Homework Assignment #1

1. KFCS 1.3.1

2. KFCS 1.4.1

3. KFCS 1.5.4, plus the following:
   (g) What is the real part of **A** = Re(**A**)?
   (h) What is the imaginary part of **B** = Im(**B**)?
   (i) What is the phase of **B**?

4. KFCS 1.5.3, plus the following:
   (f) What is the real part of **A** = Re(**A**)?

5. KFCS 1.6.1

6. KFCS 1.7.1. plus the following:

(c) Is the angular frequency that maximizes the acceleration calculated in part (b) the resonant frequency?

(d) What is the angular frequency that maximizes particle velocity? Is this the resonance frequency?

Hint: Prior to maximizing the velocity and acceleration by setting their derivatives w.r.t. frequency equal to zero, find the real magnitude of the velocity and acceleration. You don't need the phase because that term and the time dependent exponential can be no bigger than one.

SHM PROBLEM 1.3.1

Note Title 8/22/2010

GIVEN: SHO FREQUENCY $= \omega_0$, AT $t = \frac{T}{2}$ $u(t) = +U$

FIND: $x(t)$

SOLUTION 1: $x(t) = A\cos(\omega_0 t + \phi)$ (EQN 1.3.2)

$$u(t) = \frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \phi)$$

$t = \frac{T}{2}$

$$x\left(\frac{T}{2}\right) = A\cos\left(\omega_0 \frac{T}{2} + \phi\right) = A\cos\left(\frac{2\pi}{T}\frac{T}{2} + \phi\right)$$

$$= A\cos(\pi + \phi) = \text{NOT SPECIFIED (DIRECTLY)}$$

$$u\left(\frac{T}{2}\right) = -A\omega_0 \sin\left(\omega_0 \frac{T}{2} + \phi\right) = -A\omega_0 \sin\left(\frac{2\pi}{T}\frac{T}{2} + \phi\right)$$

$$= -A\omega_0 \sin(\pi + \phi) = +U$$

$$\sin(\pi + \phi) = -1$$

(PICK ONE) $\pi + \phi = -\frac{\pi}{2}, \boxed{\frac{3\pi}{2}}, \frac{7\pi}{2} \ldots$

$$\phi = \frac{\pi}{2}$$

(OTHER VALUES FOR ϕ ARE ALSO CORRECT)

$$A\omega_0 = U$$

$$A = \frac{U}{\omega_0}$$

$$\therefore \boxed{x(t) = \frac{U}{\omega_0}\cos\left(\omega_0 t + \frac{\pi}{2}\right)}$$

SOLUTION 2: $x(t) = A_1 \cos\omega_0 t + A_2 \sin\omega_0 t$ (EQN 1.2.8)

$$u(t) = -A_1\omega_0 \sin\omega_0 t + A_2\omega_0 \cos\omega_0 t$$

$$u\left(\frac{T}{2}\right) = -A_1\omega_0 \sin\left(\omega_0 \frac{T}{2}\right) + A_2\omega_0 \cos\left(\omega_0\frac{T}{2}\right)$$

$$= -A_1\omega_0 \sin\left(\frac{2\pi}{T}\frac{T}{2}\right) + A_2\omega_0 \cos\left(\frac{2\pi}{T}\frac{T}{2}\right)$$

$$= -A_1\omega_0 \underbrace{\cancel{\sin}}_{0}(\pi) + A_2\omega_0 \underbrace{\cancel{\cos}}_{-1}(\pi) = U$$

$$A_2 = \frac{-U}{\omega_0}$$

BUT WHAT IS A_1 ? EARLIER I SAID $x\left(\frac{T}{2}\right)$ WAS NOT SPECIFIED. THIS IS NOT REALLY TRUE. IF $u\left(\frac{T}{2}\right) =$ MAX, $x\left(\frac{T}{2}\right) = 0$. SO IN SOLUTION 1,

$$\cos(\pi + \phi) = 0 \Rightarrow \pi + \phi = -\frac{\pi}{2}, \frac{\pi}{2}, \boxed{\frac{3\pi}{2}}, \frac{5\pi}{2}, \ldots$$

$$\phi = \frac{\pi}{2} \quad \text{AGAIN} \checkmark$$

IN SOLUTION 2,

$$x(t) = A_1 \cos\omega_0 t - \frac{U}{\omega_0}\sin\omega_0 t$$

$$x\left(\frac{T}{2}\right) = A_1 \cos\left(\frac{2\pi}{T}\frac{T}{2}\right) - \frac{U}{\omega_0} \underbrace{\cancel{\sin\left(\frac{2\pi}{2}\frac{T}{2}\right)}}_{0} = 0$$

$$A_1 \cos(\pi) = 0 \Rightarrow A_1 = 0$$

$$\boxed{x(t) = -\frac{U}{\omega_0}\sin\omega_0 t}$$

THIS SHOWS $\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$!!

SHM ENERGY 1.4.1

Note Title 8/22/2010

SHOW: $E_{k(max)} = E_{p(max)}$ FOR UNDAMPED SHO

$$E_k = \frac{1}{2} m u^2 \qquad E_p = \frac{1}{2} s x^2$$

USE $x = A\cos(\omega_0 t + \phi)$

$$u = \frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \phi)$$

$$E_k = \frac{1}{2} m \left(-A\omega_0 \sin(\omega_0 t + \phi)\right)^2 \qquad E_p = \frac{1}{2} s \left(A\cos(\omega_0 t + \phi)\right)^2$$

$$= \frac{1}{2} m A^2 \omega_0^2 \sin^2(\omega_0 t + \phi) \qquad E_p = \frac{1}{2} s A^2 \cos^2(\omega_0 t + \phi)$$

$$E_{k(max)} = \frac{1}{2} m A^2 \omega_0^2 \qquad E_{p(max)} = \frac{1}{2} s A^2$$

(SINCE THE MAX VALUE OF cos & sin ARE 1)

BUT $\omega_0^2 = \frac{s}{m}$

$$E_{k(max)} = \frac{1}{2} \cancel{m} A^2 \left(\frac{s}{\cancel{m}}\right) = \frac{1}{2} s A^2 = E_{p(max)}$$

Q.E.D.

COMPLEX NUMBERS CARTESIAN 1.5.4

Note Title 8/22/2010

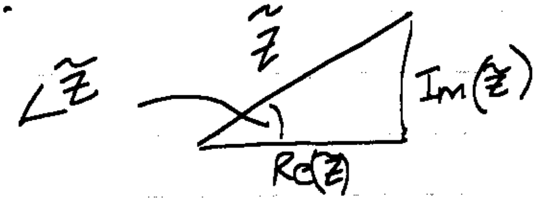
GIVEN: $\tilde{A} = x + j y \qquad \tilde{B} = X + j Y$

FIND: $|\tilde{A}|$, $|\tilde{B}|$, $|\tilde{A}\tilde{B}|$, $Re(\tilde{A}\tilde{B})$, PHASE OF $\tilde{A}\tilde{B}$, $\angle\tilde{A}\tilde{B}$

$Re\left(\frac{\tilde{A}}{\tilde{B}}\right)$, $Re(\tilde{A})$, $Im(\tilde{B})$, $\angle\tilde{B}$

$$\boxed{|\tilde{A}| = (x^2 + y^2)^{1/2}}$$

$$\boxed{|\tilde{B}| = (X^2 + Y^2)^{1/2}}$$



$$\tilde{A}\tilde{B} = (x + jy)(X + jY) = xX + jxY + jyX - yY$$

$$= xX - yY + j(xY + yX)$$

$$|\tilde{A}\tilde{B}| = \left((xX - yY)^2 + (xY + yX)^2\right)^{1/2}$$

$$= \left(x^2X^2 + y^2Y^2 - \cancel{2xyXY} + x^2Y^2 + y^2X^2 + \cancel{2xyXY}\right)^{1/2}$$

$$= \boxed{\left(x^2X^2 + y^2Y^2 + x^2Y^2 + y^2X^2\right)^{1/2}}$$

$$\boxed{Re(\tilde{A}\tilde{B}) = xX - yY}$$

$$\angle\tilde{A}\tilde{B} = \tan^{-1}\left(\frac{Im(\tilde{A}\tilde{B})}{RE(\tilde{A}\tilde{B})}\right) = \boxed{\tan^{-1}\left(\frac{xY + yX}{xX - yY}\right)}$$

$$\frac{\tilde{A}}{\tilde{B}} = \frac{x + jy}{X + jY}\left(\frac{X - jY}{X - jY}\right) = \frac{xX + yY + j(yX - xY)}{X^2 + Y^2}$$

$$\boxed{\mathrm{Re}\left(\frac{\tilde{A}}{\tilde{B}}\right) = \frac{xX + yY}{X^2 + Y^2}}$$

$$\boxed{\mathrm{Re}(\tilde{A}) = x}$$

$$\boxed{\mathrm{Im}(\tilde{B}) = Y}$$

$$\boxed{\angle\tilde{B} = \tan^{-1}\left(\frac{Y}{X}\right)}$$

COMPLEX NUMBERS POLAR 1.5.3

Note Title 8/22/2010

GIVEN: $\tilde{A} = Ae^{j(\omega t+\theta)}$ $\tilde{B} = Be^{j(\omega t+\phi)}$

FIND: $Re(\tilde{A}\tilde{B})$, $Re\left(\frac{\tilde{A}}{\tilde{B}}\right)$, $Re(\tilde{A})Re(\tilde{B})$, $\angle\tilde{A}\tilde{B}$, $\angle\frac{\tilde{A}}{\tilde{B}}$, $|\tilde{A}|$,

$$\tilde{A}\tilde{B} = ABe^{j(\omega t+\theta)}e^{j(\omega t+\phi)} = ABe^{j(2\omega t+\theta+\phi)}$$

$$\boxed{Re(\tilde{A}\tilde{B}) = AB\cos(2\omega t+\theta+\phi)}$$

$$\frac{\tilde{A}}{\tilde{B}} = \frac{Ae^{j(\omega t+\theta)}}{Be^{j(\omega t+\phi)}} = \frac{A}{B}e^{j(\theta-\phi)}$$

$$\boxed{Re\left(\frac{\tilde{A}}{\tilde{B}}\right) = \frac{A}{B}\cos(\theta-\phi)}$$

$$Re(\tilde{A})Re(\tilde{B}) = A\cos(\omega t+\theta)B\cos(\omega t+\phi)$$
$$= \boxed{AB\cos(\omega t+\theta)\cos(\omega t+\phi)} \neq Re(\tilde{A}\tilde{B})$$

$$\boxed{\angle\tilde{A}\tilde{B} = 2\omega t+\theta+\phi}$$

$$\boxed{\angle\frac{\tilde{A}}{\tilde{B}} = \theta-\phi}$$

$$\boxed{|\tilde{A}| = A}$$

$$\mathrm{Re}(\tilde{A}) = \boxed{A\cos(\omega t + \theta)}$$

DAMPED OSCILLATOR 1.6.1

Note Title 8/22/2010

GIVEN: $m = .5\,kg$ $\quad \Delta m = .2\,kg \Leftrightarrow \Delta x = .04\,m$

$T = 1\,s$ $\quad x(t=0) = .04\,m \quad u(t=0) = 0$

FIND: $R_m,\ \omega_d,\ A\ \&\ \phi$

WE COULD JUST START WITH EQNS 1.6.8 & 1.6.9 BUT IT IS HELP FULL TO SEE WHERE THEY CAME FROM.

THE EOM FROM N2L $\quad m\dfrac{d^2\tilde{x}}{dt^2} + R_m\dfrac{d\tilde{x}}{dt} + s\tilde{x} = 0$

ASSUME $\quad \tilde{x} = \tilde{A}e^{\gamma t}$

$$\frac{d\tilde{x}}{dt} = \gamma \tilde{A} e^{\gamma t}$$

$$\frac{d^2 x}{dt^2} = \gamma^2 \tilde{A} e^{\gamma t}$$

$$m\gamma^2\cancel{\tilde{A}e^{\gamma t}} + R_m\gamma\cancel{\tilde{A}e^{\gamma t}} + s\cancel{\tilde{A}e^{\gamma t}} = 0$$

$$m\gamma^2 + R_m\gamma + s = 0$$

$$\gamma = \frac{-R_m \pm \sqrt{R_m^2 - 4ms}}{2m} = \frac{-R_m}{2m} \pm \sqrt{\left(\frac{R_m}{2m}\right)^2 - \frac{s}{m}}$$

LET $\boxed{\beta = \dfrac{R_m}{2m}}$ & RECALL $\boxed{\omega_0^2 = \dfrac{s}{m}}$

$$\gamma = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

FOR THE UNDERDAMPED CASE, $\omega_0 > \beta$

$$\gamma = -\beta \pm j\sqrt{\omega_0^2 - \beta^2} = -\beta \pm j\omega_d$$

WHERE $\boxed{\omega_d = \sqrt{\omega_0^2 - \beta^2}}$

$$\tilde{x} = \tilde{A}_1 e^{-\beta t} e^{+j\omega_d t} + \tilde{A}_2 e^{-\beta t} e^{-j\omega_d t}$$

$$\tilde{A}_1 = \frac{A}{2} e^{j\phi} \qquad \tilde{A}_2 = \frac{A}{2} e^{-j\phi}$$

$$= A e^{-\beta t}\left[\frac{e^{j\phi} e^{j\omega_d t} + e^{-j\phi} e^{-j\omega_d t}}{2}\right]$$

$$= \boxed{A e^{-\beta t} \cos(\omega_d t + \phi)} \qquad \text{(EQN 1.6.13)}$$

$$s = \frac{\Delta f}{\Delta x} = \frac{\Delta m g}{\Delta x} = \frac{(.2\,\text{kg})(9.8\,\text{m/s}^2)}{.04\,\text{m}} = 49\,\frac{\text{N}}{\text{m}}$$

$$\beta = \frac{1}{\tau} = \frac{1}{1\,\text{s}} = 1\,\text{s}^{-1}$$

$$\beta = \frac{R_m}{2m} \Rightarrow R_m = 2m\beta = 2(.5\,\text{kg})(1\,\text{s}^{-1}) = \boxed{1\,\frac{\text{kg}}{\text{s}}}$$

$$\omega_0^2 = \frac{s}{m} = \frac{49\,\frac{\text{N}}{\text{m}}}{.5\,\text{kg}} = 98\,\text{s}^{-2}$$

$$\omega_d^2 = \omega_0^2 - \beta^2 = 98\,\text{s}^{-2} - 1\,\text{s}^{-2} = 97\,\text{s}^{-2}$$

$$\omega_d = \sqrt{97\,\text{s}^{-2}} = \boxed{9.85\,\text{s}^{-1}}$$

$$x(t=0) = .04\,\mathrm{m} = A\left(e^{0}\right)\cos(0+\phi)$$

$$.04 = A\cos\phi$$

$$u = \frac{dx}{dt} = Ae^{-\beta t}(-\omega_d)\sin(\omega_d t+\phi) + A(-\beta)e^{-\beta t}\cos(\omega_d t+\phi)$$

$$u(t=0) = 0 = \cancel{A}e^{0}(-\omega_d)\sin\phi + \cancel{A}(-\beta)e^{0}\cos\phi = 0$$

$$-\omega_d\sin\phi = \beta\cos\phi$$

$$\frac{\sin\phi}{\cos\phi} = \tan\phi = \frac{-\beta}{\omega_d} = \frac{-1\,\cancel{s^{-1}}}{9.85\,\cancel{s^{-1}}}$$

$$\phi = \tan^{-1}\left(\frac{1}{9.85}\right) = \boxed{-.101\,\mathrm{rad} = -5.8^{\circ}}$$

$$A = \frac{.04}{\cos(-.101\,\mathrm{rad})} = \boxed{.0402\,\mathrm{m}}$$

DAMPED FORCED OSCILLATOR 1.7.1

Note Title 8/22/2010

GIVEN: DAMPED FORCED OSCILLATOR $F_{APP} = F\cos\omega t$

REAL PART

FIND: $a(t)$, ω FOR a_{max}

ω FOR u_{max}

$\omega_{resonance}$

YOU COULD START WITH EQN 1.7.5 BUT LET'S REVIEW IT'S ORIGIN

E.O.M FROM N2L $\quad m\frac{d^2\tilde{x}}{dt^2} + R_m\frac{d\tilde{x}}{dt} + s x = Fe^{j\omega t}$

ASSUME

$$\tilde{x} = \tilde{A}e^{j\omega t}$$

$$\frac{d\tilde{x}}{dt} = j\omega\tilde{A}e^{j\omega t}$$

$$\frac{d^2\tilde{x}}{dt^2} = -\omega^2\tilde{A}e^{j\omega t}$$

$$\left(-m\omega^2\tilde{A} + jR_m\omega\tilde{A} + s\tilde{A}\right)e^{j\omega t} = Fe^{j\omega t}$$

$$\tilde{A} = \frac{F}{-m\omega^2 + s + jR_m\omega}$$

REARRANGING $$\tilde{A} = \frac{F}{j\omega\left(R_m + j\omega m - j\frac{s}{\omega}\right)}$$

$$= \frac{F}{j\omega\left(R_m + j\left(\omega m - \frac{s}{\omega}\right)\right)}$$

$$\tilde{x} = \tilde{A}e^{j\omega t} = \frac{Fe^{j\omega t}}{j\omega\left(R_m + j\left(\omega m - \frac{s}{\omega}\right)\right)}$$

$$\tilde{u} = \frac{d\tilde{x}}{dt} = \frac{Fe^{j\omega t}}{R_m + j\left(\omega m - \frac{s}{\omega}\right)}$$

IDENTIFY $Z_m = R_m + j\left(\omega m - \frac{s}{\omega}\right)$

BY DEFINITION $\omega_{res} m - \frac{s}{\omega_{res}} = 0$

$$\omega_{res} = \sqrt{\frac{s}{m}}$$

$$\boxed{\tilde{a} = \frac{d\tilde{u}}{dt} = \frac{j\omega Fe^{j\omega t}}{R_m + j\left(\omega m - \frac{s}{\omega}\right)}}$$

THIS IS A SIMPLE LOOKING RESULT, BUT WE WANT A REAL FUNCTION TO MAXIMIZE

i.e. $\tilde{a} = a_o e^{j\theta_{a_o}} e^{j\omega t}$

THEN $\frac{da_o}{d\omega} = 0$ WILL GIVE THE FREQ FOR MAX a

$$a_o = \left(\left[\frac{j\omega F}{R_m + j\left(\omega m - \frac{s}{\omega}\right)} \right] \left[\frac{j\omega F}{R_m + j\left(\omega m - \frac{s}{\omega}\right)} \right]^* \right)^{1/2}$$

$$= \left(\frac{(j\omega F)(-j\omega F)}{\left[R_m + j\left(\omega m - \frac{s}{\omega}\right)\right]\left[R_m - j\left(\omega m - \frac{s}{\omega}\right)\right]} \right)^{1/2}$$

$$= \left(\frac{\omega^2 F^2}{R_m^2 + \left(\omega m - \frac{s}{\omega}\right)^2} \right)^{1/2} = \frac{\omega F}{\left(R^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right)^{1/2}}$$

$$= \omega F \left(R_m^2 + \left(\omega m - \frac{s}{\omega}\right)^2 \right)^{-1/2}$$

$$\frac{da_0}{d\omega} = \omega \cancel{F}\left(-\frac{1}{\cancel{2}}\right)\left(R_m^2 + \left(\omega m - s\omega^{-1}\right)^2\right)^{-3/2} \cancel{2}\left(\omega m - s\omega^{-1}\right)\left(m + s\omega^{-2}\right)$$

$$+ \cancel{F}\left(R_m^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right)^{-1/2} = 0$$

$$= \frac{-\omega\left(\omega m - \frac{s}{\omega}\right)\left(m + \frac{s}{\omega^2}\right)}{\left(R_M^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right)^{3/2}} + \frac{1}{\left(R_M^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right)^{1/2}}$$

$$= \frac{-\omega\left(\omega m - \frac{s}{\omega}\right)\left(m + \frac{s}{\omega^2}\right) + R_m^2 + \left(\omega m - \frac{s}{\omega}\right)^2}{\left(R_m^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right)^{3/2}}$$

$$= \frac{\left(\omega m - \frac{s}{\omega}\right)\left(-\omega m - \frac{s}{\omega}\right) + R_m^2 + \left(\omega m - \frac{s}{\omega}\right)^2}{\left(R_m^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right)^{3/2}}$$

$$0 = -\cancel{\omega^2 m^2} - \cancel{ms} + \cancel{ms} + \frac{s^2}{\omega^2} + R_m^2 + \cancel{\omega^2 m^2} - 2ms + \frac{s^2}{\omega^2}$$

$$0 = \frac{2s^2}{\omega^2} - 2ms + R_m^2$$

$$\frac{2s^2}{\omega^2} = 2ms - R_m^2 \qquad \omega^2 = \frac{2s^2}{2ms - R_m^2}$$

$$\omega = \sqrt{\frac{2s^2}{2ms - R_m^2}} \quad \left(\begin{array}{l}\text{NOT } \omega_{res} \\ \text{UNLESS } R_m = 0\end{array}\right)$$

$$\omega = \sqrt{\frac{\cancel{2}s\cancel{2}}{\cancel{2}m\cancel{s}\left(1 - \frac{R_m^2}{2ms}\right)}} = \sqrt{\frac{s}{m}}^{1}\left(1 - \frac{R_m^2}{2ms}\right)^{-1/2}$$

$$= \omega_{res}\left(1 - \frac{R_m^2}{2ms}\right)^{-1/2}$$

FOLLOWING THE SAME APPROACH FOR VELOCITY

$$\tilde{u} = \frac{Fe^{j\omega t}}{R_m + j\left(\omega m - \frac{s}{\omega}\right)} = u_o e^{j\Theta_{u_o}} e^{j\omega t}$$

$$u_o = \left[\left(\frac{F}{R_m + j\left(\omega m - \frac{s}{\omega}\right)}\right)\left(\frac{F}{R_m - j\left(\omega m - \frac{s}{\omega}\right)}\right)\right]^{1/2} = \left[\frac{F^2}{R_m^2 + \left(\omega m - \frac{s}{\omega}\right)^2}\right]^{\frac{1}{2}}$$

$$= F\left(R_m^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right)^{-1/2}$$

$$\frac{du_o}{d\omega} = F\left(-\frac{1}{\cancel{2}}\right)\left(R_m^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right)^{-3/2} \cancel{2}\left(\omega m - \frac{s}{\omega}\right)\left(m + \frac{s}{\omega^2}\right)$$

$$0 = \left(\omega m - \frac{s}{\omega}\right)\left(m + \frac{s}{\omega^2}\right) = \omega m^2 + \frac{s\cancel{m}}{\cancel{\omega}} - \frac{s\cancel{m}}{\cancel{\omega}} - \frac{s^2}{\omega^3}$$

$$\omega m^2 = \frac{s^2}{\omega^3}$$

$$\omega^4 = \frac{s^2}{m^2} \Rightarrow \omega^2 = \frac{s}{m} = \omega_{res}^2$$