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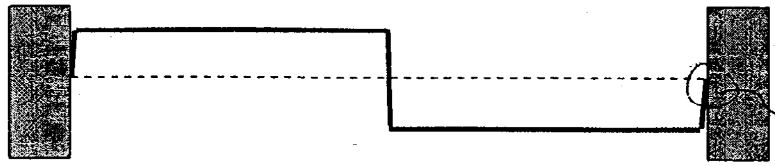
1. (25) A stretchy string of length, L, is suspended between two fixed posts. A template is created that stretches the string into the initial position shown below:



BC: $y(L,t)=0$

$kL = n\pi$

$k_n = \frac{n\pi}{L}$

A reasonable approximation for this initial condition is:

$$y(x,t=0)=\begin{cases} +h & 0<x<\frac{L}{2} \\ -h & \frac{L}{2}<x<L \end{cases}$$

If the template is rapidly removed from this initial rest position, the string vibrates according to the general result:

$$y(x,t)=\sum_{n=1}^{\infty}(A_n \cos\omega_n t + B_n \sin\omega_n t)\sin k_n x$$

a. (15) Find the expression for the coefficients A_n and B_n.

b. (5) If h = 1 cm, find the amplitude of the fundamental mode and the first five overtones (n=1 to n=6). Sketch the non-zero modes.

$c = \sqrt{\frac{T}{\rho}} = \sqrt{\frac{5N}{.003kg/.314m}} = 22.9 m/s$

c. (5) If the mass of the string is .0030 kg and the length is 0.314 m and the tension is 5.0 N, How much energy is in the n=2 mode?

$B_n = \frac{2}{\omega_n L}\int_0^L u(x,0)\sin k_n x\,dx \qquad u(x,t)=0 \Rightarrow \boxed{B_n = 0}$

$A_n = \frac{2}{L}\int_0^L y(x,0)\sin k_n x\,dx = \frac{2}{L}\left\{\int_0^{L/2} h\sin\left(\frac{n\pi}{L}x\right)dx + \int_{L/2}^{L}(-h)\sin\left(\frac{n\pi}{L}x\right)dx\right\}$

TI-92 $\rightarrow = \frac{2}{L}\left\{\frac{hL}{n\pi}\left(1-2\cos\frac{n\pi}{2}+\cos n\pi\right)\right\} = \boxed{\frac{2h}{n\pi}\left(1-2\cos\frac{n\pi}{2}+\cos n\pi\right)}$

$A_1 = \frac{2h}{\pi}(1-0+(-1)) = 0$

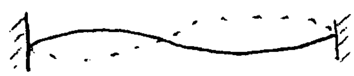
$A_2 = \frac{2h}{2\pi}(1-2(-1)+1) = \frac{4h}{\pi} = \frac{4(.01m)}{\pi} = \boxed{.0127m}$

$A_3 = \frac{2h}{3\pi}(1-0+(-1)) = 0$

$A_4 = \frac{2h}{4\pi}(1-2(+1)+1) = 0$

$A_5 = \frac{2h}{5\pi}(1-0+(-1)) = 0$

$A_6 = \frac{2h}{6\pi}(1-2(-1)-(-1)) = \frac{8h}{6\pi} = \frac{4(.01m)}{3\pi} = \boxed{.00424m}$

n = 2 

n = 6 

$E_n = \frac{1}{4}m_s\omega_n^2 A_n^2$

$k_2 = \frac{2\pi}{.314m}$

$\omega_2 = ck_2 = \frac{(22.9m/s)(2\pi)}{.314m} = 458 \frac{r}{s}$

$E_2 = \frac{1}{4}(.003kg)(458 r/s)^2(.0127m)^2 = \boxed{.0253 J}$

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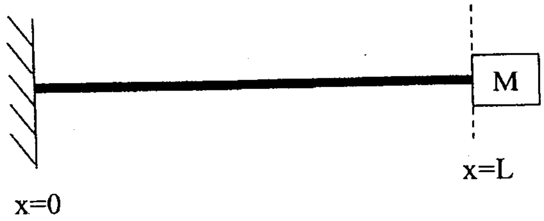
2. (25) A long, Aluminum bar is fixed at one end (x=0) and loaded with a large mass, M, at the other end (x=L). The bar has cross sectional area, S.



The bar vibrates in the longitudinal mode subject to the wave equation,

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2}$$

where $c^2 = \frac{Y}{\rho}$. We showed that the complex harmonic solution of the wave equation is,

$$\tilde{\xi}(x,t) = \tilde{A}e^{j(\omega t - kx)} + \tilde{B}e^{j(\omega t + kx)}.$$

a. (10) What are the boundary conditions for this system.

$$\boxed{\begin{aligned} \xi(x=0) &= 0 \\ -SY\frac{\partial \xi}{\partial x}\Big|_{x=L} &= M\frac{\partial^2 \xi}{\partial t^2}\Big|_{x=L}\end{aligned}}$$

b. (5) Use your boundary condition at x=0 to show that: $\tilde{\xi}(x,t) = -j2\tilde{A}e^{j\omega t}\sin kx$

$\xi(x=0) = \tilde{A}e^{j(\omega t)} + \tilde{B}e^{j(\omega t)} = 0 \Rightarrow \tilde{A} = -\tilde{B}$; $\xi(x,t) = \tilde{A}e^{j\omega t}(e^{-jkx} - e^{jkx}) = -2j\tilde{A}e^{j\omega t}\sin kx$ Q.E.D

c. (5) Combine with your boundary condition at x=L to show that: $\cot kL = \frac{M\omega^2}{SYk}$

d. (5) Since $Y = \rho c^2 = \frac{m_{bar}}{SL}c^2$, show this can be rewritten: $\cot kL = \frac{M}{m_{bar}}kL$

$$-SY\frac{\partial \xi}{\partial x}\Big|_{x=L} = M\frac{\partial^2 \xi}{\partial t^2}\Big|_{x=L}$$

$$+SY\left(+2j\tilde{A}e^{j\omega t}k\cos kL\right) = M\left(+2j\tilde{A}(j\omega)^2 e^{j\omega t}\sin kL\right)$$

$$SYk\cos kL = M\omega^2\sin kL$$

$$\cot kL = \frac{M\omega^2}{SYk} \qquad \text{Q.E.D}$$

$$\cot kL = \frac{M\omega^2}{S\left(\frac{M_{bar}}{SL}\right)c^2 k} = \frac{M}{m_{bar}}\frac{\omega}{c}L = \frac{M}{m_{bar}}kL \qquad \text{QED.}$$

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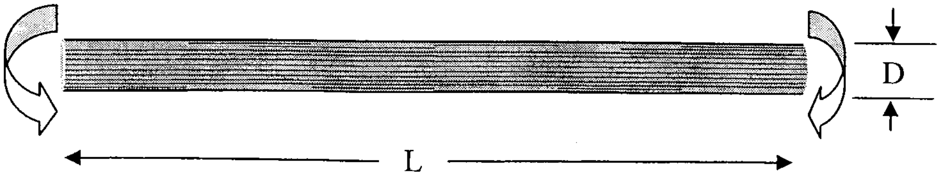
3. (20) An Aluminum bar is 75.0 cm long with a circular cross section and a diameter of 1.00 cm. The bar is excited to vibrate in the bending mode where both ends are free.



We showed that Helmholtz equation for this bending mode is

$$\frac{\partial^4 \Psi}{\partial x^4} = g^4 \Psi$$

with wave number, $g = \frac{\omega}{v}$ and phase speed, $v^2 = \omega \kappa c_{long}$. Further we showed that applying free-free boundary conditions to the solution to Helmhotz equation resulted in the following relation

$$\cosh gL \cos gL = 1$$

where

$$gL = m\frac{\pi}{2} \quad \text{with } m = 3.011, 4.9994, 7.00, 9.00, 11.0\ldots$$

a. (5) What are the numeric values of the wave numbers for the first 3 normal modes? **Include units.**

b. (5) What are the first 3 normal mode wavelengths?

c. (5) What are the first 3 normal mode frequencies (in Hz)? 3.12.2

d. (5) What are the phase speeds for the first 3 normal modes?

$$gL = m\frac{\pi}{2}$$

$$g_1 = \frac{3.011\ \pi}{2(.75m)} = 2.007\ \pi\ \text{rad}/m = 6.31\ \frac{\text{rad}}{m}$$

$$g_2 = \frac{5\pi}{2(.75m)} = 3.33\ \pi\ \text{rad}/m = 10.5\ \frac{\text{rad}}{m}$$

$$g_3 = \frac{7\pi}{2(.75m)} = 4.67\ \pi\ \text{rad}/m = 14.7\ \frac{\text{rad}}{m}$$

$g_1 = \frac{2\pi}{\lambda}$

$$\lambda_1 = \frac{2\pi}{g_1} = \frac{2\cancel{\pi}}{2.007\cancel{\pi}\ \text{rad}/m} = .996\ m$$

$$\lambda_2 = \frac{2\pi}{g_2} = \frac{2\pi}{3.33\pi\ \text{rad}/m} = .6\ m$$

$$\lambda_3 = \frac{2\pi}{g_3} = \frac{2\cancel{\pi}}{4.67\cancel{\pi}\ \text{rad}/m} = .429\ m$$

$$g = \frac{m\pi}{2L}$$

$$\omega = g v = g\sqrt{\omega K c_l} = g\sqrt{\omega}\sqrt{K c_l}$$

$$\frac{\omega}{\sqrt{\omega}} = \sqrt{\omega} = g\sqrt{K c_l}$$

$$K = \frac{d}{4} = \frac{.01m}{4}$$

$$= .0025m$$

$$\omega = g^2 K c_l = \frac{m^2\pi^2}{4L^2} K c_l$$

$$c_l = 5050\ m/s$$

$$f = \frac{m^2\pi^{\cancel{2}}}{2\cancel{\pi}(4L^2)} K c_l$$

$$= \frac{m^2\pi K c_l}{8L^2}$$

$$f_1 = \frac{(3.011)^2 \pi (.0025\cancel{m}) 5050 \frac{m}{s}}{8(.75\cancel{m})^2} = 79.9 \text{ Hz}$$

$$f_2 = \frac{(4.9994)^2 \pi (.0025m)(5050 \frac{m}{s})}{8(.75m)^2} = 220 \text{ Hz}$$

$$f_3 = \frac{(7)^2 \pi (.0025m)(5050 \frac{m}{s})}{8(.75m)^2} = 432 \text{ Hz}$$

$$v_1 = f_1 \lambda_1 = (79.9 \text{ Hz})(.996 m) = 79.6 \frac{m}{s}$$

$$v_2 = f_2 \lambda_2 = (220 \text{ Hz})(.6 m) = 132 \frac{m}{s}$$

$$v_3 = f_3 \lambda_3 = (432 \text{ Hz})(.429 m) = 185 \frac{m}{s}$$

OR

$$v_1 = \sqrt{(2\pi f_1) K C_e} = \sqrt{2\pi (79.9 \text{ Hz})(.0025m)(5050 \frac{m}{s})} = 79.6 \frac{m}{s}$$

$$v_2 = \sqrt{2\pi f_2 K C_e} = 132 \frac{m}{s}$$

$$v_3 = \sqrt{2\pi f_3 K C_e} = 185 \frac{m}{s}$$

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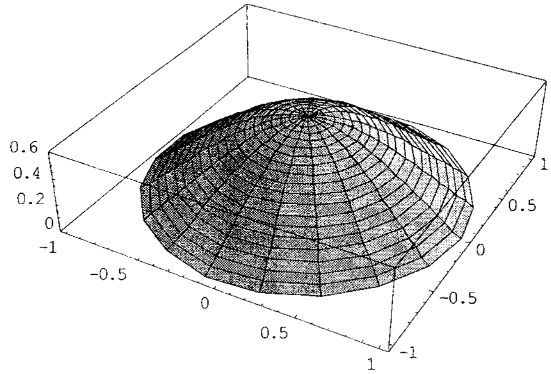
3. (25) An elastic membrane is stretched and clamped on a rigid circular frame with uniform tension. The fundamental frequency shown below is 70 Hz. The radius of the membrane is 0.10 m and the area density is 0.20 kg/m^2.



$$f_{01} = \frac{j_{01} c}{2\pi a}$$

a. (5) What is the phase speed of the standing wave on the membrane?

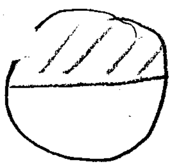
$$c = \frac{2\pi f_{01} a}{j_{01}} = \frac{2\pi(70\,\mathrm{Hz})(.1\,\mathrm{m})}{2.4} = \boxed{18.3\ \mathrm{m/s}}$$

b. (5) What is the tension per unit length (assumed to be isotropic) of the elastic membrane?

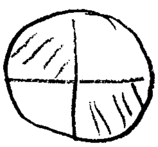
$$\mathcal{T} = \rho_s c^2 = (0.2\,\tfrac{\mathrm{kg}}{\mathrm{m}^2})(18.3\,\mathrm{m/s})^2 = 67.2\ \mathrm{N/m}$$

c. (10) What are the next 5 lowest frequencies at which this membrane can vibrate?

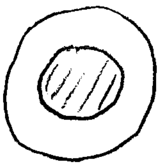
d. (5) Include a rough sketch of the oscillation spatial pattern similar to figure 4.4.1 in your textbook for each frequency.



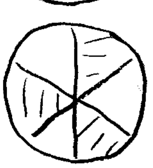
$$f_{1,1} = \frac{j_{11} c}{2\pi a} = (70\,\mathrm{Hz})\left(\frac{j_{11}}{j_{01}}\right) = 70\,\mathrm{Hz}\left(\frac{3.83}{2.4}\right) = \boxed{112\ \mathrm{Hz}}$$



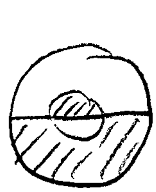
$$f_{2,1} = 70\,\mathrm{Hz}\left(\frac{j_{21}}{j_{01}}\right) = 70\,\mathrm{Hz}\left(\frac{5.14}{2.4}\right) = \boxed{150\ \mathrm{Hz}}$$



$$f_{0,2} = 70\,\mathrm{Hz}\left(\frac{j_{02}}{j_{01}}\right) = 70\,\mathrm{Hz}\left(\frac{5.52}{2.4}\right) = \boxed{161\ \mathrm{Hz}}$$



$$f_{3,1} = 70\,\mathrm{Hz}\left(\frac{j_{31}}{j_{01}}\right) = 70\,\mathrm{Hz}\left(\frac{6.38}{2.4}\right) = \boxed{186\ \mathrm{Hz}}$$



$$f_{1,2} = 70\,\mathrm{Hz}\left(\frac{j_{12}}{j_{10}}\right) = 70\,\mathrm{Hz}\left(\frac{7.02}{2.4}\right) = \boxed{205\ \mathrm{Hz}}$$