Homework Assignment #2

1. KFCS 1.8.2(a) Hint: $Re(e^{j\omega t}) = \cos(\omega t)$. In this problem you want the forcing function to be $\sin(\omega t)$. What complex exponential function has $\sin(\omega t)$ as its real part?

2. KFCS 1.10.4

3. KFCS 1.12.2

4. Given the following periodic function:

$$f(t) = \begin{cases} t & 0 < t < \frac{T}{2} \\ t - T & \frac{T}{2} < t < T \end{cases}$$

   a. Draw a graph of the function over several cycles.
   b. Represent this function by a Fourier series

(Hint: You can integrate over any period you choose. It does not have to be 0->T.

5. A Triangle wave is represented by the following periodic function:

$$f(t) = \begin{cases} -t & -\frac{T}{2} < t < 0 \\ t & 0 < t < \frac{T}{2} \end{cases}$$

   a. Draw a graph of the function over several cycles.
   b. Represent this function by a Fourier series

6. Verify Figure 1.15.2(b) by performing the Fourier Transform of the function in 1.15.2(a). What is the maximum value of the spectrum and at what frequencies does this maximum value occur?
Hint: Figure 1.15.2(a) is a train of 4 cosine functions with period T spanning from -2T to +2T and centered on t = 0. I recommend writing this cosine using complex notation prior to integrating.

1.8.2 GIVEN: $f = F \sin \omega t \qquad \omega \neq \omega_0$ UNDAMPED

FIND: a) $u(t)$ FOR NO DAMPING

c) FOR SMALL DAMPING SHOW $u = \frac{\omega F}{s} \cos \omega t$

STEADY STATE:
$$m \frac{d^2 \tilde{x}}{dt^2} + s\tilde{x} = -jFe^{j\omega t}$$

$$x = \tilde{A} e^{j\omega t}$$

$$\ddot{x} = -\omega^2 \tilde{A} e^{j\omega t}$$

$$-m\omega^2 \tilde{A} e^{j\omega t} + s\tilde{A} e^{j\omega t} = -jFe^{j\omega t}$$

$$\left(-m\omega^2 \tilde{A} + s\tilde{A} + jF\right) e^{j\omega t} = 0$$

$$\tilde{A}\left(s - m\omega^2\right) + jF = 0$$

$$\tilde{A} = \frac{-jF}{s - m\omega^2} = \frac{F}{j\omega\left(\frac{s}{\omega} - m\omega\right)}$$

$$= \frac{-F}{j\omega\left(m\omega - \frac{s}{\omega}\right)}$$

$$x = \frac{-F e^{j\omega t}}{j\omega\left(m\omega - \frac{s}{\omega}\right)}$$

$$u_{ss} = \frac{-F \cancel{j\omega} e^{j\omega t}}{\cancel{j\omega}\left(m\omega - \frac{s}{\omega}\right)} = -\frac{Fe^{j\omega t}}{m\omega - \frac{s}{\omega}}$$

TRANSIENT SOLN:
$$m \frac{d^2 \tilde{x}}{dt^2} + s\tilde{x} = 0$$

$$\tilde{x} = \tilde{A} e^{j\omega_0 t}$$

$$\ddot{\tilde{x}} = -\omega^2 \tilde{A} e^{j\omega_0 t}$$

$$-m\omega^2 \tilde{A} e^{j\omega_0 t} + s\tilde{A} e^{j\omega_0 t} = 0$$

$$\omega^2 = \frac{s}{m}$$

$$\tilde{x} = A e^{j(\omega_0 t + \phi)}$$

$$\dot{\tilde{x}} = u_t = jA\omega_0 e^{j(\omega_0 t + \phi)}$$

$$\tilde{x} = A e^{j(\omega_0 t + \phi)} - \frac{F e^{j\omega t}}{j\omega\left(m\omega - \frac{s}{\omega}\right)}$$

$$Re\left(\tilde{x}(0)\right) = Re\left(A e^{j\phi} - \frac{F}{j\omega\left(m\omega - \frac{s}{\omega}\right)}\right) = 0$$

$$A\cos\phi = 0 \Rightarrow \boxed{\phi = -\frac{\pi}{2}}$$

$$\tilde{u} = jA\omega_0 e^{j(\omega_0 t + \phi)} - \frac{F e^{j\omega t}}{m\omega - \frac{s}{\omega}}$$

$$Re\left(\hat{u}(0)\right) = -A\omega_0 \sin(\phi) - \frac{F}{m\omega - \frac{s}{\omega}} = 0$$

$$-A\omega_0(-1) = \frac{F}{m\omega - \frac{s}{\omega}}$$

$$A = \frac{F/\omega_0}{m\omega - \frac{s}{\omega}}$$

$$u = Re(\hat{u}) = -\frac{F/\cancel{\omega_0}}{m\omega - \frac{s}{\omega}}\cancel{\omega_0} \sin\left(\omega_0 t - \frac{\pi}{2}\right) - \frac{F}{m\omega - \frac{s}{\omega}} \cos\omega t$$

$$= \left(\frac{F}{m\omega - \frac{s}{\omega}}\right)\left[\cos\omega_0 t - \cos\omega t\right]$$

c. $\omega \ll \omega_0$ $R_m =$ SMALL STEADY STATE

$$m\frac{d^2\tilde{x}}{dt^2} + R_m\frac{d\tilde{x}}{dt} + s\tilde{x} = -jFe^{j\omega t}$$

$$\tilde{x} = \tilde{A}e^{j\omega t}$$

$$\dot{\tilde{x}} = j\omega\tilde{A}e^{j\omega t}$$

$$\ddot{\tilde{x}} = -\omega^2\tilde{A}e^{j\omega t}$$

$$-m\omega^2\tilde{A}e^{j\omega t} + j\omega R_m\tilde{A}e^{j\omega t} + s\tilde{A}e^{j\omega t} = -jFe^{j\omega t}$$

$$\tilde{A}\left(-m\omega^2 + j\omega R_m + s\right) + jF = 0$$

$$\tilde{A}\,j\omega\left(j\omega m + R_m - j\frac{s}{\omega}\right) = -jF$$

$$\tilde{A} = \frac{-\cancel{j}F}{\cancel{j}\omega\left[R_m + j\left(\omega m - \frac{s}{\omega}\right)\right]}$$

$$\tilde{x} = \frac{-F\,e^{j\omega t}}{\omega\left[R_m + j\left(\omega m - \frac{s}{\omega}\right)\right]}$$

$$\tilde{u} = \frac{-j\cancel{\omega}F\,e^{j\omega t}}{\cancel{\omega}\left[R_m + j\left(\omega m - \frac{s}{\omega}\right)\right]}$$

$R_m \rightarrow$ small $\quad \tilde{u} \approx \dfrac{-jFe^{j\omega t}}{j\left(\omega m - \frac{s}{\omega}\right)} \approx \dfrac{-Fe^{j\omega t}}{\omega m - \frac{s}{\omega}}$

$\omega \ll \omega_0$

$\omega^2 \ll \omega_0^2$

$\omega^2 \ll \frac{s}{m}$

$\omega m \ll \frac{s}{\omega}$

$\Rightarrow$

$$\tilde{u} \approx \frac{-Fe^{j\omega t}}{-\frac{s}{\omega}} \approx \frac{F\omega}{s}e^{j\omega t}$$

$$\boxed{u = Re(\tilde{u}) = \frac{F\omega}{s}\cos\omega t}$$

Q.E.D.

1.10.4 SHOW $\omega_{1/2} \approx \omega_0 \pm \frac{R_m}{2m}$

$$\Pi = \frac{F^2 R_m}{2 Z_m^2}$$

WITH $Z_m^2 = R_m^2 + \left(\omega m - \frac{s}{\omega}\right)^2$

AT RESONANCE $Z_m^2 = R_m^2$

AT ½ PWR POINT $Z_m^2 = 2 R_m^2$

$$\therefore \quad R_m^2 = X_m^2$$

$$\pm R_m = X_m = \omega m - \frac{s}{\omega}$$

$$\pm R_m \omega = \omega^2 m - s$$

$$m\omega^2 \mp R_m \omega - s = 0$$

$$\omega = \frac{\pm R_m \pm \sqrt{R_m^2 + 4ms}}{2m}$$

$$\omega = \pm \frac{R_m}{2m} \pm \sqrt{\frac{R_m^2}{4m^2} + \frac{\cancel{4}\cancel{m}s}{\cancel{4}m^{\cancel{2}}}}$$

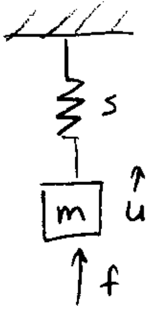
$$= \pm \frac{R_m}{2m} \pm \sqrt{\beta^2 + \omega_0^2}$$

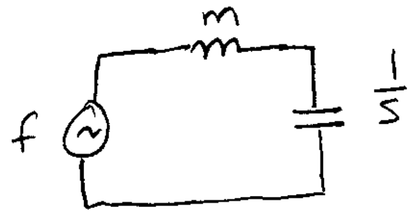
$$= \sqrt{\omega_0^2 + \beta^2} \pm \frac{R_m}{2m}$$

$\omega_0 \gg \beta$ $\approx \omega_0 \pm \frac{R_m}{2m}$ Q.E.D.

1.12.2

a)





$$m\frac{d^2x}{dt^2} + sx = f$$

$$-m\omega^2\tilde{A}e^{j\omega t} + s\tilde{A}e^{j\omega t} = fe^{j\omega t}$$

$$(-m\omega^2 + s)\tilde{A} = F$$

$$\tilde{A} = \frac{F}{-m\omega^2 + s}$$

$$x = \left(\frac{F}{-m\omega^2 + s}\right)e^{j\omega t}$$

$$\tilde{u} = \frac{j\omega F}{-m\omega^2 + s}e^{j\omega t}$$

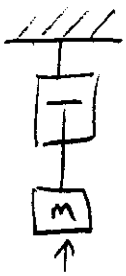
$$\frac{1}{\tilde{Z}} = \frac{j\omega}{-m\omega^2 + s} = \frac{j\cancel{\omega}}{\cancel{\omega}\left(-m\omega + \frac{s}{\omega}\right)} = \frac{-1}{j\left(-m\omega + \frac{s}{\omega}\right)}$$

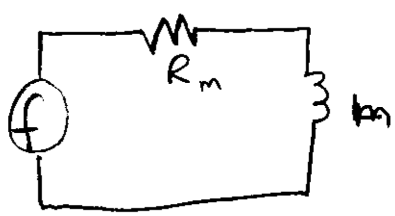
$$\boxed{\tilde{Z} = j\left(m\omega - \frac{s}{\omega}\right)}$$

$$\boxed{\omega_0^2 = \frac{s}{m}}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{s}{m}}$$

b)



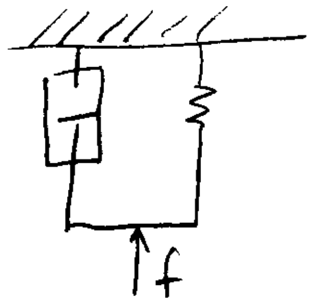


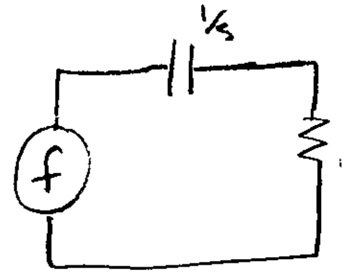
$$\boxed{Z_m = R_m + jm\omega}$$

THIS TIME, JUST WRITE THE IMPEDANCE BY INSPECTION

LR CIRCUIT DOES NOT VIBRATE $(\omega \to 0)$

c)



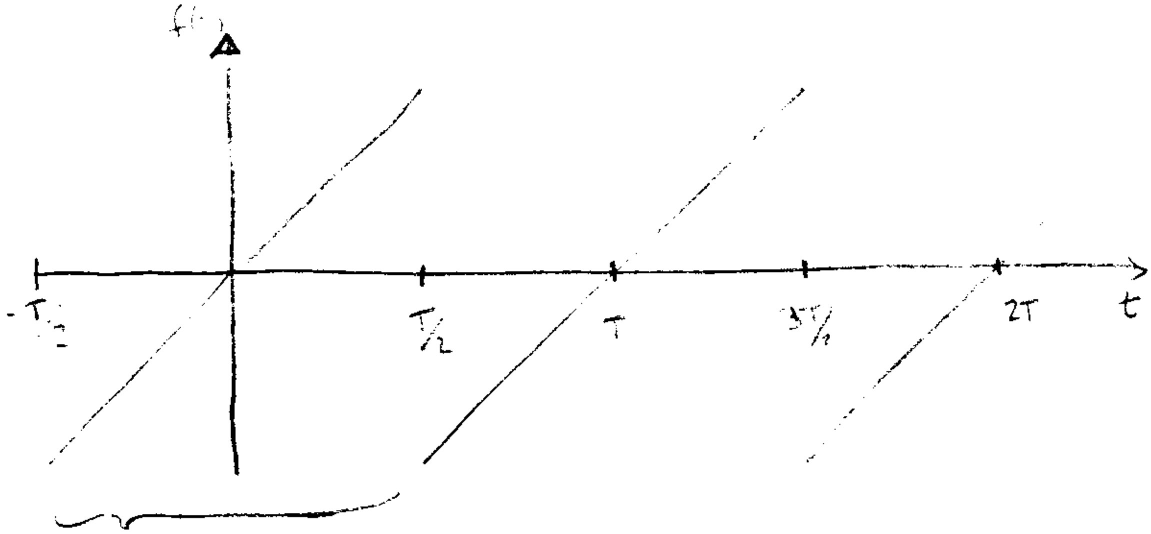


$$Z_m = R_m + \frac{s}{j\omega}$$

$$= R_m - j\frac{s}{\omega} \quad (\omega \to \infty)$$

RC CIRCUIT DOES NOT VIBRATE

4. $f(t) = \begin{cases} t & 0 < t < \frac{T}{2} \\ t - T & \frac{T}{2} < t < T \end{cases}$



Use Period From
$-T/2 \rightarrow T/2$

$$A_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} t\, dt = 0$$

$$A_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} t \cos(n\omega t)\, dt = 0$$

$$B_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} t \sin(n\omega t)\, dt \qquad \begin{matrix} u = t & dv = \sin n\omega t\, dt \\ du = dt & v = -\frac{1}{n\omega}\cos n\omega t \end{matrix}$$

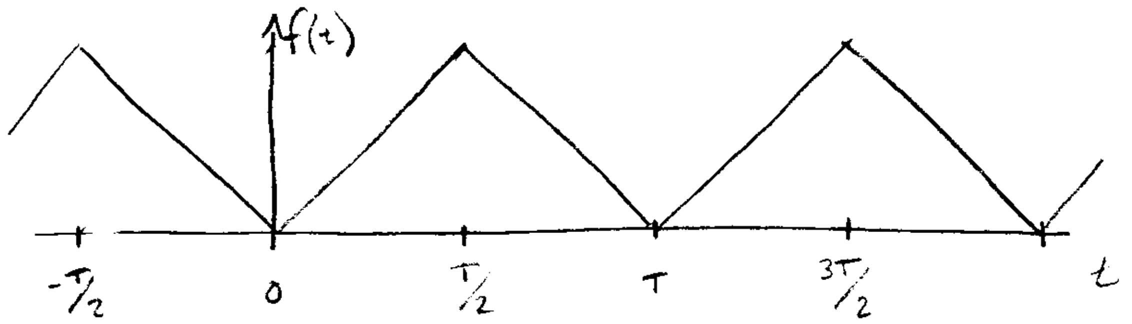
$$= \frac{2}{T}\left\{\left[-\frac{t}{n\omega}\cos n\frac{2\pi}{T}t\right]_{-\frac{T}{2}}^{\frac{T}{2}} + \frac{1}{n\omega}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos n\omega t\, dt\right\}$$

$$= \frac{2}{T}\left[-\frac{T}{2n\omega}\cos n\frac{2\pi}{T}\frac{T}{2} + \frac{-T}{2n\omega}\cos n\frac{2\pi}{T}\left(-\frac{T}{2}\right)\right]$$

$$= \frac{2}{T}\left[-\frac{T}{n\omega}\cos n\pi\right] = -\frac{2}{n\omega}\cos n\pi = -\frac{2T}{n2\pi}\cos n\pi$$

$$f(t) = \frac{T}{\pi}\left[\sin\omega t - \frac{\sin 2\omega t}{2} + \frac{\sin 3\omega t}{3} - \dots\right]$$

5. $f(t) = \begin{cases} -t & -\frac{T}{2} < t < 0 \\ t & 0 < t < \frac{T}{2} \end{cases}$



$$A_0 = \frac{2}{T}\left[\int_{-\frac{T}{2}}^{0} -t\, dt + \int_0^{T/2} t\, dt\right] = \frac{2}{T}\left[\left(-\frac{t^2}{2}\right)\Big|_{-T/2}^{0} + \left(\frac{t^2}{2}\right)\Big|_0^{T/2}\right]$$

$$= \frac{2}{T}\left(0 - \frac{-T^2/4}{2} + \frac{T^2/4}{2}\right) = \frac{2}{T}\,\frac{T^2}{4\cdot 2} = \frac{T}{2}$$

$$A_n = \frac{2}{T}\left[-\int_{-\frac{T}{2}}^{0} t \cos n\omega t\, dt + \int_0^{T/2} t \cos n\omega t\, dt\right]$$

$$= \frac{2}{T}\left[2\int_0^{T/2} t \cos n\omega t\, dt\right] \qquad \begin{matrix} u = t & dv = \cos n\omega t\, dt \\ du = dt & v = \frac{1}{n\omega}\sin n\omega t \end{matrix}$$

$$= \frac{4}{T}\left[\frac{t}{n\omega}\sin n\frac{2\pi}{T}t\,\Big|_0^{T/2} - \frac{1}{n\omega}\int_0^{T/2} \sin n\omega t\, dt\right]$$

$$= \frac{4}{T}\left[0 - 0 - \frac{1}{n\omega}\left(-\frac{1}{n\omega}\right)\cos n\omega t\,\Big|_0^{T/2}\right]$$

$$= \frac{4}{T}\,\frac{1}{n^2\frac{(2\pi)^2}{T^2}}\left[\cos n\left(\frac{2\pi}{T}\right)\frac{T}{2} - 1\right] = \frac{T}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$n = 0, 2, 4, \dots \qquad A_n = 0$

$n = 1, 3, 5, \dots \qquad A_n = -\frac{2T}{n^2\pi^2}$

$B_n = 0$ BECAUSE $f(t)$ IS AN <u>EVEN</u> FUNCTION

$$f(t) = \frac{T}{4} - \sum_{n = \text{ODD}} \frac{2T}{n^2\pi^2}\cos n\omega t$$

FOURIER TRANSFORM VERIFY 1.15.2(b)

Note Title 8/22/2010

SHOW 1.15.2(b) IS THE FT OF 1.15.2(a)

$$f(t) = \cos(\omega_0 t) \qquad \text{WITH } \omega_0 = \frac{2\pi}{T}$$

FOR $-2T < t < 2T$

$$\therefore \quad g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-2T}^{2T} \cos(\omega_0 t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-2T}^{2T} \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-j\omega t}\, dt$$

$$= \frac{1}{4\pi}\int_{-2T}^{2T} \left[e^{-j(\omega-\omega_0)t} + e^{-j(\omega+\omega_0)t}\right] dt$$

$$= \frac{1}{4\pi}\left[\frac{e^{-j(\omega-\omega_0)t}}{-j(\omega-\omega_0)} + \frac{e^{-j(\omega+\omega_0)t}}{-j(\omega+\omega_0)}\right]_{-2T}^{2T}$$

$$= \frac{1}{4\pi}\left[\frac{e^{-j(\omega-\omega_0)2T} - e^{j(\omega-\omega_0)2T}}{-j(\omega-\omega_0)} + \frac{e^{-j(\omega+\omega_0)2T} - e^{j(\omega+\omega_0)2T}}{-j(\omega+\omega_0)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j(\omega-\omega_0)2T} - e^{-j(\omega-\omega_0)2T}}{2j\,(\omega-\omega_0)} + \frac{e^{j(\omega+\omega_0)2T} - e^{-j(\omega+\omega_0)2T}}{2j\,(\omega+\omega_0)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\sin[(\omega-\omega_0)2T]}{\omega-\omega_0} + \frac{\sin[(\omega+\omega_0)2T]}{\omega+\omega_0} \right]$$

$$= \frac{2T}{2\pi}\left[\frac{\sin[(\omega-\omega_0)2T]}{(\omega-\omega_0)2T} + \frac{\sin[(\omega+\omega_0)2T]}{(\omega+\omega_0)2T} \right]$$

$$= \frac{T}{\pi}\left[\operatorname{sinc}[\omega-\omega_0]2T + \operatorname{sinc}[(\omega+\omega_0)2T] \right]$$

SINC CENTERED AT $\omega = \omega_0$

SINC CENTERED AT $\omega = -\omega_0$

MAX AMPLITUDE OF SINC