 20 ms

$$\mathrm{DT} = 5\log\left(\frac{\mathrm{d}}{\mathrm{T}\Delta\mathrm{f}}\right) = 5\log\left(\frac{18}{20\mathrm{x}10^{-3}\mathrm{s}\ 400\mathrm{Hz}}\right) = 1.8\mathrm{dB}$$

Part 3

- What are the angles for the nulls of the Seawolf's 30 m long hull array (only give from 0° to 90°)?

$$\sin\theta = \mathrm{m}\frac{\lambda}{\mathrm{L}}$$

$$\lambda = \frac{\mathrm{c}}{\mathrm{f}} = \frac{1500\frac{\mathrm{m}}{\mathrm{s}}}{400\mathrm{Hz}} = 3.75\mathrm{m}$$

	2-element array	continuous line array	circular piston
defining parameters	element separation distance – d	array length – L	array diameter - D
beam pattern function $b(\theta)$ =	$\cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right)$	$\left(\frac{\sin\left[\frac{\pi L}{\lambda}\sin\theta\right]}{\frac{\pi L}{\lambda}\sin\theta}\right)^2$	$\left[\frac{2J_1\left(\frac{\pi D}{\lambda}\sin\theta\right)}{\frac{\pi D}{\lambda}\sin\theta}\right]^2$
directivity index DI	$10\log\left[\frac{2}{1+\left(\frac{\sin\left(2\pi d/\lambda\right)}{2\pi d/\lambda}\right)}\right]$	$10\log\frac{2L}{\lambda}$ for L >> λ	$10\log\left(\frac{\pi D}{\lambda}\right)^2$ for D >> λ
null angles $b(\theta) = 0$ θ_{null}	$\sin\theta = (m)\frac{\lambda}{2d}$ $m = 1, 3, 5, \ldots$	$\sin\theta = (m)\frac{\lambda}{L}$ $m = 1, 2, 3, \ldots$	$\sin\theta = (z)\frac{\lambda}{D}$ $z = 1.22, 2.23, 3.24, 4.24, \ldots$ roots of $J_1\left(\frac{\pi D}{\lambda}\sin\theta\right) = 0$
side lobes $b(\theta)=1$ θ_{max}	$\sin\theta = m\frac{\lambda}{d}$ $m = 0, 1, 2, 3\ldots.$	$\tan\left(\frac{\pi L\sin\theta}{\lambda}\right) = \left(\frac{\pi L\sin\theta}{\lambda}\right)$ $\sin\theta = y\left(\frac{\lambda}{L}\right)$ where $y = 1.43, 2.46, 3.47, 4.48, \ldots$	$\sin\theta_{hp} = 0.51\frac{\lambda}{D}$
half power angles $b(\theta)=0.5$ θ_{hp} $\theta_{BW}=2\theta_{hp}$ (only for beam about array axis)	$\sin\theta_{hp} = \frac{n\lambda}{4d}$ $n = 1, 3, 5, 7, \ldots$	$\sin\theta_{hp} = 0.442\frac{\lambda}{L}$	$\sin\theta = w\frac{\lambda}{D}$ where $w = 1.64, 2.68, 3.70, \ldots.$

The Angles

$$\theta = \sin^{-1}\left(m\frac{\lambda}{L}\right) = \sin^{-1}\left(m\frac{3.75\text{m}}{30\text{m}}\right) = \sin^{-1}(0.125\text{m})$$

$$\theta_1 = \sin^{-1}(0.125(1)) = 7.2^\circ \qquad \theta_5 = \sin^{-1}(0.125(5)) = 38.7^\circ$$

$$\theta_2 = \sin^{-1}(0.125(2)) = 14.5^\circ \qquad \theta_6 = \sin^{-1}(0.125(6)) = 48.6^\circ$$

$$\theta_3 = \sin^{-1}(0.125(3)) = 22.0^\circ \qquad \theta_7 = \sin^{-1}(0.125(7)) = 61.0^\circ$$

$$\theta_4 = \sin^{-1}(0.125(4)) = 30.0^\circ \qquad \theta_8 = \sin^{-1}(0.125(8)) = 90.0^\circ$$

Part 4

- What is the Directivity Index for this 30 m long array at the frequency of the principle tonal?

$$DI = 10\log\left(\frac{2L}{\lambda}\right) = 10\log\left(\frac{2\text{x}30\text{m}}{3.75\text{m}}\right) = 12\text{dB}$$

Part 5

- What is the attenuation coefficient at the frequency of the principle tonal?

$$\alpha = \left(0.003 + \frac{0.1f^2}{1+f^2} + \frac{40f^2}{4100+f^2} + 2.75\times10^{-4}f^2\right)\ \text{dB}/\text{kyd}$$

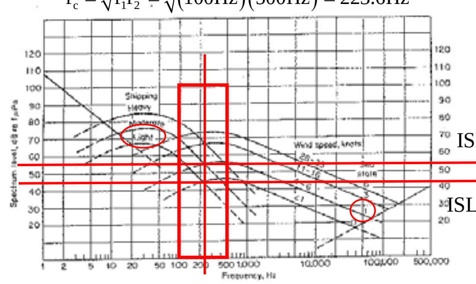
f in kHz

$$\alpha = \left(0.003 + \frac{0.1(0.4)^2}{1+(0.4)^2} + \frac{40(0.4)^2}{4100+(0.4)^2} + 2.75\times10^{-4}(0.4)^2\right) = 0.0184\ \text{dB}/\text{kyd}$$

Part 6

- What is the Ambient Noise Level (Sea State and Shipping)?

$$f_c = \sqrt{f_1 f_2} = \sqrt{(100\text{Hz})(500\text{Hz})} = 223.6\text{Hz}$$



$ISL_{ss} = 55\text{dB}$

$ISL_{ship} = 44\text{dB}$

Ambient Noise Band Levels

$$NL_{ss} = ISL_{ss} + 10\log(\Delta f) = 55\text{dB} + 10\log(400\text{Hz}) = 81\text{dB}$$

$$NL_{ship} = ISL_{ship} + 10\log(\Delta f) = 44\text{dB} + 10\log(400\text{Hz}) = 70\text{dB}$$

$$NL_{Amb} = NL_{ss} \oplus NL_{ship} = 81\text{dB} \oplus 70\text{dB} = 81\text{dB}$$

Part 7

- What is the Total Noise Level?

$$NL = NL_{Amb} \oplus NL_{Self}$$

$$NL = 81\text{db} \oplus 81\text{dB} = 84\text{dB}$$

Part 8

- If the Memphis were at 16,000 yds from the Seawolf, what would be the Transmission Loss (include attenuation)?

transition range = r_o = 14 Kyds

$$TL = 20\log r_0 + 10\log\frac{r}{r_0} + \alpha\left(r \times 10^{-3}\right)$$

$$TL = 20\log(14000) + 10\log\frac{16000}{14000} + \alpha\left(16000 \times 10^{-3}\right) = 83.7\text{dB}$$

Part 9

- If the Memphis were at 16,000 yds from the Seawolf, what would be the signal-to-noise level?

$$L_{S/N} = (SL - TL) - (NL - DI)$$

$$L_{S/N} = (180\text{dB} - 83.7\text{dB}) - (84\text{dB} - 12\text{dB})$$

$$L_{S/N} = 96.3\text{dB} - 72\text{dB} = 24.3\text{dB}$$

Part 10

- Is Memphis detectable? If so, what is the Signal Excess?

$$L_{S/N} = 24.3\text{dB} \quad > \quad DT = 1.8\text{dB}$$

$$SE = L_{S/N} - DT = 24.3\text{dB} - 1.8\text{dB} = 22.5\text{dB}$$

Part 11

- What is the max detection range of the Memphis by the Seawolf (this time you can ignore attenuation)?

$$L_{S/N} = DT = (SL - TL) - (NL - DI)$$

$$TL = (SL - DT) - (NL - DI) = (180\text{dB} - 1.8\text{dB}) - (72\text{dB}) = 106.2\text{dB}$$

$$TL = 20\log r_0 + 10\log\frac{r}{r_0} = 106.2\text{dB}$$

$$TL = 20\log 14000 + 10\log\frac{r}{14000} = 106.2\text{dB}$$

$$r = 2980\text{kyds}$$

NAME______________________

Passive Sonar Homework

Attacking Platform Data
A passive continuous line array sonar 40 m long receives signals in a one half octave bandwidth centered on a frequency of 400 Hz.
The Integration time of the sonar suite = 2.0 s.
Want P(D) = 50%, P(FA) = 0.1% assume ideal sonar processor
$ISL_{\text{self noise}}$ = 52 dB

Target Data
ISL_{target} = 120 dB

All spectrum levels are constant in the range of frequencies in the sonar's receiver bandwidth.

Environment
Wind Speed = 5 knots
Shipping = moderate to heavy (split the difference)
transition range = 5000 yds

1. What are the upper and lower frequencies in the half octave band and what is the band width?

2. What is the detection index required for detection of the target?

3. What would be the detection threshold for detecting the target using passive sonar?

4. What are the first three angles for the side lobes of the sonar's 40 m long linear array (only give from 0° to 90°)?

5. What is the Directivity Index for this 40 m long array at the center frequency?

6. What is the attenuation coefficient at the center frequency?

7. What is the Ambient Noise Level (Sea State and Shipping) in the half octave band?

8. What is the Self Noise Level in the half octave band?

9. What is the Total Noise Level in the half octave band?

10. What is the Source Level, SL, of the target in the half octave band?

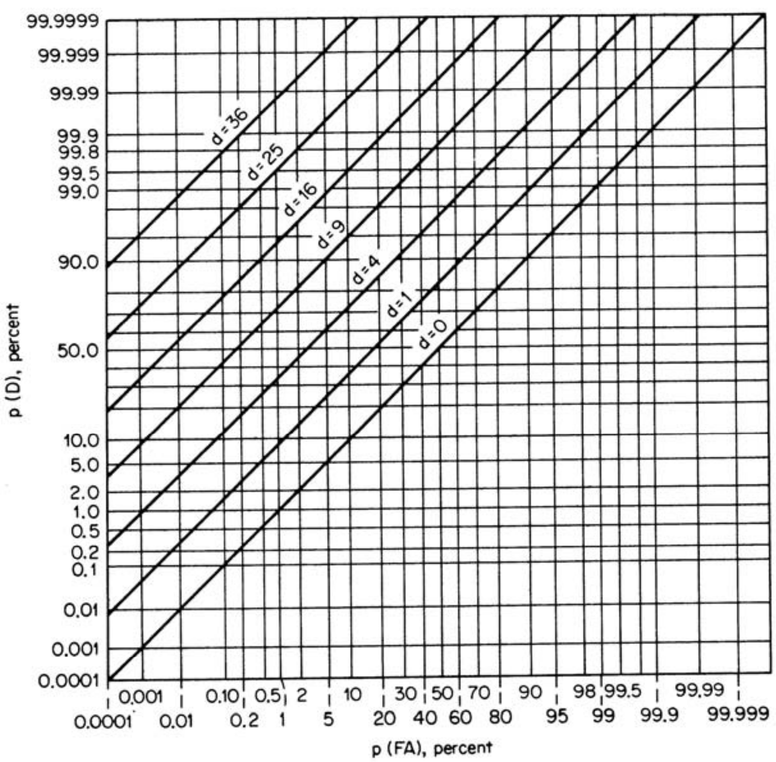
11. If the target is at 10000 yds, what would be the Transmission Loss (include attenuation)?

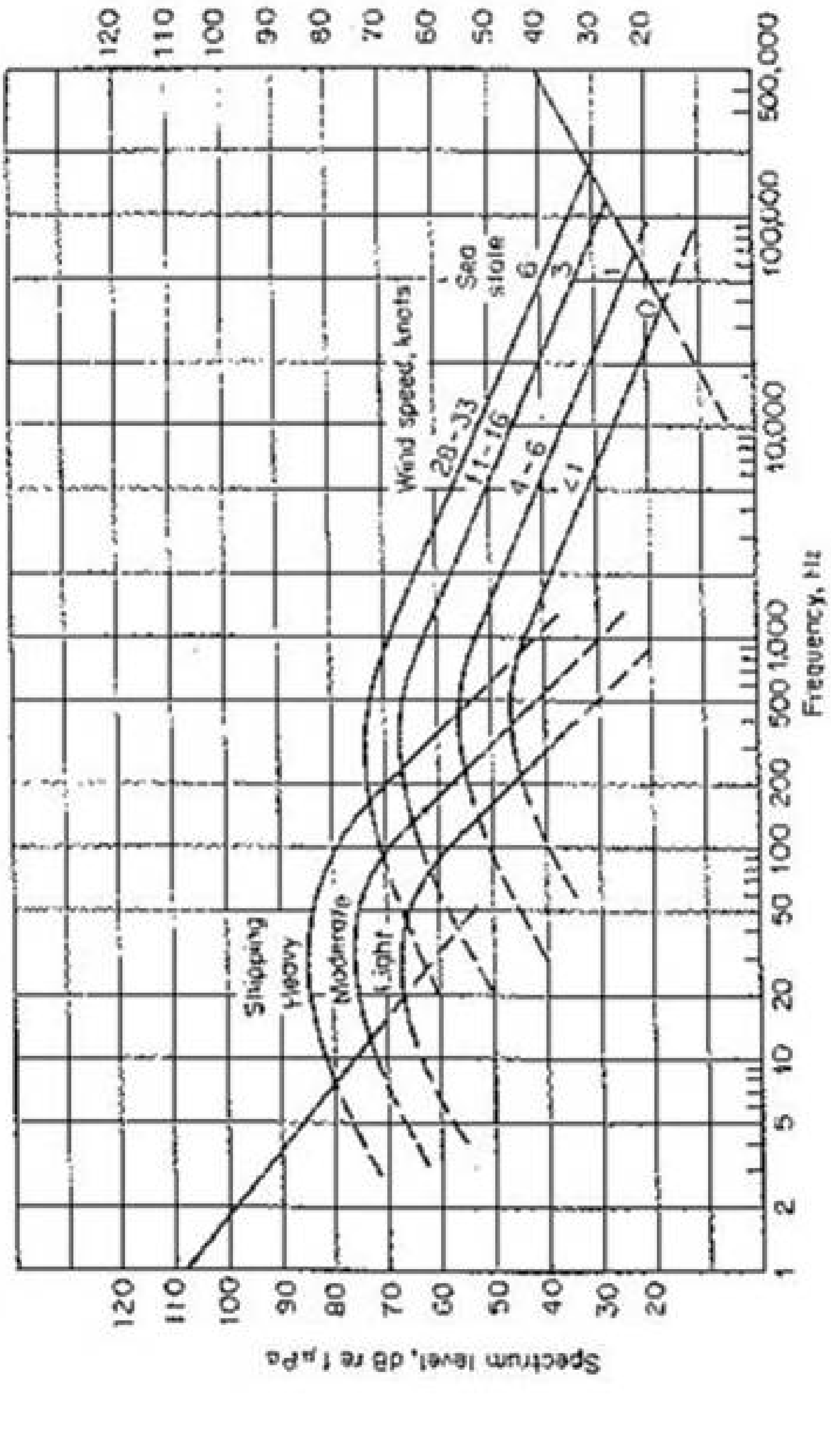
12. If the target is at 10,000 yds, what would be the signal-to-noise level?

13. Is the target detectable? If so, what is the Signal Excess?

14. What is the Figure of Merit (this time you can ignore attenuation)?

15. What is the max detection range of the target (this time you can ignore attenuation)?





Spectrum level, dB re 1 μPa
Frequency, Hz
Shipping
Heavy
Moderate
Light
Wind speed, knots
28-33
11-16
4-6
<1
Sea state
6
3
1
0

Active Sonar Equation and Projector Source Level

Active Sonar Equation

We are now going to shift from the case where a sonar system is designed to detect acoustic energy emitted from a target but masked by the background noise of the ocean to the case where the acoustic energy originates from our own sonar system, travels to the target and is reflected back to our system (or perhaps another system). Active sonar functions in a manner analogous to RADAR. The sonar system must act as both a transmitter and a receiver. Recall the passive sonar equation,

$$L_{S/N} = SL - TL - (NL - DI) > DT$$

The equation tells us if the signal received at our detector in the bandwidth of our detector divided by the noise received at our detector in the same bandwidth it greater than a threshold ratio, we should be able to detect the target with some established certainty and some acceptable probability of false alarms. The detection threshold is typically due to energy detection.

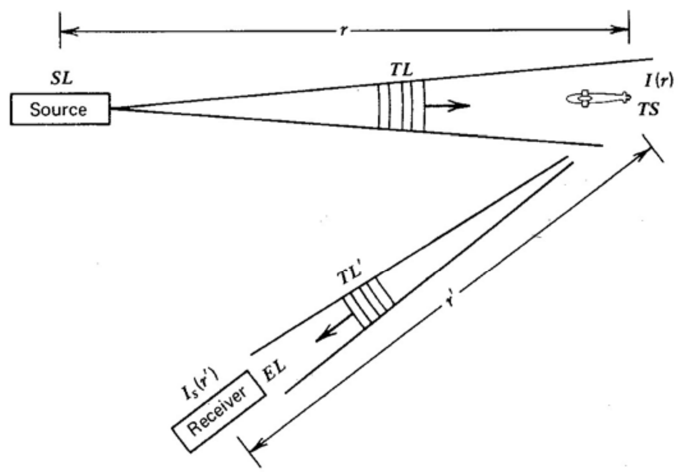


Fig. 15.10. Diagram used in deriving the expression for target strength.

For the case of active sonar, there must be a fundamental change to the signal terms. Specifically, the source level refers to the source level of our projector. The transmission loss is necessarily twice that of the passive case. Additionally only some fraction of the energy that reaches the target is actually reflected back to our system. The passive terms SL-TL are replaced by the terms SL-2TL+TS, where TS is called "Target Strength" and represents the ability of the target to reflect energy. These terms are collectively referred to as the "echo level," often abbreviated, EL. With these alterations, the active sonar equation becomes:

$$L_{S/N} = SL - 2TL + TS - (NL - DI) > DT$$

Here the detection threshold is due to correlation detection.

Active sonar is more complicated than the passive case because as an emitter of acoustic energy, our system adds to the background noise masking the reflected signal. This is particularly true if there are other non-target items that reflect sound back to our system at about the same time as the target reflection is detected. Possible sources of reflection are the surface and bottom, fish, other biologics, air bubbles, and dust or dirt.

$$L_{S/N} = SL - 2TL + TS - RL > DT$$

These reflections are in combination referred to as reverberation. The term that describes the ability of these unwanted reflections to mask the target signal is called "Reverberation Level."

The first active sonar equation is the case when the received noise level only limits the detectability of the return reflection from the target. The second is used when reverberation of the outgoing pulse, limits the detectability of the return reflection. We will discuss these equations further during the next few weeks. Unfortunately, much like income tax calculations, there is often no way to know which method to use until both calculations are done and we see which is more limiting.

Projector Source Level

Before we delve further into the active sonar equation though, let's start with a revisit and redefinition of the source level term, SL. In the active sonar equation, the source level is no longer the level of the contact or target, but rather the source level of the projector from the active sonar system. This source level is the level (in dB re 1μPa) of the projector, 1 yard from the projector.

To solve for the source level, we can start with the definition of passive source level:

$$SL = 10\log\frac{I_{1yd}}{I_{ref}}$$

Substituting in the equation for intensity:

$$I_{1\,yd} = \frac{p_{rms}^2}{\rho c} = \frac{Pwr}{Area_{at\ 1\ yd}}$$

$$\text{where } Area_{at\ 1\ yd} = 4\pi(1\ yd)^2$$

so the SL becomes:

$$SL = 10\log\frac{\left(Pwr \big/ 4\pi(1\ yd)^2\right)}{\left(p_{ref}^2 \big/ \rho c\right)} = 10\log\frac{Pwr \cdot \rho c}{4\pi(1\ yd)^2\, p_{ref}^2}$$

We can substitute in the nominal values for the density and speed of sound of seawater (ρ_{SW}=1000 kg/m^3 and c_{SW}=1500 m/s), knowing p_{ref}=1μPa and converting yards to meters we get:

$$SL = 10\log\left(Pwr\right) + 10\log\frac{\left(1000\ \text{kg/m}^3\right)\left(1500\ \text{m/s}\right)}{4\pi\left(1\ \text{yd}\right)^2\left(0.9144\ \text{m}/1\ \text{yd}\right)^2\left(1\text{x}10^{-6}\ \text{Pa}\right)^2}$$

$$SL = 10\log\left(Pwr\right) + 171.5\ \text{dB}$$

Within the sonar system, there is an efficiency at converting the electrical input power to the acoustical output power and this can further modify our results where:

$$Pwr_{acoustic} = P_E \cdot E$$

where E is the system efficiency thus:

$$SL = 171.5\ \text{dB} + 10\log\left(P_E \cdot E\right)$$

$$SL = 171.5\ \text{dB} + 10\log P_E + 10\log E$$

BUT, this is only for an omni-directional hydrophone. We must now account for the directionality of our transducer.

Directionality of Transducer

Our latest result assumes that the active source is omni directional (all power is transmitted equally in all directions.) An omni-directional transducer is nearly impossible to build though and may not be the best option. To account for the directionality of the transducer, we must add in a directionality term, DI_T, the directivity index for the active transducer. The directivity index is defined as it was for the passive sonar equation, the only difference is that the intensities would now be the intensities of the active transmission from the transducer.

$$DI_T = 10\log\frac{I_{non-directional}}{I_{directional}}$$

A well known theorem in acoustics called the Principle of Reciprocity states as one of its conclusions that under certain conditions the beam pattern $b(\theta,\phi)$ of a receiving array is the same as that for a transmitting array. This means that the receiving directional properties of n-element arrays, line arrays, and circular piston arrays will be useful in describing the directional properties of transmitting arrays.

We can show that the source level of the sound within the main beams of the transducer becomes:

$$SL = 171.5\ \text{dB} + 10\log P_E + 10\log E + DI_T$$

Just as passive directivity index prevented us from listening to noise from unimportant directions and effectively reducing the noise, transmitting directivity index prevents us from directing sound into unwanted directions, effectively increasing the source level.

Transducer Sensitivity

We next define transducer sensitivity. This quantifies the quality of the electro-acoustic conversion. It expresses the relation between the input and output values of the transducer (acoustic pressure to electric voltage).

$$SV = 10\log\left(\frac{I_{1V}}{I_{ref}}\right) = 20\log\left(\frac{p_{1V}}{p_{ref}}\right)$$

Where p_{1v} is the acoustic pressure 1 m away from the transducer in a given direction for a voltage of 1 V.

For an input voltage of 1 V, recalling that electric power, $P_E = \frac{V^2}{R}$, we get:

$$SL = SV = 171.5 - 10\log R_P + 10\log E + DI$$

Where R_P is the real part of the input electrical impedance. Manufacturers typically use SV to allow consumers to compare systems with the same 1 V input. To convert SV to actual SL, simply add 20logV.

Acoustic Cavitation

The maximum transmission power is limited by two physical constraints:

1. If too large a voltage is applied to the transducer, it leads to a non-linear response of the materials, followed by degradation and failure.

2. Limits of the propagation medium – cavitation.

Cavitation occurs when the local low pressure caused by the acoustic pressure wave causes gas bubbles to form in front of the transducer, thereby limiting the electro-acoustic efficiency. The bubbles act as little shock absorbers damping effect of the motion of the transducer face on the surrounding water. This effect doesn't occur when the acoustic pressure on the projector wall is greater than or equal to p_{cav}.

$$p_{cav} = p_{atm} + 10^4 z \text{, where z is depth in meters}$$

In terms of power that causes cavitation:

$$P_{cav} = S\frac{|p_{cav}|^2}{2\rho c} \text{, where S is transmitting surface}$$

Therefore:

$$SL_{cav} = 186 + 10\log S + DI + 20\log(10 + z)$$

Problems

1. Given the peak electric power of an active sonar system as 850 W, system efficiency as 27%, I_{ND} = 1, and I_D = 18, determine:
   a) The Source Level***?***
   b) The acoustic power level?

2. A sound projector is a plane circular piston of diameter 50 cm and operates at a frequency of 15 kHz with a power output of 2500 W. The speed of sound is 1500 m/s.
   a) What is the source level of the projector on the beam axis
   b) What is the plane wave rms acoustic pressure at one yard from the acoustic center (i.e. on the beam axis)?

3. An acoustic homing torpedo transducer is a plane circular array of diameter 25 cm. It operates at 15 kHz in water where c = 1500 m/s. If the efficiency of converting electrical energy into acoustic energy is 60%, and a source level of 220 dB is required, what must be the electric power input?

Active Sonar Equation

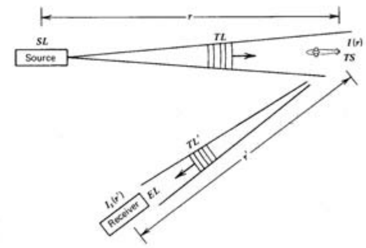


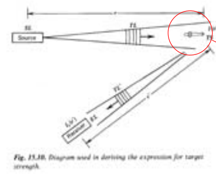
Fig. 15.10. Diagram used in deriving the expression for target strength.

Adapting Passive Ideas

Passive Case: $L_{S/N} = SL - TL - (NL - DI) > DT$

From our Sonar

$TL + TL' \approx 2TL$



Target Strength describes the fraction of energy reflecting back from the target

$$L_{S/N} = SL - 2TL + TS - (NL - DI) > DT$$

Reverberation Noise

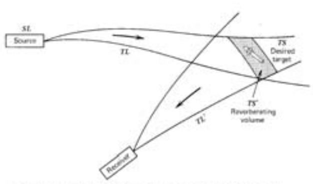


Fig. 15.11. Diagram used in deriving the reverberation level for volume scatterers.

Reflections from non-target objects is greater that noise.

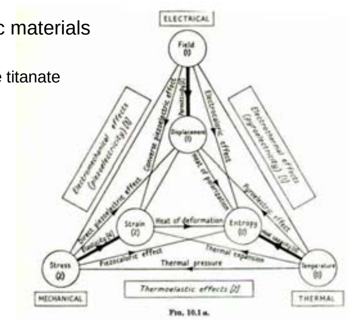
Reverberation limited

$$RL > NL - DI$$

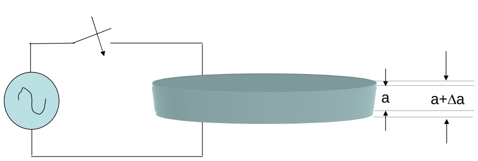
$$L_{S/N} = SL - 2TL + TS - RL > DT$$

Active Sonar – Materials

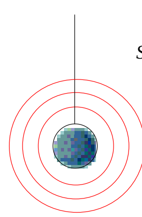
- Typical piezoelectric materials
  - Quartz
  - PZT -Lead zirconate titanate
  - Barium Titanate



Piezoelectricity



Source Level for an Omni-directional projector



$$SL = 10\log\frac{I_{1yd}}{I_{ref}}$$

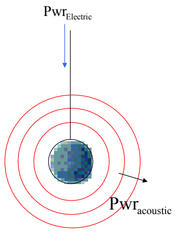
$$I_{1\,yd} = \frac{p_{rms}^2}{\rho c} = \frac{Pwr}{Area_{at\ 1\ yd}}$$

where $Area_{at\ 1\ yd} = 4\pi(1\ yd)^2$

$$SL = 10\log\frac{\left(\frac{Pwr}{4\pi(1\ yd)^2}\right)}{\left(\frac{p_{ref}^2}{\rho c}\right)} = 10\log\frac{Pwr\cdot\rho c}{4\pi(1\ yd)^2\,p_{ref}^2}$$

$$SL = 10\log(Pwr) + 10\log\frac{(1000\ kg/m^3)(1500\ m/s)}{4\pi(1\ yd)^2\left(0.9144\ \frac{m}{1\ yd}\right)^2(1x10^{-6}\ Pa)^2} = 10\log(Pwr) + 171.5\ dB$$

Electrical Efficiency



$$Pwr_{acoustic} = Pwr_{Electric} \cdot E$$

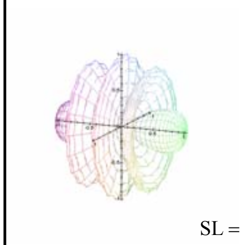
where E is the system efficiency thus:

$$SL = 171.5 \text{ dB} + 10\log\left(P_E \cdot E\right)$$

$$SL = 171.5 \text{ dB} + 10\log P_E + 10\log E$$

Efficiency may range from 20% to 70% for most sonar applications

Directional Arrays



$$DI_T = 10\log\frac{I_{non-directional}}{I_{directional}}$$

Principle of Reciprocity

$$b(\theta,\phi)_{receiving} = b(\theta,\phi)_{Transmitting}$$

$$SL = 171.5 \text{ dB} + 10\log P_E + 10\log E + DI_T$$

Transducer Sensitivity

How many dB for 1 volt input?

$$SV = 10\log\left(\frac{I_{1V}}{I_{ref}}\right) = 20\log\left(\frac{p_{1V}}{p_{ref}}\right)$$

$$P_E = \frac{V^2}{R}$$

$$SL(@1V) = SV = 171.5 - 10\log R_P + 10\log E + DI$$

Input impedance

Manufacturers typically advertise based on SV. To find SL, add 20logV.

Example

- Compute the source level for an circular piston projector of diameter = 1 meter radiating 10 kW acoustic power at a frequency of 15 kHz in water

$$SL = 171.5 \text{ dB} + 10\log P_E + 10\log E + DI_T$$

Piston array: $DI_T = 10\log\left(\frac{\pi D}{\lambda}\right)^2 = 10\log\left(\frac{\pi(1m)}{.1m}\right)^2 = 29.94\text{dB}$

$$SL = 171.5 \text{ dB} + 10\log 10^4 + 29.94 \text{ dB} = 241.5 \text{ dB}$$

Cavitation

Pressure Threshold $\quad p_{cav} = p_{atm} + 10^4 z \quad$ (z in meters)

Power Threshold $\quad P_{cav} = S\frac{|p_{cav}|^2}{2\rho c} \quad$ S = Tranducer Surface Area

$$SL_{cav} = 186 + 10\log S + DI + 20\log(10+z)$$

Target Strength

When an active sonar pulse is transmitted into the water, some of the sound reflects off of the target. The ratio of the intensity of the reflected wave at a distance of 1 yard to the incident sound wave (in decibels) is the target strength, TS.

$$TS = 10\log\left(\frac{I_r}{I_i}\right) = 10\log\left[\frac{\sigma}{4\pi}\right]$$

$I_r \equiv$ Intensity reflected from target

$I_i \equiv$ Intensity incident on target

$\sigma \equiv$ Backscattering cross-section

I_r depends on the physical characteristics of the target and characteristics of the signal (angle and frequency). The result in the square brackets comes from the fact that if all the energy reflects from the target, the Power striking the target and the power leaving the target must be equal.

$$I_i\sigma = 4\pi r^2 I_r$$

The ratio of reflected to incident intensity is simply

$$\frac{I_r}{I_i} = \frac{\sigma}{4\pi r^2}$$

where r is 1 yard. The backscatter cross section is a number that represents the degree to which sound is scattered off a target. It is related to the size, shape and reflectivity of a target.

Can the quantity, target strength be solved for analytically? Yes, but only for simple geometric shaped objects. We will present how this can be done for a convex object and a simple sphere. For more complicated geometric objects, I have included a table from Urick, **Principles of Underwater Sound**, which gives the formula to calculate the target strength for many other shaped objects. For any irregularly shaped object, we may be able to model them as a simple geometric object but for a precise value, we would have to use empirical data.

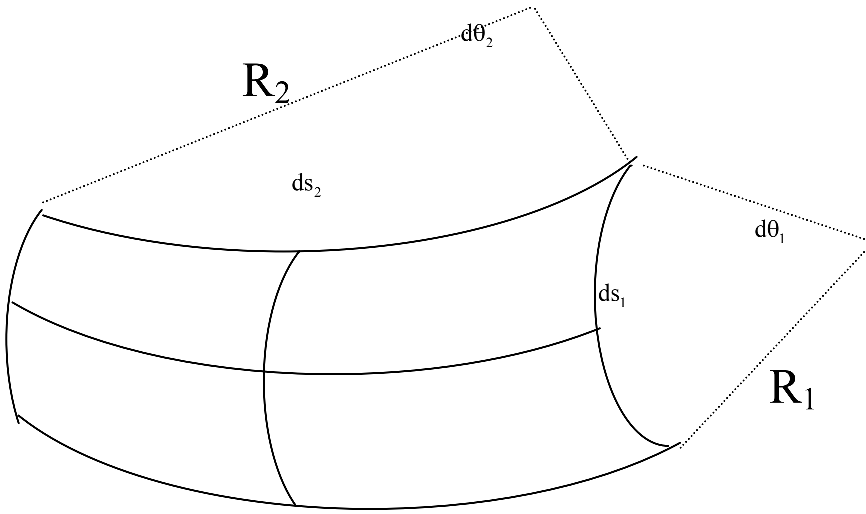
For analysis, assume that the incident wave is a plane wave (valid if source far from target) and that the scattered wave is spherical originating from the target. I_r is measured 1 yd (or 1 m) from the target.

Target Strength of an arbitrary convex object

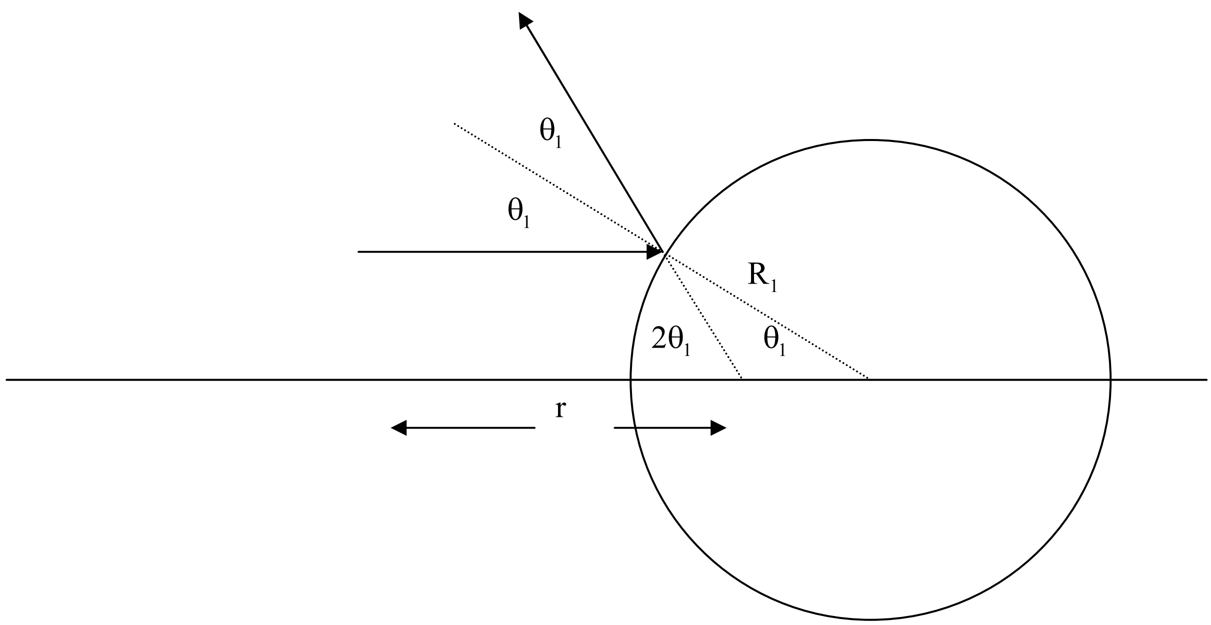
In the diagram below, let the surface area of the arbitrary convex surface be $dA=ds_1ds_2$. If the sound incident on the surface has an intensity, I_i, then the power striking the surface is

$$dP = I_i ds_1 ds_2 = I_i R_1 d\theta_1 R_2 d\theta_2$$

since $ds=Rd\theta$. The centers of curvature for the two sides of the surface are not in general the same point.



The pivotal question when examining the reflected intensity is what angles $d\theta_1$' and $d\theta_2$' does the sound energy bounce of the surface into. Examination of the ray diagram below shows that sound hitting the surface within an angle, θ_1, of the equator, bounce of the surface following the law of reflection. As such the ray departs the surface with an angle, $2\theta_1$, twice the incident angle. We notice that the exiting rays appear to emanate from a point half way between the center of curvature and the surface. In General Physics we called this a "focal point" and for a spherical mirror we recall that it was located at one half the radius of curvature.



With this in mind, we identify the surface the energy leaving the surface must pass through is

$$dA = ds_1' ds_2' = r2d\theta_1 r2d\theta_2 = 4r^2 d\theta_1 d\theta_2$$

The reflected intensity is then:

$$I_r = \frac{dP}{dA} = \frac{I_i R_1 d\theta_1 R_2 d\theta_2}{4r^2 d\theta_1 d\theta_2} = \frac{I_i R_1 R_2}{4r^2}$$

The resulting Target Strength follows from the definition:

$$TS = 10\log\left(\frac{I_r}{I_i}\right) = 10\log\left(\frac{\frac{I_i R_1 R_2}{4r^2}}{I_i}\right) = 10\log\left(\frac{R_1 R_2}{4r^2}\right)_{r=1yd} = 10\log\left(\frac{R_1 R_2}{4}\right)$$

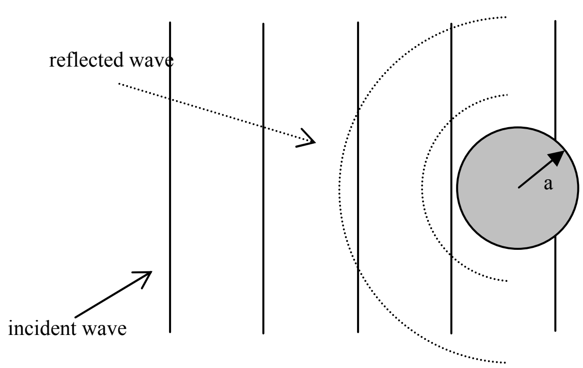
As a special case, let us look at a simple rigid sphere. In this case, $R_1=R_2=a$, the radius of the sphere. The Target Strength then becomes

$$TS = 10\log\left(\frac{a^2}{4}\right)$$

Let's see if this result makes physical sense.

Target Strength of Simple Rigid Sphere

Case I: (ka>>1 {or ka > 10} or in other words, when the radius of the sphere is much larger than the wavelength of the incident wave.)



If the rigid sphere is large compared with the wavelength of the incident sound wave and the sphere is an isotropic reflector (reflects sound equally in all directions), we can use the diagram at right:

The power of the incident wave that will be reflected is that power of the wave incident on a cross-section of the sphere where:

$$P_i = I_i \pi a^2$$
$$\text{where } \sigma = \pi a^2$$

Since the power of the incident wave is all reflected back, we find that:

$$
\begin{aligned}
&\text{power reflected} = \text{power incident} \\
&P_r = P_i \\
&I_T 4\pi r^2 = I_i \pi a^2 \\
&\frac{I_T}{I_i} = \frac{a^2}{4r^2}
\end{aligned}
$$

Then using the definition of target strength, we find:

$$TS = 10\log\frac{I_T}{I_i} = 10\log\left.\frac{a^2}{4r^2}\right|_{r=1\,yd}$$

$$TS = 10\log\frac{a^2}{4}$$

This is exactly the same result we obtained above as a special case of an arbitrary convex surface. Note that the above target strength result is independent of frequency (as long as ka>10). Target strength just depends on the radius, a. For a 1 cm radius rigid sphere, $\sigma_{bs} = 2.5\times10^{-5}\ \text{m}^2$ and TS=-46 dB. A 2 m radius sphere however would have a TS= 0 dB. This simple approximation is only meaningful for high frequencies where the wave effects can be averaged. For lower frequencies (longer wavelengths), the wave effects must be taken into account.

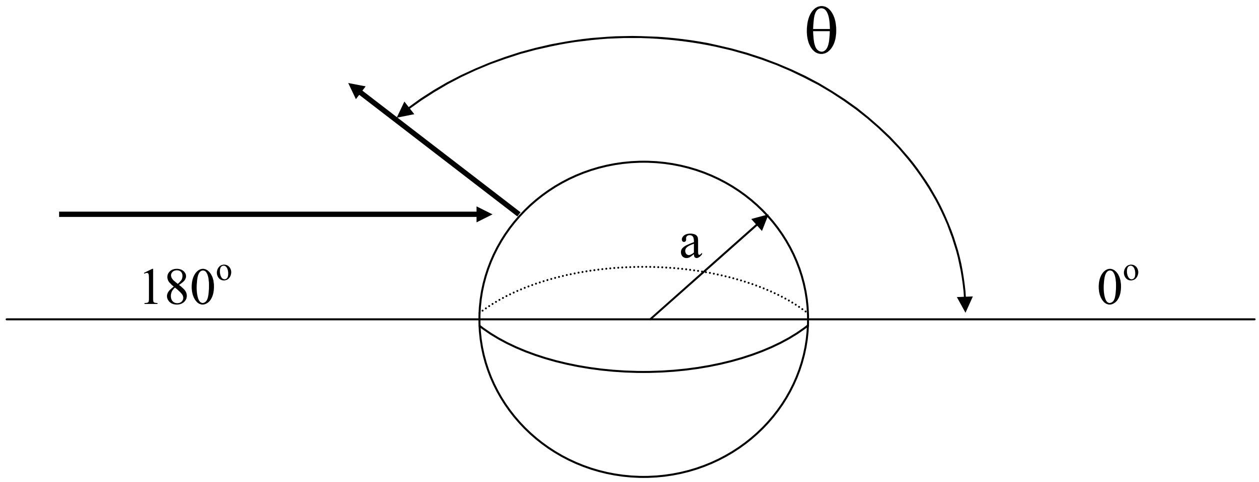
Case II: (ka<1)

When the wavelength of the incident wave is large compared to the size of the sphere, some of the wave will appear to continue past the ball as if it did not exist. There will actually be very little backscattering. This case, Lord Rayleigh showed that:

$$\frac{I_T}{I_i} = \frac{\pi^2 V^2}{\lambda^4 r^2}\left[\frac{3}{2}\cos\theta - 1\right]^2$$

where:

$V \equiv$ volume of the sphere



For the target strength, μ=-1 (cos 180° = -1, straight backscatter) and r = 1 yd. The above then becomes:

$$TS = 10\log\left\{(ka)^4\left(\frac{25}{36}\right)[a]^2\right\}$$

$$\Rightarrow \sigma_{bs} = 4\pi\frac{25}{36}k^4a^6$$

For Case I, one of the major assumptions was that the entire cross-sectional area (σ) contributed to the backscattering of the incident sound energy. For this case, the ratio of the effective backscattering cross-section to the geometric cross-section would be:

$$\frac{\sigma_{bs}}{\pi a^2} = 2.8(ka)^4$$

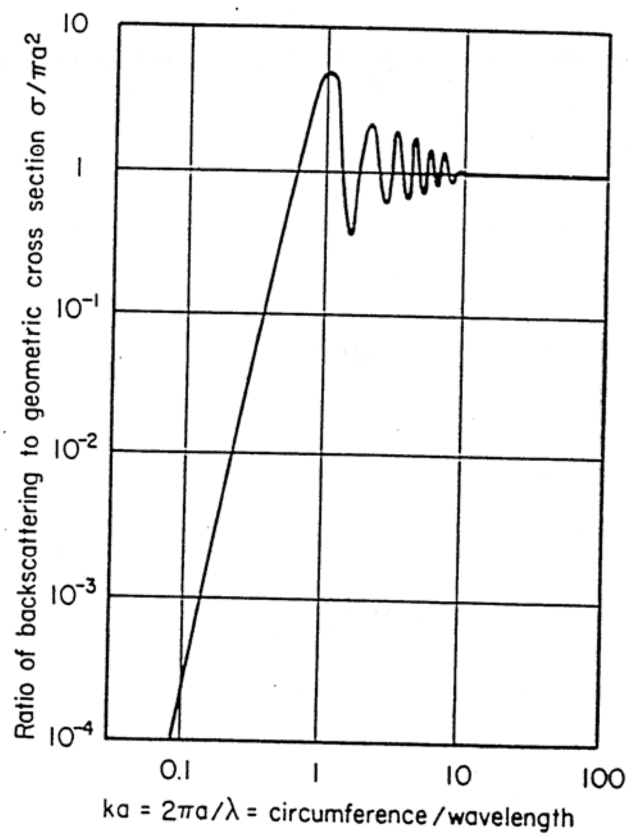
Notice that $\sigma/\pi a^2$ increases very rapidly with frequency $\left(\propto f^4\right)$, therefore target is barely detectable when size is much smaller than the wavelength. As frequency increases there is a limit to Rayleigh scattering:

$$ka = \frac{2\pi a}{\lambda} = 1$$

Occurs when $\lambda=2\pi a$

Case III: If 1<ka<10

For this exceptional case, we can use the plot given below which was taken from Urick, **Principles of Underwater Sound**, p. 299. This plot shows the ratio of the backscattering cross-section to the geometric cross-section as a function of ka, which can be used to calculate a value for the target strength. Target response in this range is dominated by interference between reflected wave and "creeping waves" refracted around the surface of the sphere.



Fluid Sphere

Spherical target is no longer ideally rigid, therefore in the Rayleigh regime:

$$\sigma_{bs} = k^4 a^6 \left[\frac{1}{3} - \frac{\rho_1 c_1^2}{3\rho_2 c_2^2} + \frac{\rho_2 - \rho_1}{2\rho_2 + \rho_1} \right]^2$$

$\rho_1, c_1 \equiv$ density and sound speed in water

$\rho_2, c_2 \equiv$ density and sound speed in target

When $\rho_2 > \rho_1$ and $c_2 > c_1$, therefore σ_{bs} approaches that of ideal rigid sphere. When $\rho_2 < \rho_1$ and $c_2 < c_1$, σ_{bs} is dominated by the compressibility of the sphere:

$$\sigma_{bs} = k^4 a^6 \left[\frac{\rho_1 c_1^2}{3\rho_2 c_2^2} \right]^2$$

σ_{bs} is much higher than for a rigid sphere of the identical radius. For example, the target strength of an air bubble is 75 dB higher than the target strength of rigid sphere with same radius.

Scattered Gas Bubbles

Backscatter of gas bubbles in sea water is widely studied because of the important acoustic implications. Air bubble clouds can create undesirable reverberation from the sea surface. Gas bubbles are also present in sediment and are an essential component of seafloor backscattering. Effects of random populations on the acoustic propagation and backscattering are difficult to predict accurately other than statistically. Gas bubble acoustic behavior is dominated by resonance. For frequencies near the resonance frequency (f_0 depends on bubble size), backscattering and absorption are enhanced;

$$\sigma_{bs} = \frac{a^3}{\left(\left(\frac{f_0}{f}\right)^2 - 1\right)^2 + \delta^2}$$

$$f_0 \equiv \text{ resonant frequency}$$

$$\delta \equiv \text{ damping term}$$

Resonant frequency can be approximated as:

$$f_0 = \frac{1}{2\pi a}\sqrt{\frac{3\gamma P_w}{\rho_w}} \approx \frac{3.25}{a}\sqrt{1+0.1z}$$

$$\rho_w = 1000 \text{ kg/m}^3$$

$$P_w \equiv \text{ hydrostatic pressure in Pa } \left(\approx 10^5\left(1+0.1z\right)\right)$$

$$z \equiv \text{ depth in meters}$$

$$\gamma \equiv \text{ adiabatic constant for air (} \approx 1.4\text{)}$$

Damping effect is due to the combined effects of radiation, shear viscosity and thermal conductivity. A good approximation is $\delta \approx 0.03 f_k^{0.3}$ for $1 \text{ kHz} < f_k < 100 \text{ kHz}$, where f_k is the frequency in kHz.

Fish Target Strength

Main contribution for fish target strength comes from the swim bladder. This gas-filled bladder shows a very strong impedance contrast with the water and fish tissues. It behaves either as a resonator (frequencies of 500 Hz-2 kHz depending on fish size and depth) or as a geometric reflector (> 2 kHz). This swim bladder behaves very similar to gas bubbles. The difference in target strength between fish with and without swim bladder can be 10-15 dB.

A semi-empirical model most often used is:

$$TS_{fish} = 19.1\log L + 0.9\log f_k - 24.9$$

Love (1978)

This formula is valid for dorsal echoes at wavelengths smaller than fish length L.

A more detailed model is:

$$TS_{fish} = 20\log L - TS_{spec}$$

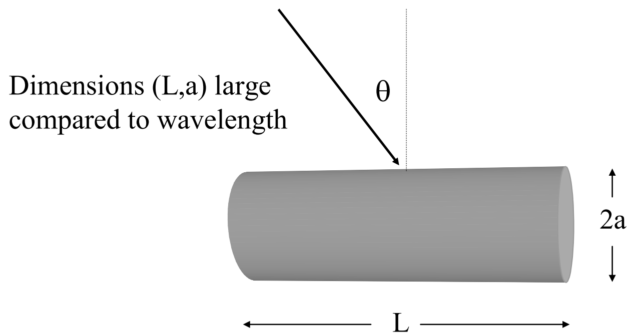
McLennan and Simmonds (1992)

TS_{spec} is given in Table 3.1 of Lurton, p.77. Note the lowest TS_{spec} is for mackerel which has no swim bladder. As frequencies approach the resonant frequency around 1 kHz, the target strength increases and can reach -25 to -20 dB.

For many other geometric shapes:

Use the tables given at the end of this lesson. Below are the equations and definition of terms for a cylinder.

Scattering from Cylinders



$$\mathrm{TS} = 10\log\left[\left(\frac{\mathrm{aL}^2}{2\lambda}\right)\left(\frac{\sin\alpha}{\alpha}\right)^2 \frac{\cos^2\theta}{(1\mathrm{yd})^2}\right]$$

$$\alpha = \frac{2\pi \mathrm{L}}{\lambda}\sin\theta$$

$$\theta = 0^{\circ} \Rightarrow \mathrm{TS} = 10\log\left[\left(\frac{\mathrm{aL}^2}{2\lambda}\right)\frac{1}{(1\mathrm{yd})^2}\right]$$

Conclusion

One of the main points of this section is that it is extremely difficult to get an accurate value for the target strength of a complex target but, if we can approximate the target as a simple geometric shape, we can calculate a value that sould be sufficient.

For the wavelengths that we typically use for active sonar systems though, a rough approximation that can often be used is that the target strength will be directly related to the cross-sectional area of the target.

Form	*t* *TS=10log(t)*	*Symbols*	*Direction of incidence*	*Conditions*
Any convex surface	$\frac{a_1 a_2}{4}$	$a_1 a_2$ = principal radii of curvature r = range k = 2π/wavelength	Normal to surface	ka_1, ka_2 >>1 r>a
Large Sphere	$\frac{a^2}{4}$	a = radius of sphere	Any	ka>>1 r>a
Small Sphere	$61.7\frac{V^2}{\lambda^4}$	V = vol. of sphere λ = wavelength	Any	ka<<1 kr>>1
Infinitely long thick cylinder	$\frac{ar}{2}$	a = radius of cylinder	Normal to axis of cylinder	ka>>1 r > a
Infinitely long thin cylinder	$\frac{9\pi^4 a^4}{\lambda^2} r$	a = radius of cylinder	Normal to axis of cylinder	ka<<1
Finite cylinder	$\frac{aL^2}{2\lambda}$	L = length of cylinder a = radius of cylinder	Normal to axis of cylinder	ka>>1 r > L^2/λ
	$\frac{aL^2\left(\sin\beta/\beta\right)^2\cos^2\theta}{2\lambda}$	a = radius of cylinder β = kLsinθ	At angle θ with normal	
Infinite Plane surface	$\frac{r^2}{4}$		Normal to plane	
Rectangular Plate	$\left(\frac{ab}{\lambda}\right)^2\left(\frac{\sin\beta}{\beta}\right)^2\cos^2\theta$	a,b = sides of rectangle β = ka sinθ	At angle θ to normal in plane containing side a	r > a^2/λ kb >> 1 a > b
Ellipsoid	$\left(\frac{bc}{2a}\right)^2$	a, b, c = semimajor axis of ellipsoid	parallel to axis of a	ka, kb, kc >>1 r >> a, b, c
Circular Plate	$\left(\frac{\pi a^2}{\lambda}\right)\left(\frac{2J_1(\beta)}{\beta}\right)^2\cos^2\theta$	a = radius of plate β = 2kasinθ	At angle θ to normal	r > a^2/λ ka>>1
Circular Plate	$\left(\frac{4}{3\pi}\right)^2 k^4 a^6$	a = radius k = $2\pi//\lambda$	Perpendicular to plate	ka<<1

Problems

1. Johns Hopkins Applied Physics Lab is researching active, mine mapping sonar. The sonar they are using uses a frequency of 40 kHz. The mines they are trying to detect are spherical balls that are 1.4 m in diameter.

Which of the following is true:

a) The TS of the mines can be approximated using the large sphere formula ($TS = 10\log\frac{a^2}{4}$) since ka>>1.
b) the TS of the mines can be approximated using the small sphere formula ($\mathrm{TS} = 10\log\left(61.7\frac{\mathrm{V}^2}{\lambda^4}\right)$) since ka<<1.
c) The target strength of the mines does not depend on the frequency of the sonar system.
d) Lower frequency sonar should be used to get better spatial resolution of the mines.

2. If the target strength of the mines in problem 1 is found to be –9.1 dB, what would be the intensity of a return wave if the incident wave had an intensity of 21 $\mathrm{W/m^2}$?

3. What would be the best approximation of the target strength of a submarine that is 300 meters long, and 30 meters in diameter? Assume the frequency of the active sonar is 40 kHz.

4. Given a sphere of radius 1.0 m in water (c = 1500 m/s) for what range of frequencies is the sphere considered to be
   a) A “large perfectly rigid” sphere (corresponding to specular or geometrical scattering).
   b) A “small fixed rigid” sphere (corresponding to Rayleigh scattering).

5. A modern torpedo is roughly 65 cm in diameter and 6 m long. An active sonar of frequency 20 kHz is used to measure the target strength when c = 1500 m/s. For each case take r = 1000 m.
   a) Why is range, r, given in this problem?
   b) If from the beam aspect, we consider the torpedo to be a cylinder, what target strength is expected.
   c) If from head-on we take the nose to be spherical, what target strength is to be expected?

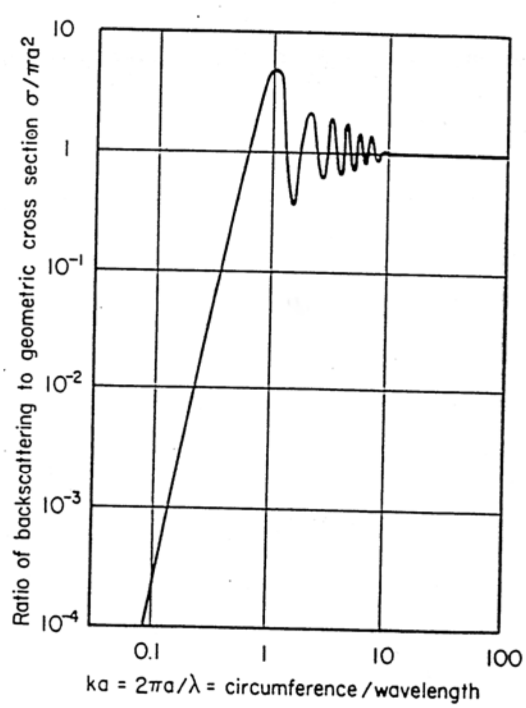
6. The first teardrop shaped submarine was USS Albacore, shown below at its museum site in Portsmouth NH.

Image courtesy of the Historic Naval Ships Association

Consider USS Albacore to be an ellipsoid of length 68 m and diameter 9.0 m at the midpoint. Calculate the target strength for active sonar at a beam aspect.

7. A sound beam of frequency 15 kHz is being used to search for a thick rectangular flat plate with dimensions 5.0 m x 3.0 m dropped from an oil rig at a depth of 100 m. Calculate the target strength of the plate:
   a) At normal incidence, and
   b) At an angle of 30^{o} from the normal in the plane of the longer axis of the plate.

8. Given a sphere of radius 0.20 m in seawater where c = 1500 m/s, use the below figure to determine:
   a) The ratio of backscattering to geometric cross section for 10 Hz, 100 Hz, 1000 Hz, 10 kHz.
   b) The target strength for frequencies of 10 Hz, 100 Hz, 1000 Hz, 10 kHz.



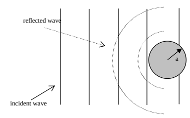
9. An acoustic pulse has an intensity of 10 W/m^2 incident 100 m from the center of an underwater target. The intensity of the 180^o reflected pulse has an average intensity of 3.16 $\mu W/m^2$ also measured 100 m from the target center. If spherical spreading is the only transmission loss, find the target strength of the object. Hint: EL = SL – 2 TS - +TS

Target Strength

$$TS = 10\log\left(\frac{I_r}{I_i}\right)$$

At r = 1 yd.

σ = scattering cross section



$$I_i\sigma = 4\pi r^2 I_r$$

$$TS = 10\log\left(\frac{\sigma}{4\pi r^2}\right) = 10\log\left(\frac{\sigma}{4\pi}\right)$$

Factors Determining Target Strength

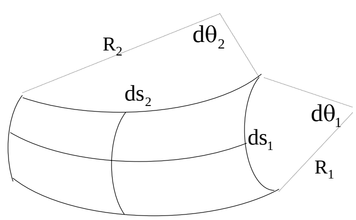
- the shape of the target
- the size of the target
- the construction of the walls of the target
- the wavelength of the incident sound
- the angle of incidence of the sound

Target Strength of a Convex Surface

Incident Power

$$dP = I_i ds_1 ds_2$$

$$dP = I_i R_1 d\theta_1 R_2 d\theta_2$$



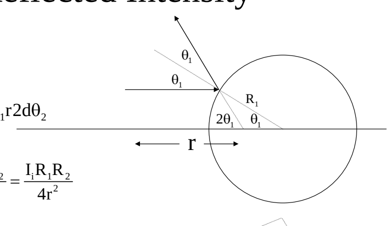
Large objects compared to the wavelength

Reflected Intensity

$$ds_1' = r2d\theta_1$$

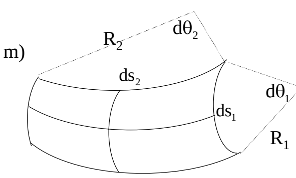
$$dA = ds_1' ds_2' = r2d\theta_1 r2d\theta_2$$

$$I_r = \frac{dP}{dA} = \frac{I_i R_1 d\theta_1 R_2 d\theta_2}{r2d\theta_1 r2d\theta_2} = \frac{I_i R_1 R_2}{4r^2}$$



$$TS = 10\log\left(\frac{I_r}{I_i}\right) \quad \text{(At r = 1 m)}$$

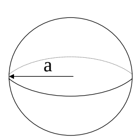
$$TS = 10\log\left(\frac{R_1 R_2}{4}\right)$$



Special Case – Large Sphere

$$R_1 = R_2 = a$$

$$TS = 10\log\left(\frac{a^2}{4}\right) = 20\log\left(\frac{a}{2}\right)$$



Note:

$$\frac{\sigma}{4\pi} = \frac{a^2}{4}$$

$$\sigma = \pi a^2$$

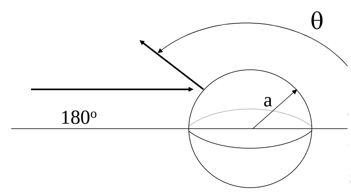
Large means circumference >> wavelength

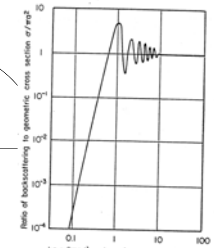
$$ka >> 1$$

TS positive only if a > 2 yds

Large Spheres (continued)

$$\frac{I_r}{I_i} = \frac{1}{4\pi r^2}\left(\pi a^2 + \pi a^2 \cot^2\left(\frac{\theta}{2}\right) J_1^2\left(ka\sin\theta\right)\right)$$





Example

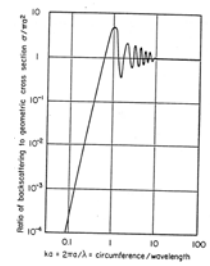
- An old Iraqi mine with a radius of 1.5 m is floating partially submerged in the Red Sea. Your minehunting sonar is a piston array and has a frequency of 15 kHz and a diameter of 5 m. 20 kW of electrical power are supplied to the transducer which has an efficiency of 40%. If the mine is 1000 yds in front of you, what is the signal level of the echo. Assume spherical spreading.

Scattering from Small Spheres (Rayleigh Scattering)

$$\frac{I_r}{I_i} = \frac{\pi^2 V^2}{\lambda^4 r^2}\left(\frac{3}{2}\cos\theta - 1\right)^2$$

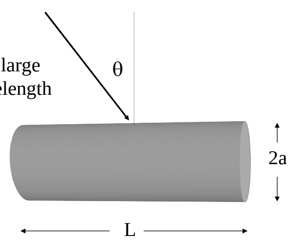
$$TS = 10\log\left[\frac{25}{36}(ka)^4 a^2\right]$$

$$ka < 1$$



Scattering from Cylinders

Dimensions (L,a) large compared to wavelength



$$TS = 10\log\left[\left(\frac{aL^2}{2\lambda}\right)\left(\frac{\sin\alpha}{\alpha}\right)^2 \frac{\cos^2\theta}{(1\text{yd})^2}\right] \quad \boxed{\theta = 0^o} \Rightarrow \quad TS = 10\log\left[\left(\frac{aL^2}{2\lambda}\right)\frac{1}{(1\text{yd})^2}\right]$$

$$\alpha = \frac{2\pi L}{\lambda}\sin\theta$$

Gas Bubbles

$$\sigma_{bs} = \frac{a^3}{\left(\left(\frac{f_0}{f}\right)^2 - 1\right)^2 + \delta^2}$$

$f_0 \equiv$ resonant frequency

$\delta \equiv$ damping term

$$f_0 = \frac{1}{2\pi a}\sqrt{\frac{3\gamma P_w}{\rho_w}} \approx \frac{3.25}{a}\sqrt{1+0.1z}$$

$\rho_w = 1000$ kg/m^3

$P_w \equiv$ hydrostatic pressure in Pa $\left(\approx 10^5(1+0.1z)\right)$

$z \equiv$ depth in meters

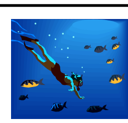
$\gamma \equiv$ adiabatic constant for air (≈ 1.4)

- Damping effect is due to the combined effects of radiation, shear viscosity and thermal conductivity. A good approximation is

$$\delta \approx 0.03 f_k^{0.3} \text{ for } 1\text{ kHz} < f_k < 100\text{ kHz}$$

- where f_k is the frequency in kHz.

Fish



- Main contribution for fish target strength comes from the swim bladder.
- This gas-filled bladder shows a very strong impedance contrast with the water and fish tissues. It behaves either as a resonator (frequencies of 500 Hz-2 kHz depending on fish size and depth) or as a geometric reflector (> 2 kHz). This swim bladder behaves very similar to gas bubbles. The difference in target strength between fish with and without swim bladder can be 10-15 dB.
- A semi-empirical model most often used is:

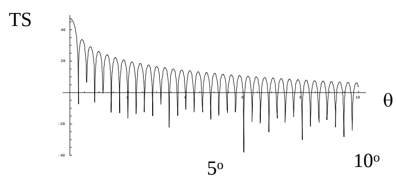
$$TS_{fish} = 19.1\log L + 0.9\log f_k - 24.9$$

- Love (1978)
- This formula is valid for dorsal echoes at wavelengths smaller than fish length L.

Form	***t*** ***TS=10log(t)***	***Symbols***	***Direction of incidence***	***Conditions***
Any convex surface	$\frac{a_1 a_2}{4}$	a_1a_2 = principal radii of curvature r = range k = 2π/wavelength	Normal to surface	ka_1, ka_2 >>1 r>a
Large Sphere	$\frac{a^2}{4}$	a = radius of sphere	Any	ka>>1 r>a
Small Sphere	$61.7\frac{V^2}{\lambda^4}$	V = vol. of sphere λ = wavelength	Any	ka<<1 kr>>1
Infinitely long thick cylinder	$\frac{ar}{2}$	a = radius of cylinder	Normal to axis of cylinder	ka>>1 r > a
Infinitely long thin cylinder	$\frac{9\pi^4 a^4}{\lambda^2} r$	a = radius of cylinder	Normal to axis of cylinder	ka<<1
Finite cylinder	$\frac{aL^2}{2\lambda}$	L = length of cylinder a = radius of cylinder	Normal to axis of cylinder	ka>>1 r > L^2/λ
	$\frac{aL^2\left(\sin\beta/\beta\right)^2\cos^2\theta}{2\lambda}$	a = radius of cylinder β = kLsinθ	At angle θ with normal	
Infinite Plane surface	$\frac{r^2}{4}$		Normal to plane	
Rectangular Plate	$\left(\frac{ab}{\lambda}\right)^2\left(\frac{\sin\beta}{\beta}\right)^2\cos^2\theta$	a,b = sides of ractangle β = ka sinθ	At angle θ to normal in plane containing side a	r > a^2/λ kb >> 1 a > b
Ellipsoid	$\left(\frac{bc}{2a}\right)^2$	a, b, c = semimajor axis of ellipsoid	parallel to axis of a	ka, kb, kc >>1 r >> a, b, c
Circular Plate	$\left(\frac{\pi a^2}{\lambda}\right)\left(\frac{2J_1(\beta)}{\beta}\right)^2\cos^2\theta$	a = radius of plate β = 2kasinθ	At angle θ to normal	r > a^2/λ ka>>1

Example

- What is the target strength of a cylindrical submarine 10 m in diameter and 100 m in length when pinged on by a 1500 Hz sonar?



Example

- What is the target strength of a single fish 1m in length if the fish finder sonar has a frequency of 5000 Hz?

Scattering and Reverberation Level

When an active sonar pulse is transmitted into the water, some of the sound reflects off of the target. Additionally, there are many other sources where the sound energy may reflect off back towards own ship. This scattering is caused by the many sources of inhomogeneities in the ocean. These sources may include fish, other biologics, air bubbles, dust or dirt as well as the ocean bottom, and surface.

There are two types of reverberation. The first is volume reverberation. This is caused primarily from biologics spread throughout the ocean. The second, surface reverberation, occurs at the two surfaces within the ocean, the surface and the bottom.

Calculating reverberation is a very difficult process that depends on many assumptions and requires that many factors be known. The reverberation level, RL, is calculated by comparing the unwanted reflected intensity to the reference intensity:

$$RL = 10\log\frac{I_{reverb}}{I_{ref}}$$

What we will do is be given an equation for each type of reverberation that satisfies the above relationship and use that to calculate the reverberation level.

Volume Reverberation

As stated previously, volume reverberation is the scattering of the active pulse back to own ship from biologics spread throughout the ocean volume. The biologics are not spread evenly throughout the ocean depths. Since the biologics are sensitive to light, the depth that they are most prevalent at will vary with the time of day. Additionally, the amount of scattering that occurs due to the biologics will vary with frequency of the active pulse. Last section we showed a model for the target strength of a fish which depends on the frequency.

To calculate the volume reverberation level, we can use the following equation:

$$RL_V = SL - 2TL + TS_{\text{false targets}} = SL - 40\log r + S_V + 10\log V$$

Hopefully this equation reminds you of the equation for the echo level from an actual target presented in section 16. The source level and the two way transmission loss are the same as the echo level. The target strength of the false targets is made up of two terms:

$$TS_{\text{false targets}} = S_V + 10\log V$$

where:

$S_V = 10\log s_V \equiv$ Volume Scattering Strength

$V = \frac{c\tau}{2}\Psi r^2 \equiv$ reverberation volume

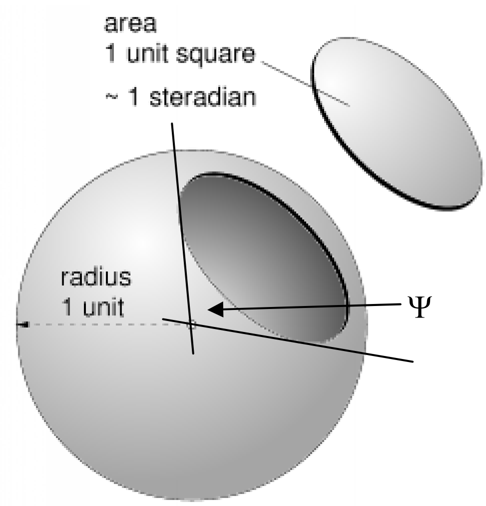
$\Psi = \int b(\theta,\phi)\,b'(\theta,\phi)\,d\Omega \equiv$ equivalent solid angle of arrays based on type of arrays

$r \equiv$ range to the target in yards

$\tau \equiv$ pulse length

Some explanation of the reverberation volume is in order. Physically, it is the volume of water around the actual target that contains false targets like bubbles and fish. Few active sonars transmit continuously, instead transmitting a pulse of known length. Even an explosive charge is of a finite duration. This pulse expands out from the source in an approximately conically shaped beam. The beam pattern function, $b(\theta,\phi)$, studied in section 13 gives the exact shape of the cone. Additionally, the sound must travel back to the array and we call the return beam pattern, $b'(\theta,\phi)$. When the outgoing and incoming beam patterns are integrated over all angles, the result is the solid angle, Ψ, of the cone. The units of solid angle are steradians and 4π steradians correspond to a solid angle covering all directions. The area intercepted by a solid angle is:

$$\text{Area} = \Psi r^2$$



As a check, note that for all directions, the area is $4\pi r^2$, the surface area of a sphere. To get a volume of water, multiply this area by the thickness required by the pulse length.

$$\text{thickness} = \frac{c\tau}{2}$$

An interesting artifact regarding volume reverberation results because the range, $r(t)=\frac{ct}{2}$. Because of this, volume reverberation decreases with time proportional to $20\log\left(\frac{t}{t_0}\right)$.

Most commonly, the volume scattering strength, S_V, will be determined from a graph based on the source depth and the time of day (night or day). An example from Principles of Underwater Sound, by Urick, p. 258 is shown at right.

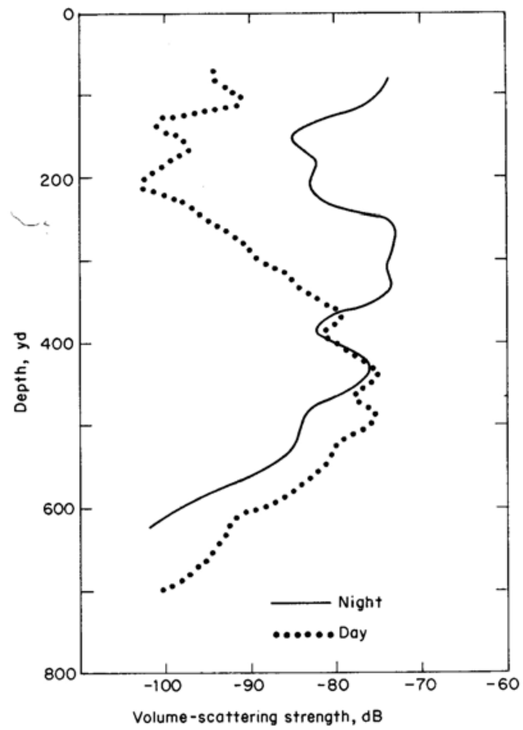


fig. 8.13 Profiles of S_v against depth showing vertical migration. Frequency 5 kHz. The column scattering strength was −54.5 dB during the day and −50.5 dB at night. (Ref. 26.)

We pointed out that reverberation volume is related to the volume of ocean that is ensonified by the active pulse and is based mostly on the beam pattern function of both the receiver and the transmitter (b and b′). The mathematics of calculating this solid angle are very challenging. As such, the value for the equivalent solid angle, Ψ, can be looked up in a table of values based on the type and dimensions of the array as well as the wavelength of the active pulse. An example table is given below for several type arrays.

ARRAY TYPE	Ψ(steradians)	Φ(radians)
Circular Plane of Diameter D	$0.60\left(\frac{\lambda}{D}\right)^2$	$1.56\left(\frac{\lambda}{D}\right)$
Horizontal Line of Length L	$1.32\left(\frac{\lambda}{L}\right)$	$1.32\left(\frac{\lambda}{L}\right)$
Non-directional Point Array	4π	2π

Caution: Remember to use a wavelength λ and the dimension D or L in the same units!

Notice also in the equation for the reverberation level that the level depends on the source level of the projector. The more sound energy that is put out in the water, the more sound energy will be reflected back to the receiver. The result is that increasing SL will increase both the echo level and the reverberation when reverberation limited. The transmission loss will be the same for both the target and the false, reverberating targets. When reverberation limited, the active sonar equation always results in a comparison between the target strength of the actual target and that of the false targets. If the difference exceeds the detection threshold, active sonar detection is possible.

$$L_{S/N} = SL - 2TL + TS - RL > DT$$

$$L_{S/N} = SL - 2TL + TS - \left(SL - 2TL + TS_{\text{false targets}}\right) > DT$$

$$TS - \left(S_V + 10\log V\right) > DT$$

Surface Reverberation Level

Surface Reverberation is due to sound waves scattering back from the surface of the ocean as well as the bottom of the ocean. We will concentrate on calculating the effect due to reverberation from the surface but the student must understand that in very shallow water, severe reverberation levels may exist due to the presence of so many surfaces for the sound to reflect off. That is why the effectiveness of active sonar is severely restricted in shallow waters.

The equation for surface reverberation level is very similar in for to that for volume reverberation:

$$RL_S = SL - 40\log r + S_s + 10\log A$$
$$S_S = 10\log s_S \equiv \text{Surface Scattering Strength}$$
$$A = \frac{c\tau}{2}\Phi r \equiv \text{reverberation area}$$
$$\Phi = \int b\left(0,\phi\right)b'\left(0,\phi\right)d\phi \equiv \text{equivalent beamwidth of arrays}$$

based on type of arrays

The equivalent beamwidth is similar to the equivalent solid angle only the integration is only over the horizontal angle, ϕ. Instead of performing this integral, the beamwidth can be looked up from the table above. Multiplying the beamwidth by a range to the target gives an arclength near the target. Multipling by the factor $c\tau/2$ gives the appropriate area on the surface around the target

The Surface Scattering Strength can be found from a graph such as the one below. Note that grazing angle is used in this chart and that 90^o is straight up and results in the highest surface scattering strength as expected. Also note that as the wind increases, the sea surface becomes rougher and allows for a more diffuse reflection back to the source and a higher surface scattering strength. A lower wind speed results in a calmer surface and more specular reflection away from the source. In this case the surface scattering strength is relatively low.

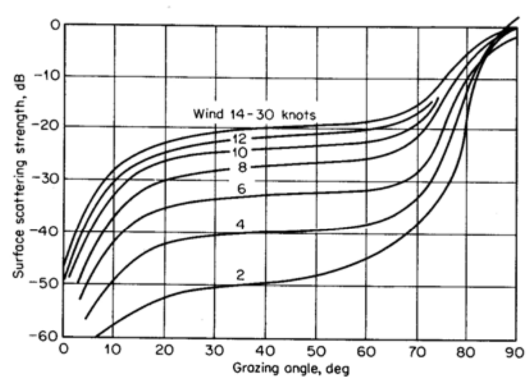


fig. 8.19 *Variation of sea-surface scattering strength at 60 kHz with angle at different wind speeds in Dabob Bay, Puget Sound, Washington. (Ref. 38.)*

Principles of Underwater Sound, Third Edition, Urick, 1983, p. 265

Surface Scattering Strength also depends on frequency, therefore you will have different graphs for different frequencies.

We can also have surface reverberation from the ocean floor. Bottom reverberation is very complex due to variations in composition, roughness and grazing angle. We can approximate the reverberation however as:

$$RL_B = SL - 40\log r + 10\log S(r) + BS_B$$

Where:

$$S(r) = \Phi \frac{c\tau}{2} \frac{r}{\sin\theta}$$

$\theta \equiv$ Bottom Grazing Angle

$BS_B \equiv$ Backscatter Strength for 1 m^2 of seafloor

Consequences of Reverberation

Reverberation acts like noise, but some differences from ambient noise:

- For a given transmission, the level of reverberation received decreases with time (although more slowly than the target echo).
- Spectral characteristics of reverberation and signal (target echo) are nearly identical except for the Doppler Effect (described in the next section). If the target is moving at sufficient speed the reverberation and the target will reflect different frequencies since bubbles and fish are moving at slow speeds.

Volume Backscattering Strength

Volume backscatter of a distribution of targets can be analyzed as the incoherent sum of contributions from each target in an average m^3 of water. For a given frequency, each contribution will depend on the size and shape of each member (shape is not important if the object is smaller than the signal wavelength) and of the composition of the material. If all scatterers are similar, therefore backscatter cross section is expressed for a certain frequency as a function of the dimension, *a*.

If we assume all targets are identical, therefore:

$$BS_V = TS + 10\log N_1$$

N_1 is the average number of targets per m^3.

Fish Schools

Fish schools are variable in shape and size (usually a couple of meters vertically and tens of meters horizontally). Fish schools are usually of one species, therefore similar size and shape (and hence target strength). If we assume fish density,

$$N_{1m^3} \approx \frac{1}{L^3} \qquad \text{L is fish length}$$

$$\Rightarrow TS_{FISH} \approx 20\log L - 25$$

$$\Rightarrow BS_{SCHOOL} = TS_{FISH} + 10\log N_{1m^3}$$

$$BS_{SCHOOL} = -25 + 20\log L - 10\log L^3$$

$$BS_{SCHOOL} = -25 - 10\log L$$

As we can see the BS decreases as fish size increases. These models and assumptions with the proportionality of backscatter strength and target strength form the basis of the *echo integration* method to assess fish school population. The number of fish present in a given area is estimated by the total energy backscattered to the source corrected for average target strength.

Deep Scattering Layer (DSL)

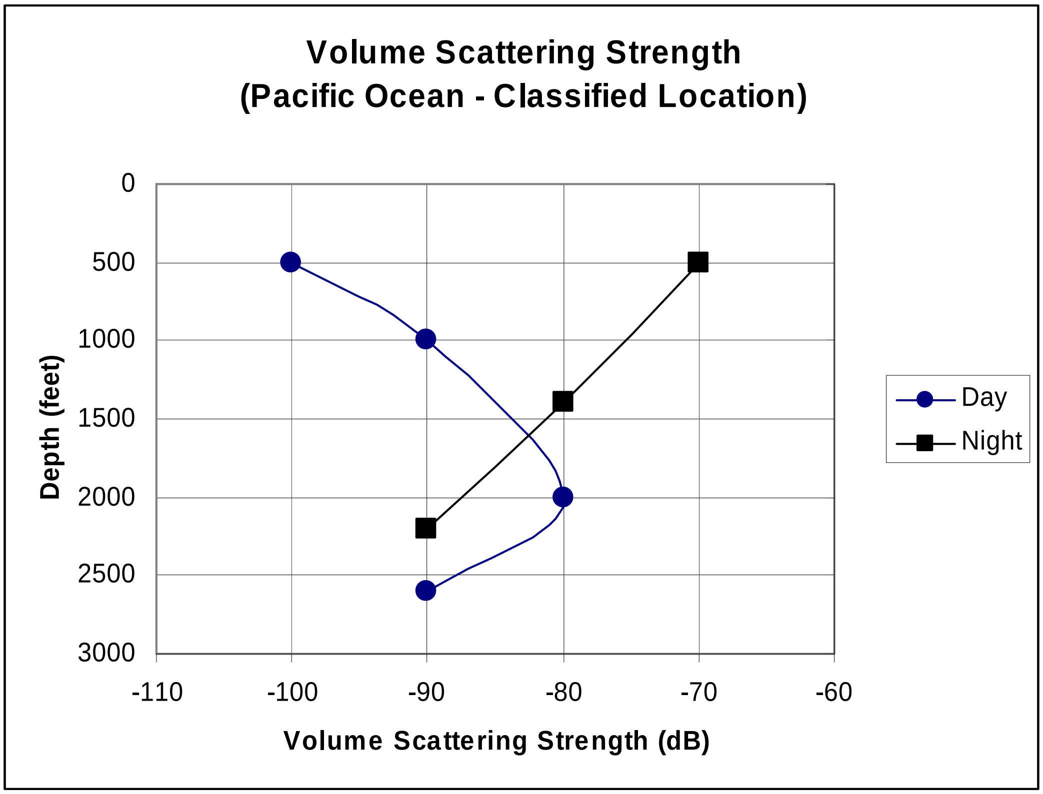
DSL is a thin layer of ocean (tens to thousands of meters) of ocean populated with plankton and other small biologics. The DSL can be found in all oceans and its depth changes with time of day. In the daytime, the layer stays at 200-600 m and at night this layer migrates to approximately 100 m. The main acoustic effect is caused by resonance of fish swim bladders (for frequencies in 1-20 kHz range). This frequency dependence changes with depth migration due to pressure effects. At higher frequencies, >20 kHz, the dominant acoustic effect comes from the scattering of plankton with an average $BS \approx -70$ dB/m^3.

Problems

1. A non-directional transducer has a source level of 200 dB and radiates 10 kHz pulses of 100 msec duration. The transducer is operating in a deep scattering layer at a depth of 800 ft at a classified location somewhere between Hawaii and California. See the figure below for the volume scattering strength. If the sound speed is 1528 m/s

a) Find the volume reverberation level for a target range of 5000 m and a day time operation.

b) Repeat for a night time operation



2. A Transducer consisting of a horizontal line array of length 3.0 m radiates 1000 Watts of acoustic power in the form of pulses. The carrier frequency is 50 kHz, the pulse duration is 20 msec and the sound speed in the ocean is 1528 m/s. It is known that the wind speed at the surface is 8 knots. The radiated sound beam impinges on the surface with an average grazing angle of 10^o.

a) Use the Figure in your course equation sheet and find the surface scattering strength S_s in dB.

b) Calculate the surface reverberation level at a distance of 500 m.

3. Given the following data for an active sonar search under conditions known to be volume reverberation limited. The sonar search is performed at night.

Plane Circular Transducer Array, Diameter = 1.2 m
Search frequency = 5 kHz
Pulse duration = 15 ms
Sound speed = 1528 m/s
Spherical spreading only, sound absorption can be neglected.
Target strength = 18 dB
Target is in a deep scattering layer at a depth of 200 yards
(use chart for S_v in your course equation sheet)
Detection Threshold is 5 dB

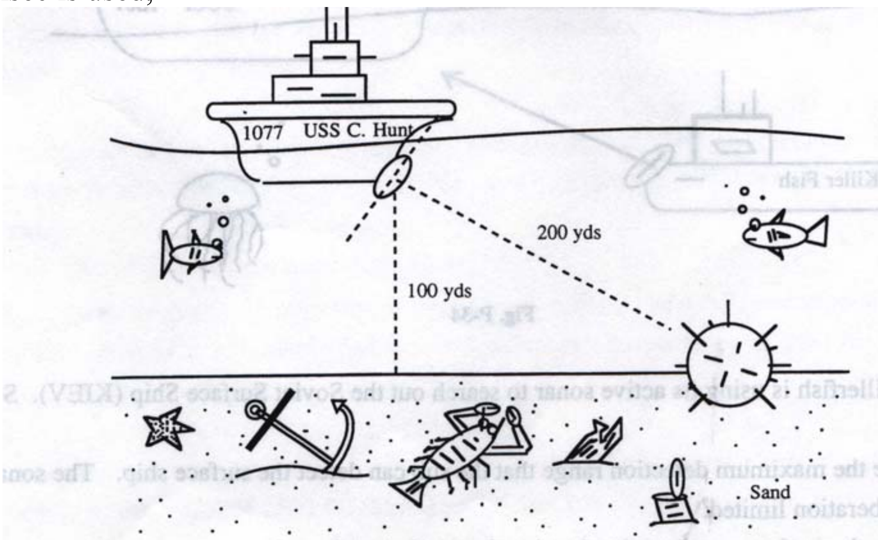
a) Find the predicted maximum detection range.
b) Compute the volume reverberation level if the Source Level is 210 dB.

4. The SSN, USS Killerfish is using its active sonar to search for a target aircraft carrier. The following information pertains to the tactical situation:

Wind speed = 10 knots
Sound speed = 1528 m/s
Transmission loss is only due to spherical spreading (TL = 20 log r)
Sonar pulse length = 20 ms
Carrier frequency = 60 kHz
Effective two way beam width is calculated for a circular plane array
    Diameter = 1.0 m
    Acoustic Power = Electric power x efficiency = 1.0 kW
The sonar's receiver is a cross correlation type
    p(D) = 50%
    p(FA) = 0.02%
Sonar receiver bandwidth = 100 Hz
Sonar beam grazing angle is 20^o with respect to the horizontal
Aircraft carrier target strength = 15 dB

a) Compute the maximum detection range that the sub can detect the surface ship. Assume the sonar is surface reverberation limited.
b) Compute the surface reverberation level at the maximum detection range.

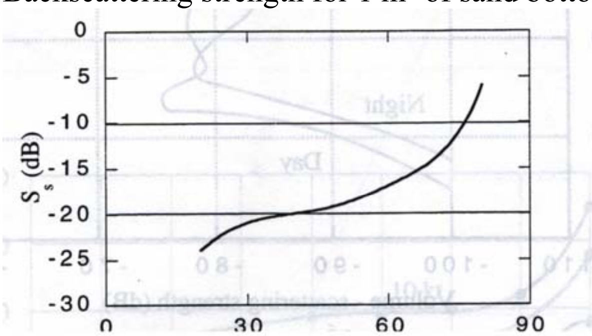
5. A mine of average aspect lies on a sand bottom. It is desired to detect the mine at a slant distance of 200 m by means of an active circular array located 100 m from the bottom. If a pulse length of 10 msec is used,



a) What horizontal effective beam width (ϕ, given in radians) will be required if the sonar is bottom reverberation limited and the detection threshold is +2 dB?
b) What is the minimum diameter of the circular array needed to detect this mine?

Useful information:

Backscattering strength for 1 m^2 of sand bottom



Sound speed = 1528 m/s
TL = 20 log r
Sonar frequency = 20 kHz
The mine is a sphere with a radius = 0.60 m
Bottom backscattering strength for a sand bottom is given in the below figure.

6. Given the following data:

Environment

c=1500 m/s
wind speed = 10 kts

Sonar

Circular plane array D = 1/3 m, f = 10 kHz, τ=5 msec
SL = 217.5 dB
DT = 5.0 dB
Beam Axis is steered to 30° above the horizon
Operating in a region where the sonar suite is surface reverberation limited

Target

TS = 15 dB

Use all data tables and graphs in this handout.

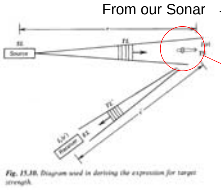
Compute the maximum range that the submarine could detect a surface ship.

Adapting Passive Ideas

Passive Case: $L_{S/N} = SL - TL - (NL - DI) > DT$

From our Sonar

$TL + TL' \approx 2TL$



Target Strength describes the fraction of energy reflecting back from the target

$$L_{S/N} = SL - 2TL + TS - (NL - DI) > DT$$

Reverberation Noise

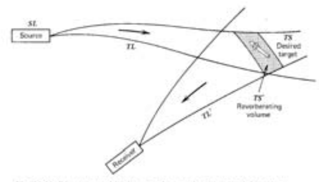


Fig. 15.11. Diagram used in deriving the reverberation level for volume scatterers.

Reflections from non-target objects is greater that noise.

Reverberation limited

$$RL > NL - DI$$

$$L_{S/N} = SL - 2TL + TS - RL > DT$$

Reverberation

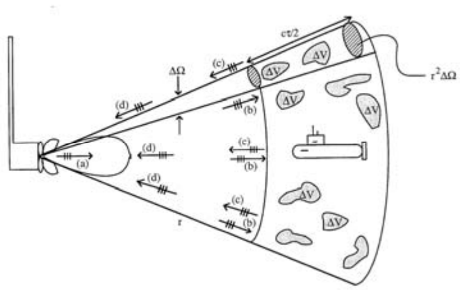


Fig 13-1 Schematic diagram of volume reverberation competing with the echoes scattered off the target sub.

Sources of Reverberation Backscatter

- Fish and smaller biological marine organisms
- Entrapped air bubbles
- Microthermal structure
- Velocity microstructure
- Variations in the characteristic impedance of the medium (e.g. dust and dirt)

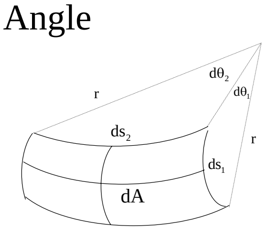
} Volume Reverberation

- Surface – air-water interface
- Ocean bottom

} Surface Reverberation

Solid Angle

$$dA = ds_1 ds_2 = r d\theta_1 r d\theta_2$$



Since $ds = r d\theta$

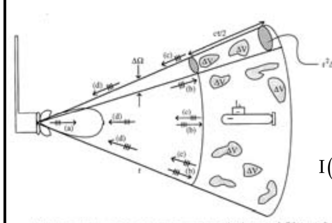
Similarly $dA = r^2 d\Omega$

Solid Angle

Units are Steradians

There are 4π steradians in all directions

Developing a Volume Reverberation Equation – Beam forming and Traveling to the Target



$$I_{source} = I_s b(\theta,\phi)$$

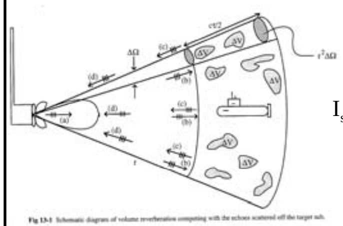
$$I_{source} r_o^2 d\Omega = I(r) r^2 d\Omega$$

$$r_o = 1yd$$

$$I(r) = \frac{I_{source} r_o^2}{r^2} = \frac{I_s b(\theta,\phi) r_o^2}{r^2}$$

Spherical Spreading

Developing a Volume Reverberation Equation – Scattering Strength



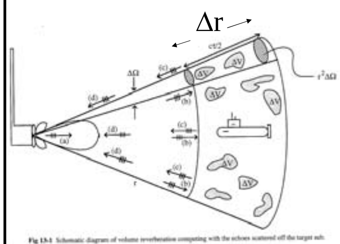
$$I_{scattered} = I(r)s_v\Delta V$$

$$I_{scattered} = \frac{I_s b(\theta,\phi) r_o^2}{r^2} s_v \Delta V$$

$$\Delta V = r^2 \Delta r d\Omega$$

Scattering Strength – fraction of energy backscattered

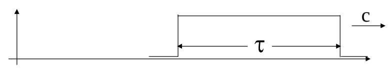
How thick is the Volume?



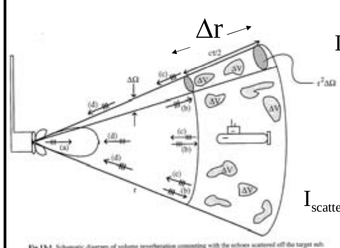
$$\Delta r = \frac{c\tau}{2}$$

$$\Delta V = r^2 \Delta r \Delta\Omega$$

$$\Delta V = r^2 \frac{c\tau}{2} \Delta\Omega$$



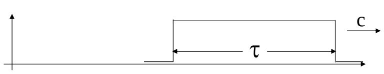
Developing a Volume Reverberation Equation – Scattering Strength



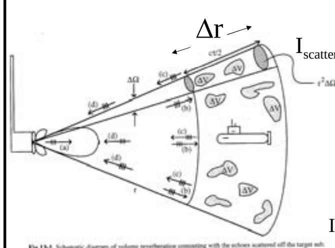
$$I_{scattered} = \frac{I_s b(\theta,\phi) r_o^2}{r^2} s_v \Delta V$$

$$\Delta V = r^2 \frac{c\tau}{2} \Delta\Omega$$

$$I_{scattered} = \frac{I_s b(\theta,\phi) r_o^2}{r^2} s_v r^2 \frac{c\tau}{2} \Delta\Omega$$



Developing a Volume Reverberation Equation – Traveling Back to Receiver

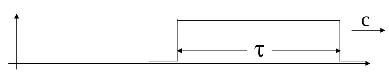


$$I_{scattered} = \frac{I_s b(\theta,\phi) r_o^2}{r^2} s_v r^2 \frac{c\tau}{2} \Delta\Omega$$

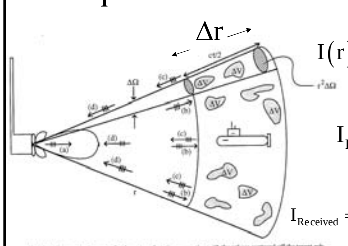
$$I_{scatter} r_o^2 d\Omega = I(r) r^2 d\Omega$$

$$I(r) = \frac{I_{scatter} r_o^2}{r^2}$$

$$I(r) = \frac{I_s b(\theta,\phi) r_o^4}{r^4} s_v r^2 \frac{c\tau}{2} \Delta\Omega$$



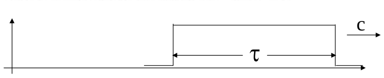
Developing a Volume Reverberation Equation – Receiver Beam Pattern



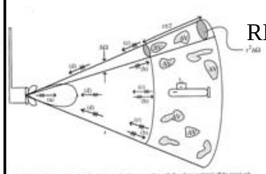
$$I(r) = \frac{I_s b(\theta,\phi) r_o^4}{r^4} s_v r^2 \frac{c\tau}{2} \Delta\Omega$$

$$I_{Received} = b'(\theta,\phi) I_{scattered}$$

$$I_{Received} = b'(\theta,\phi) \frac{I_s b(\theta,\phi) r_o^4}{r^4} s_v r^2 \frac{c\tau}{2} \Delta\Omega$$



Reverberation Level - Volume



$$RL_V = 10\log\frac{I_{reverb}}{I_{ref}} = 10\log\frac{I_{Tot\ received}}{I_{ref}}$$

$$\frac{I_{Received}}{I_{ref}} = \frac{I_s}{I_{ref}} \frac{r_o^4}{r^4} s_v \left[r^2 \frac{c\tau}{2} \iint_{Vol} b(\theta,\phi) b'(\theta,\phi) d\Omega \right]$$

$$10\log\left(\frac{I_{Received}}{I_{ref}}\right) = 10\log\left(\frac{I_s}{I_{ref}}\right) + 10\log\left(\frac{r_o^4}{r^4}\right) + 10\log(s_v) + 10\log\left[r^2 \frac{c\tau}{2} \iint_{Vol} b(\theta,\phi) b'(\theta,\phi) d\Omega \right]$$

$$RL_V = SL + 40\log\left(\frac{1}{r}\right) + 10\log(s_v) + 10\log[V] = \boxed{SL - 2TL + 10\log(s_v) + 10\log[V]}$$

Take a deep breath

$$RL_V = SL - 2TL + 10\log(s_v) + 10\log[V]$$

$S_v = 10\log(s_v)$ Volume Scattering Strength (dB)

$$V = r^2 \frac{c\tau}{2} \iint_{Vol} b(\theta,\phi) b'(\theta,\phi) d\Omega = \psi r^2 \frac{c\tau}{2}$$

$\psi = \iint_{Vol} b(\theta,\phi) b'(\theta,\phi) d\Omega$ Equivalent two way beam width – Equivalent solid angle of the sending and receiving array

Volume Scattering Strength

$S_v = 10\log(s_v)$

Diurnal Migration

Shrimp-like euphausids, squid and copepods
Fish (gas filled swim bladder) – freq differences

Higher frequencies – zoo plankton, phytoplankton fed on by small pelagic fish
(siphonphones and cephlopods)

Collectively called the DSL
(Deep scattering layer)

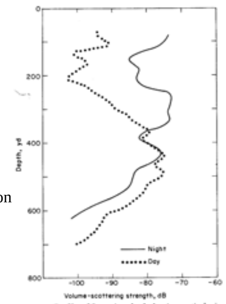


Fig. 8.13 Profiles of S_v against depth showing vertical migration. Frequency 5 kHz. The column scattering strength was −54.5 dB during the day and −50.5 dB at night. (Ref. 26.)

Equivalent two way beam width

$$\psi = \iint_{Vol} b(\theta,\phi) b'(\theta,\phi) d\Omega$$

ARRAY TYPE	Ψ(steradians)	Φ(radians)
Circular Plane of Diameter D	$0.60\left(\frac{\lambda}{D}\right)^2$	$1.56\left(\frac{\lambda}{D}\right)$
Horizontal Line of Length L	$1.32\left(\frac{\lambda}{L}\right)$	$1.32\left(\frac{\lambda}{L}\right)$
Non-directional Point Array	4π	2π

Caution: Remember to use a wavelength λ and the dimension D or L in the same units!

Reverberation Noise

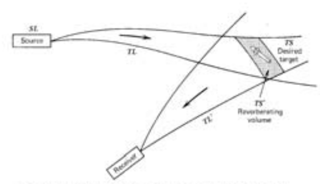


Fig. 15.11. Diagram used in deriving the reverberation level for volume scatterers.

Reflections from non-target objects is greater that noise.

Reverberation limited

$$RL > NL - DI$$

$$L_{S/N} = SL - 2TL + TS - RL > DT$$

Reverberation Level - Volume

$$\frac{I_{Received}}{I_{ref}} = \frac{I_s}{I_{ref}} \frac{r_o^4}{r^4} s_v \left[r^2 \frac{c\tau}{2} \iint_{Vol} b(\theta,\phi) b'(\theta,\phi) d\Omega \right]$$

$$RL_V = 10\log\frac{I_{received}}{I_{ref}} = SL - 2TL + 10\log(s_v) + 10\log[V]$$

$S_v = 10\log(s_v)$ Volume Scattering Strength (dB)

$$V = r^2 \frac{c\tau}{2} \iint_{Vol} b(\theta,\phi) b'(\theta,\phi) d\Omega = \psi r^2 \frac{c\tau}{2}$$

$\psi = \iint_{Vol} b(\theta,\phi) b'(\theta,\phi) d\Omega$ Equivalent two way beam width – Equivalent solid angle of the sending and receiving array

Volume Reverberation Case

$$L_{S/N} = SL - 2TL + TS - RL > DT$$

$$RL_V = SL - 2TL + 10\log(s_v) + 10\log[V]$$

$$L_{S/N} = \cancel{SL} - \cancel{2TL} + TS - (\cancel{SL} - \cancel{2TL} + 10\log(s_v) + 10\log[V]) > DT$$

$$L_{S/N} = TS - 10\log(s_v) - 10\log[V] > DT$$

Graph

$V = \psi r^2 \frac{c\tau}{2}$

Equation Sheet

Volume Reverberation Example

- Given the following data for an active sonar search under conditions known to be volume limited. The search is held at night.
  - Transducer array ifs a circular plane with diameter = 4 feet
  - Search Frequency is 5 kHz
  - Pulse Duration is 15 ms
  - Sound speed is 1670 m/s
  - Spherical spreading only
  - Target Strength = 18 dB
  - Depth = 100 fathoms
  - Detection Threshold = 5 dB
- Find the predicted maximum range
- Find the Reverberation Level if the Source Level is 210 dB

Volume Scattering Strength

$$S_v = 10\log(s_v)$$

Diurnal Migration

Shrimp-like euphausids, squid and copepods
Fish (gas filled swim bladder) – freq differences

Higher frequencies – zoo plankton, phytoplankton fed on by small pelagic fish (siphonphones and cephlopods)

Collectively called the DSL
(Deep scattering layer)

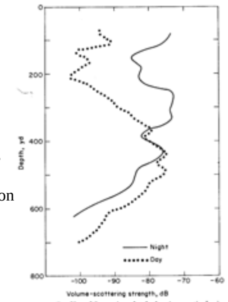


Fig. 8.13 *Profiles of S_v against depth showing vertical migration. Frequency 5 kHz. The column scattering strength was −54.5 dB during the day and −50.5 dB at night. (Ref. 26.)*

Equivalent two way beam width

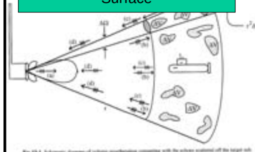
$$\psi = \iint_{Vol} b(\theta,\phi)b'(\theta,\phi)d\Omega$$

ARRAY TYPE	Ψ(steradians)	Φ(radians)
Circular Plane of Diameter D	$0.60\left(\frac{\lambda}{D}\right)^2$	$1.56\left(\frac{\lambda}{D}\right)$
Horizontal Line of Length L	$1.32\left(\frac{\lambda}{L}\right)$	$1.32\left(\frac{\lambda}{L}\right)$
Non-directional Point Array	4π	2π

Caution: Remember to use a wavelength λ and the dimension D or L in the same units!

Reverberation Level - Surface

Surface



$$RL_S = 10\log\frac{I_{reverb}}{I_{ref}} = 10\log\frac{I_{Tot\ received}}{I_{ref}}$$

$$\frac{I_{Received}}{I_{ref}} = \frac{I_s}{I_{ref}}\frac{r_o^4}{r^4}\left[s_s\, r\frac{c\tau}{2}\int_{Area} b(\theta,\phi)b'(\theta,\phi)d\phi\right]$$

$$RL_S = SL + 40\log\left(\frac{1}{r}\right) + 10\log(s_A) + 10\log[A] = SL - 2TL + \underbrace{10\log(s_s)}_{\text{Surface Scattering Strength}} + 10\log[A]$$

$$A = \Phi r\frac{c\tau}{2}$$

$$\Phi = \int_{Area} b(\theta,\phi)b'(\theta,\phi)d\phi$$

Equivalent beam width of the sending an receiving array in radians

Surface Scattering Strength

$$S_s = 10\log(s_s)$$

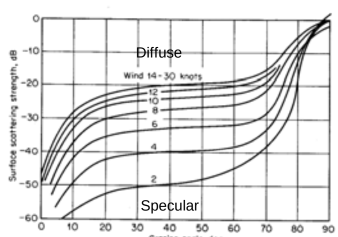
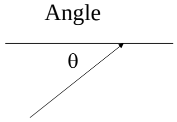


Fig. 8.19 *Variation of sea-surface scattering strength at 60 kHz with angle at different wind speeds in Dabob Bay, Puget Sound, Washington. (Ref. 38.)*

- Varies With:
  - Wind Speed (surface roughness)
  - Grazing Angle



Equivalent two way beam width

$$\psi = \iint_{Vol} b(\theta,\phi)b'(\theta,\phi)d\Omega$$

ARRAY TYPE	Ψ(steradians)	Φ(radians)
Circular Plane of Diameter D	$0.60\left(\frac{\lambda}{D}\right)^2$	$1.56\left(\frac{\lambda}{D}\right)$
Horizontal Line of Length L	$1.32\left(\frac{\lambda}{L}\right)$	$1.32\left(\frac{\lambda}{L}\right)$
Non-directional Point Array	4π	2π

Caution: Remember to use a wavelength λ and the dimension D or L in the same units!

Surface Reverberation Case

$$L_{S/N} = SL - 2TL + TS - RL > DT$$

$$RL_S = SL - 2TL + 10\log(s_S) + 10\log[A]$$

$$L_{S/N} = \cancel{SL} - 2\cancel{TL} + TS - \left(\cancel{SL} - 2\cancel{TL} + 10\log(s_A) + 10\log[A]\right) > DT$$

$$L_{S/N} = TS - 10\log(s_S) - 10\log[A] > DT$$

Graph

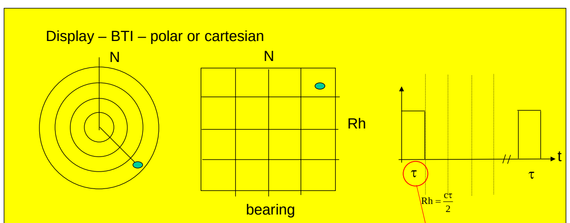
$$A = \Phi r \frac{c\tau}{2}$$

Equation Sheet

Surface Reverberation Example

- **Environment**
  - c=1500 m/s
  - wind speed = 10 kts
- **Sonar**
  - Circular plane array D = 1/3 m, f = 10 kHz, τ=5 msec
  - SL = 217.5 dB
  - DT = 5.0 dB
  - Beam Axis is steered to 30° above the horizon
  - Operating in a region where the sonar suite is surface reverberation limited
- **Target**
  - TS = 15 dB
- Use all data tables and graphs in previous slides
- Compute the maximum range that the submarine could detect a surface ship.

Misc. active sonar design ideas



ROC curves

Generally correlation detection

$$DT = 10\log\left(\frac{d}{2T\Delta f}\right)$$

Name:______________________

Active Sonar Wrap-up Exercise

(Everyone should attempt to do the following problems and
we will go over them in class.)

1. You are on a new Seawolf class submarine with the sonar system and the environment described below. Calculate the max range for detecting another submerged submarine given the following for both the noise-limited and reverberation-limited cases:

Environmental Data

c = 1500 m/s,
Wind Speed = *6 kts*
Shipping = *heavy*
Assume TL is only due to spherical spreading; neglect attenuation losses

Submarine's Sonar Data:

Linear Array = 3 m long
frequency = *10 kHz*
bandwidth = *5 Hz*
pulse length = *10 ms*
Maximum Input Electrical power to transducer 1200 W
Active Sonar system efficiency – 28%
DI_T = 16 dB
DI = 16 dB
desired p(D) = *90%*
desired p(FA) = 0.01%
assume ideal processor
NL_{self} = 45 dB

Target Data (adversary):

TS = 20 dB
depth = 300 ft @ night

2. Your ship uses active sonar in an attempt to locate a friendly 688-class submarine operating near the surface *22,000 yds* away. Given the following: *transition range = 12,000 yds,* $\alpha = 1.08$ *dB/kyd, SL = 273 dB, NL = 72 dB, DI = 10 dB,* $RL_A = 63$ *dB, TS = 14 dB and DT = 16 dB,* determine the following showing all calculations: (Note that attenuation is a consideration in this problem.)

a) The strongest type of reverberation would most likely be:

volume reverberation / surface reverberation

b) One-way total transmission loss (*TL*)?

c) Signal-to-noise level ($L_{S/N}$) received?

d) Signal excess?

e) Can your ship successfully detect the 688 sub?

Active Sonar Wrap Up Exercise

Problem #1

- You are on a new Seawolf class submarine with the sonar system and the environment described below. Calculate the max range for detecting another submerged submarine given the following for both the noise-limited and reverberation-limited cases:

- Environmental Data
  - *c = 1500 m/s,*
  - Wind Speed = *6 kts*
  - Shipping = *heavy*
  - *Assume TL is only due to spherical spreading; neglect attenuation losses*

- Submarine's Sonar Data:
  - Linear Array = 3 m long
  - frequency = *10 kHz*
  - bandwidth = *5 Hz*
  - pulse length = *10 ms*
  - Maximum Input Electrical power to transducer 1200 W
  - Active Sonar system efficiency – 28%

    DI_T = 16 dB

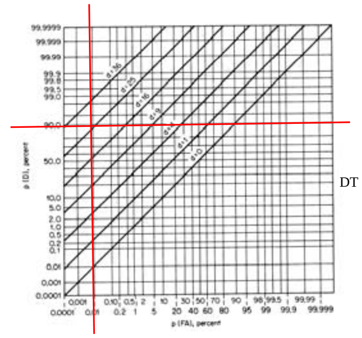
    DI = 16 dB
  - desired p(D) = *90%*

    desired p(FA) = 0.01%

    assume ideal processor
  - NLself = 45 dB

- Target Data (adversary):
  - TS = 20 dB
  - depth = 300 ft @ night

Detection Threshold



$$d = 26$$

$$DT = 10\log\left[\frac{d}{2T\Delta f}\right]$$

$$DT = 10\log\left[\frac{26}{2(0.010s)(5Hz)}\right] = 24.1dB$$

Volume Reverberation Case

$$L_{S/N} = SL - 2TL + TS - RL > DT$$

$$RL_V = SL - 2TL + 10\log(s_v) + 10\log[V]$$

$$L_{S/N} = \cancel{SL} - \cancel{2TL} + TS - \left(\cancel{SL} - \cancel{2TL} + 10\log(s_v) + 10\log[V]\right) > DT$$

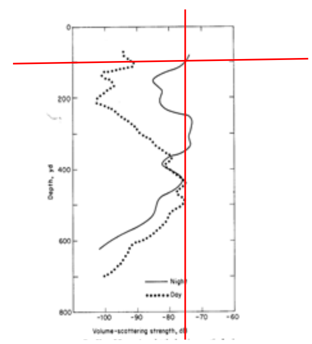
$$L_{S/N} = TS - 10\log(s_v) - 10\log[V] > DT$$

$$L_{S/N} = 20dB - (-76dB) - 10\log(V) > 24.1$$

$$V = \psi r^2 \frac{c\tau}{2}$$

$$10\log(V) = 71.9dB$$

Volume Reverberation



Reverberation Volume

$$V = 10^{7.19} = \psi r^2 \frac{c\tau}{2}$$

$$\lambda = \frac{c}{f} = \frac{1500m/s}{10000Hz} = 0.15m \qquad \psi = 1.32\left(\frac{\lambda}{L}\right) = 1.32\left(\frac{.15m}{3m}\right) = 0.066$$

$$V = 10^{7.19} = 0.066r^2 \frac{(1500m/s)(0.01s)}{2}$$

$$r = 5600m$$

Noise Limited Case

$$L_{S/N} = SL - 2TL + TS - (NL - DI) > DT$$

$$SL = 171.5\ dB + 10\log P_E + 10\log E + DI_T$$

$$SL = 171.5\ dB + 10\log(1200W) + 10\log(.28) + 16dB = 212.8dB$$

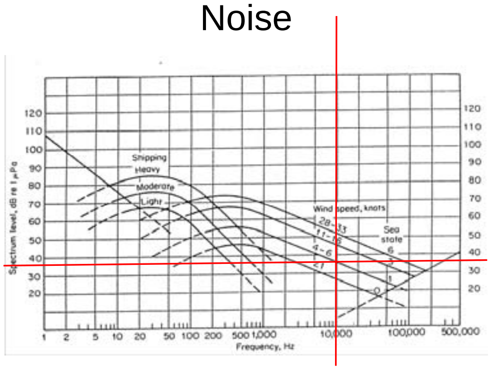
$$NL = NL_{self} \oplus NL_{amb}$$

$$NL_{sea\ state} = 36 + 10\log(5) = 42dB$$

$$NL = 45dB \oplus 42db = 10\log\left(10^{4.5} + 10^{4.2}\right) = 46.8dB$$

Active Sonar Wrap Up Exercise

Noise



Finding the Range

$$L_{S/N} = SL - 2TL + TS - (NL - DI) > DT$$

$$212.8 - 40\log r + 20 - (46.8 - 16) = 24.1$$

$$40\log r = 177.9$$

$$r = 28000m$$

Problem #2

- Your ship uses active sonar in an attempt to locate a friendly 688-class submarine operating near the surface *22,000 yds* away.
- Given the following:
  - *transition range = 12,000 yds,*
  - *α = 1.08 dB/kyd,*
  - *SL = 273 dB,*
  - *NL = 72 dB,*
  - *DI = 10 dB,*
  - *RL_A = 63 dB,*
  - *TS = 14 dB*
  - *and DT = 16 dB,*
- determine the following showing all calculations: (Note that attenuation is a consideration in this problem.)
  - The strongest type of reverberation would most likely be:
    - volume reverberation / surface reverberation
  - One-way total transmission loss (*TL*)?
  - Signal-to-noise level (*LS/N*) received?
  - Signal excess?
  - Can your ship successfully detect the 688 sub?

Transmission Loss

$$TL = 20\log r_o + 10\log\left(\frac{r}{r_o}\right) + \alpha\left(rx10^{-3}\right)$$

$$TL = 20\log 12000 + 10\log\left(\frac{22000}{12000}\right) + 1.08db/kyd\left(22kyd\right) = 108dB$$

Noise/Reverb

$$NL - DI = 72dB - 10dB = 62dB$$

$$RL_s = 63dB$$

Neither is more significant

$$Noise = 62dB \oplus 63dB = 65.5dB$$

Signal Excess

$$L_{S/N} = SL - 2TL + TS - NOISE > DT$$

$$L_{S/N} = 273 - 2(108) + 14 - 65.5 = 5.5dB > 16dB$$

Not dedectable at 22000 yds

$$SE = L_{S/N} - DT = 5.5dB - 16dB = -10.5dB$$

Names:____ Section:_______

Active Sonar Homework

All data given purely for test purposes and do not necessarily reflect reality.

You are on a deep submergence vehicle searching the ocean floor for a Russian torpedo that was lost at sea. You have lost electrical power in your DSV and the Mother Ship is searching for you with the active sonar described below to recover you before you run out of oxygen.

Environmental Data:
Wind speed = 6 kts
Shipping - Heavy
Assume TL is only due to spherical spreading; neglect attenuation losses
During daytime

DSV Data:
TS = + 1.0 dB
Depth = 1500 feet, 2500 feet above ocean floor

Active Sonar Data:
Circular plane/piston array radius=2.4 m
θ = 0.2 radians
Operating frequency = 25 kHz
Bandwidth = 5 Hz
Pulse length = 6 ms
NL_{self} = 15 dB
P(FA) = 1%
P(D) = 75%
(Assume ideal receiver)
Efficiency = 90%
P_E = 750 W
Beam axis can be steered from 30^o above to 60^o below the horizontal

1. What is the directivity index of the mother ship's active sonar's array?

2. What is the Detection Threshold?

3. What is the Volume Scattering Strength for this daytime rescue?

4. What is the maximum detection range if the active sonar system is reverberation limited?

5. Find the projector source level of the Mother Ship's active sonar?

6. What is the reverberation level if the Mother Ship is 8,000 yards from the DSV?

7. What is the total Noise Level due to both ambient and self noise?

8. Is the Mother Ship's active sonar reverberation or noise limited if range is 8,000 yards?

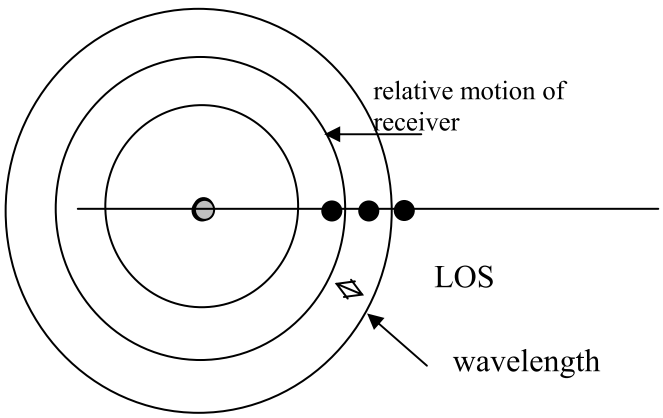
9. Calculate the surface reverberation level if the Mother Ship transmits such that grazing angle with surface is 40º and range is 6,000 yds?

Doppler Effect

The Doppler Effect is the change in the **observed frequency** of a source due to the **relative motion** between the source and the receiver. The relative motion that affects the observed frequency is only the motion **in the Line-Of-Sight (LOS)** between the source and the receiver.

Relative motion of the receiver.

If a source is stationary, as the one below, it will emit sound waves that propagate out from the source as shown below.



As the receiver moves towards the source, it will detect the sound coming from the source but each successive sound wave will be detected earlier than it would have if the receiver were stationary, due to the motion of the receiver in the LOS. Thus the frequency that each successive wave front would be detected would be changed by this relative motion where:

$$\Delta f = \frac{v_r}{\lambda_0}$$

λ_0 is the original wavelength of the source

Δf is the change in the observed frequency

v_r is the velocity of the reciever in the LOS

Since the original frequency of the source can be expressed in terms of the wavelength where $f_0 = {}^{c}\!/_{\lambda_0}$, the observed frequency becomes:

$$f' = f_0 + \Delta f$$

$$f' = \frac{c}{\lambda_0} + \frac{v_r}{\lambda_0}$$

$$f' = f_0\left(\frac{c + v_r}{c}\right)$$

Note that this equation only works if the relative velocity of the receiver, v_r is towards the source. If the motion is away from the source, the relative velocity would be in the opposite direction and the equation would become:

$$f' = f_0\left(\frac{c - v_r}{c}\right)$$

The two equations are usually combined and expressed as:

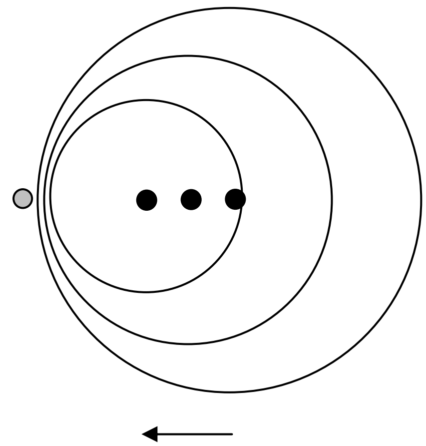
$$f' = f_0\left(\frac{c \pm v_r}{c}\right)$$

Relative motion of the source

If the source is moving towards the receiver, the effect is slightly different. The spacing between the successive wave fronts would be less as seen in the diagram below. This would be expressed as:

$$\Delta\lambda = \frac{v_s}{f_0}$$

v_s is the relative velocity of the source



To calculate the observed frequency:

$$f' = \frac{c}{(\lambda_0 + \Delta\lambda)}$$

$$f' = f_0\left(\frac{c}{c - v_s}\right)$$

Note that this is only when the source is moving **towards** the receiver. If the source is moving away, the equation would be changed to:

$$f' = f_0\left(\frac{c}{c + v_s}\right)$$

When combined with the previous result, the equation would be expressed as:

$$f' = f_0\left(\frac{c}{c \mp v_s}\right)$$

Notice that this time, the plus/minus symbol is inverted because the sign on top is to be used for relative motion of the source **towards** the receiver.

Doppler Equation

By combining the previous results, we can derive one equation to use as the Doppler Equation. This is usually expressed as:

$$f' = f_0\left(\frac{c \pm v_r}{c \mp v_s}\right)$$

The student must be careful that the quantities for the velocity of the receiver, $v_{r,}$ and the velocity of the source, v_s, are only the **magnitudes** of the relative velocities **in (or along) the LOS**. In other words, the component of the velocity of the source and the receiver, that are perpendicular to the LOS do not change the received frequency. Secondly, the top sign in the numerator and the denominator are the sign convention to be used when the relative velocities are **towards** the other. If the source were moving towards the receiver, the sign to use in the denominator would be the minus sign. If the source were moving away from the receiver, the sign to use would be the plus sign.

Active Sonar Problem

One interesting Doppler problem is the active sonar problem. In this problem, one must define a "source" and "receiver" for both the outgoing active pulse and the returning signal.

For the outgoing active pulse, the Doppler shifted frequency of the active pulse when it hits the target would be:

$$f' = f_0\left(\frac{c \pm v_r}{c \mp v_s}\right)$$

For the return pulse, there would be a similar shift but now the "source" would be the target, the "receiver" would be the ship sending out the original active pulse and the base frequency, f_0 would be the Doppler shifted frequency from above. If we redefine the velocity of the target as, v_t, and the velocity of the source of the active pulseas v_s, we get:

$$f'' = f'\left(\frac{c \pm v_s}{c \mp v_t}\right)$$

Substituting in the equation from above for f′ and changing the subscripts for the appropriate terms:

$$f'' = f_0\left(\frac{c \pm v_t}{c \mp v_s}\right)\left(\frac{c \pm v_s}{c \mp v_t}\right)$$

Again, the velocities are only the magnitudes of the velocity **in the LOS** and one must take care to pick the correct sign to use in front of each velocity.

Problems:

1. A surface ship is traveling on a course of 045°T and is conducting an active sonar search. The sonar frequency is 4 kHz. At the same time a submarine is located on a bearing of 060°T from the surface ship. The submarine is traveling due north (000°τ at a speed of 10 kts. The surface ship is traveling at 15 kts. (Useful information: c=2950 kts)
   a) What is the frequency that the submarine receives?
   b) What is the frequency of the active return from the submarine?
   c) If the submarine emits a strong tonal at 415 Hz, what is the frequency received at the surface ship?
   d) Assuming that there is a large school of shrimp near the submarine, what is the frequency of the active return from the school of shrimp? (Assume that the shrimp are not moving.)

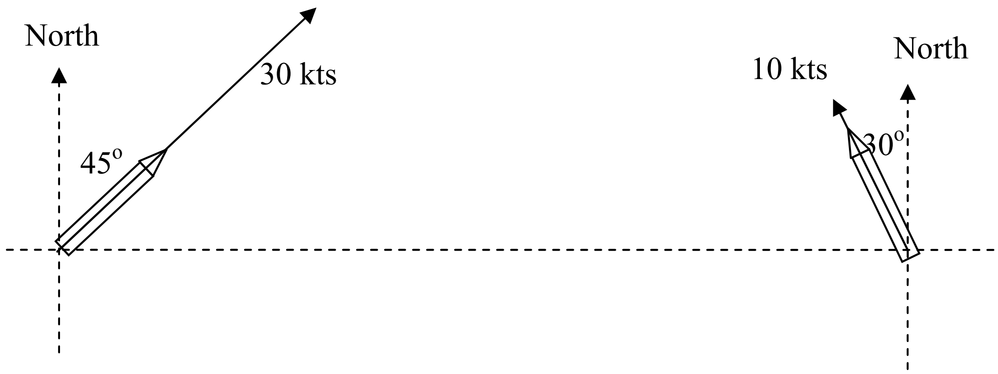
2. A US submarine is trailing a new Afghani diesel submarine to gain tonal intelligence on the Afghani sub. Both subs were going 5 knots on course north when the Afghani sub turns due west.
   a) If the tonal the US sub was originally tracking was 250Hz, what is the received frequency after the Afghani sub turns?
   b) If the US sub goes active with a frequency of 18,000 Hz to get an exact range on the Afghani sub, what is the frequency of the received return?

3. A sound source emits a sound frequency of 1000 Hz on a day when the speed of sound in air is 340 m/s and there is no wind. What is the frequency you will receive if:
   a) You move toward the source at 34 m/s?
   b) You are stationary and the source moves towards you at 34 m/s?
   c) Repeat part a) with a speed of 68 m/s instead of 34 m/s.
   d) Repeat part b) with a speed of 68 m/s instead of 34 m/s.

4. Two submarines are moving as shown in the figure, where the speeds are in knots. The speed of sound in knots is 2912 kts. Sub A is pinging on B with an active sonar frequency of 10 kHz.

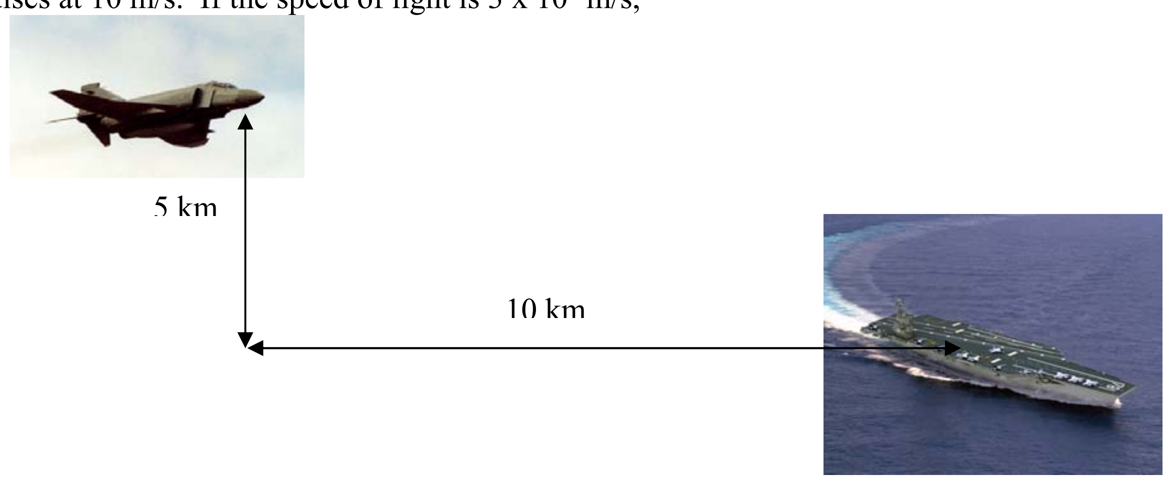
   a) What frequency will Breceive from A's sonar?
   b) What is the frequency of the echo A receives from B?

5. Ship S is on course 045 T with speed 30 kts. Target T is on course 330 T with speed 10 kts. S uses active sonar to ping on T with frequency 10 kHz. The speed of sound is 3000 kts. When T is due east of S (as shown in the below sketch):



a) What frequency will T receive from S?
b) What will be the echo frequency that S receives back from T?
c) What is the frequency of the reverberation received by S?

6. A phantom jet flying at an altitude of 5 km is directly behind and closing at a horizontal range of 10 km from the carrier. The jet is tracking the ship with an active radar unit of source frequency 400 MHz. The jet's speed is 200 m/s parallel to the ground. The ship cruises at 10 m/s. If the speed of light is 3×10^8 m/s,



a) Compute the radar frequency detected by the aircraft carrier.
b) Compute the echo frequency detected by the jet.
c) Compute the Doppler shift between the echo and source frequency.

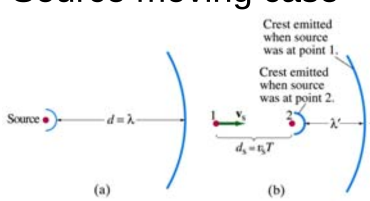
Doppler Effect

(a) At rest

Doppler Effect – 4 cases

- Source moving toward receiver
- Source moving away from receiver
- Receiver (observer) moving towards source
- Receiver (observer) moving away from source.

Source moving case



$$\lambda' = \lambda - v_s T$$

Away:

$$f' = f\left(\frac{1}{1+\frac{v_s}{v}}\right)$$

$$c = \frac{\lambda}{T}$$

$$\lambda' = \lambda - v_s\frac{\lambda}{c} = \lambda\left(1-\frac{v_s}{c}\right)$$

Towards:

$$f' = \frac{c}{\lambda'} = \frac{c}{\lambda\left(1-\frac{v_s}{c}\right)} = f\left(\frac{1}{1-\frac{v_s}{c}}\right)$$

Receiver (observer) moving case

Towards:

$$f' = \frac{c+v_r}{\lambda} = f\left(1+\frac{v_r}{c}\right)$$

Away:

$$f' = \frac{c-v_r}{\lambda} = f\left(1-\frac{v_r}{c}\right)$$

Source and receiver moving

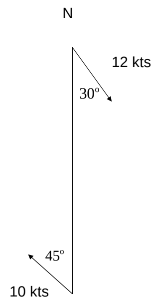
$$f' = f\left(\frac{1 \pm \frac{v_r}{c}}{1 \mp \frac{v_s}{c}}\right) = f\left(\frac{c \pm v_r}{c \mp v_S}\right)$$

- Numerator – Receiver (observer)
  - Toward +
  - Away –
- Denominator – Source
  - Toward –
  - Away +

Doppler Example

- Intelligence tells you that a particular piece of machinery in the engine room of a Soviet Victor III submarine emits a frequency of 320 Hz. Your sonar operator hears the machinery but reports the frequency is 325 Hz. Assume you have slowed to a negligible speed in order to better hear the Russian.
  - Is the VIII coming toward you or moving away from you?
  - Assuming the Victor is either moving directly toward or away from you, what is his speed in m/s?

Line of sight diagrams



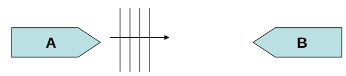
$$f' = f\left(\frac{1 \pm \frac{v_r}{c}}{1 \mp \frac{v_s}{c}}\right) = f\left(\frac{c \pm v_r}{c \mp v_S}\right)$$

$$v_s = 12\text{kts}\sin 30 = 6\text{kts}$$

$$v_r = 10\text{kts}\sin 45 = 7.07\text{kts}$$

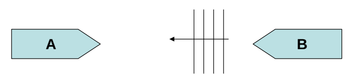
1 knot = 0.5144 m/s

Active Case



$$f_B = f_A\left(\frac{1+\frac{v_r}{c}}{1-\frac{v_s}{c}}\right)$$

$$\Delta f = 2f_A\left(\frac{v_r}{c} + \frac{v_s}{c}\right)$$



$$f_{B\to A} = f_B\left(\frac{1+\frac{v_s}{c}}{1-\frac{v_r}{c}}\right) = f_A\left(\frac{1+\frac{v_r}{c}}{1-\frac{v_s}{c}}\right)\left(\frac{1+\frac{v_s}{c}}{1-\frac{v_r}{c}}\right) = f_A + 2f_A\left(\frac{v_r}{c} + \frac{v_s}{c}\right)$$

Example

- A ship moving north at 20 kts pings on a target (that bears 030 and is) moving east at 10 kts. The source frequency is 4000 Hz and the sound speed is 3000 kts.
  - What is the frequency received by the target?
  - What is the echo frequency received by the ship?
  - What is the doppler shift of the echo received by the ship relative to the ship's source?
  - What is the frequency heard by reverberation near the target
  - What is the echo frequency received by the ship from reverberation?
  - What is the doppler shift of the echo received by the ship from reverberation relative to the ships source?