SP212 Blurb 11 2010

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Serway Chapter 32- Self Inductance

We know that a strong magnetic field develops inside a current carrying solenoid. Now suppose we decide to change the current in the solenoid, which will start to change the magnetic field causing a changing flux in each turn of the solenoid. This causes an emf to be generated on each turn of the solenoid that opposes the change in magnetic flux accordingto Lenz' law. Thus, the emf generated in the coil is proportional to the rate of change of current.

$$\mathcal{E}_L \propto \left(-\frac{dI}{dt}\right) = \mathcal{E}_L = -L\frac{dI}{dt} \quad \left(\text{Compare to } \mathcal{E} = \frac{-d\Phi_B}{dt}\right)$$

where L is a proportionality constant that depends on geometry and material properties.

$$-L\frac{dI}{dt} = \frac{-Nd\Phi_B}{dt} \Rightarrow \frac{d}{dt}(LI) = \frac{d}{dt}(N\Phi)$$

$$\Rightarrow \boxed{L = \frac{N\Phi_B}{I}}$$

L is called the self inductance of the coil and has units, $Tm^2/Amp \equiv$ Henry (1H)

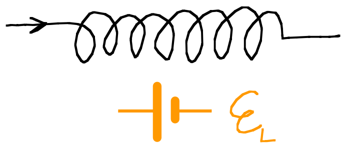
Example: The self inductance of a solenoid can be found using the solenoid design equation, $B = \mu_0 nI$

$$L = \frac{N\Phi_B}{I} = \frac{NBA}{I} = \frac{N\mu_0\frac{N}{\ell}IA}{I}$$

$$\Rightarrow \boxed{L_{solenoid} = \frac{\mu_0 N^2 A}{\ell}} = \frac{\mu_0 n^2 \ell^2 A}{\ell} = \mu_0 n^2 A\ell = \mu_0 n^2 \text{Vol}$$

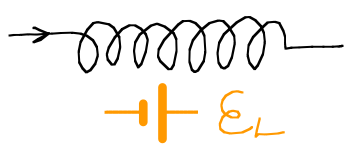
Inductance Interpretation

I (increasing)



$\mathcal{E}_L$ opposes increasing

I (decreasing)

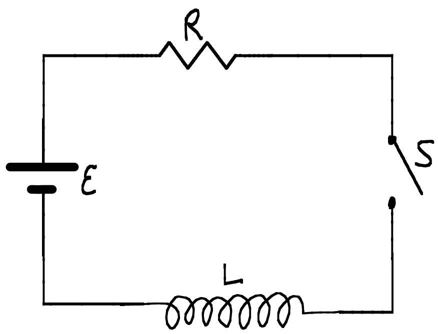


$\mathcal{E}_L$ supports decreasing current

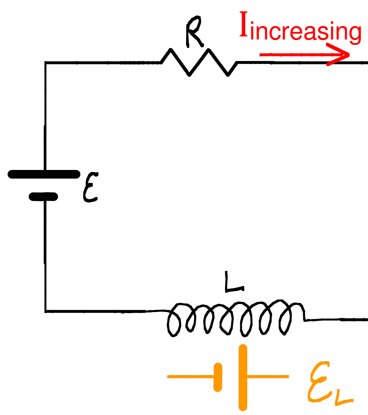
So, the self inductance adds a kind of inertia to the circuit.

While the solenoid has a large magnetic flux inside it, the rest of the circuit also has a magnetic field. Thus, every circuit has self inductance, though a coil in a circuit will probably have most of it. The self inductance makes it impossible to instantly change current in any circuit because very rapid changes in current cause large changes in magnetic flux, which causes large emfs to oppose the

The LR Circuit



If we put a coil in a circuit most of the self inductance is in the coil. Now, close the switch S at t = 0 and let's find out how the circuit behaves. That is, let's



Just after the switch is closed, the current is increasing, causing an emf to appear across the inductor that opposes the battery emf.

Using Kirchhoff's Loop Rule,

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0 \Rightarrow \mathcal{E} - IR = L\frac{dI}{dt} \Rightarrow dt = \frac{L\,dI}{\mathcal{E} - IR}$$

$$\Rightarrow dt = \frac{-L}{R}\left(\frac{dI}{I - \mathcal{E}/R}\right) \Rightarrow \frac{-dt}{(L/R)} = \frac{dI}{I - \mathcal{E}/R} \Rightarrow \int_0^I \frac{dI}{I - \mathcal{E}/R} = \frac{-1}{(L/R)}\int_0^t dt$$

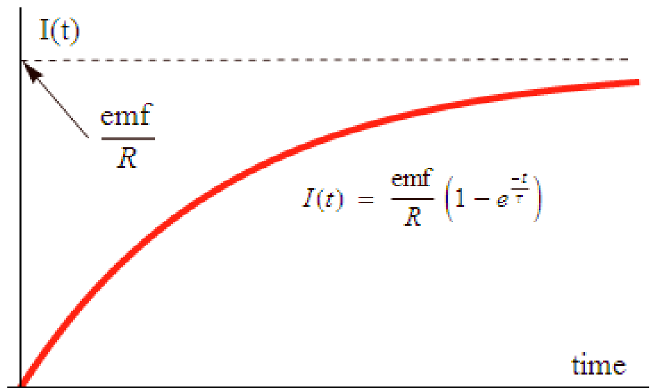
$$\Rightarrow \ln(I - \mathcal{E}/R)\Big|_0^I = -t/(L/R) \Rightarrow \ln(I - \mathcal{E}/R) - \ln(-\mathcal{E}/R) = -t/(L/R)$$

$$\Rightarrow \ln\left(\frac{I - \mathcal{E}/R}{-\mathcal{E}/R}\right) = \ln\left(\frac{\mathcal{E}/R - I}{\mathcal{E}/R}\right) = -t/(L/R)$$

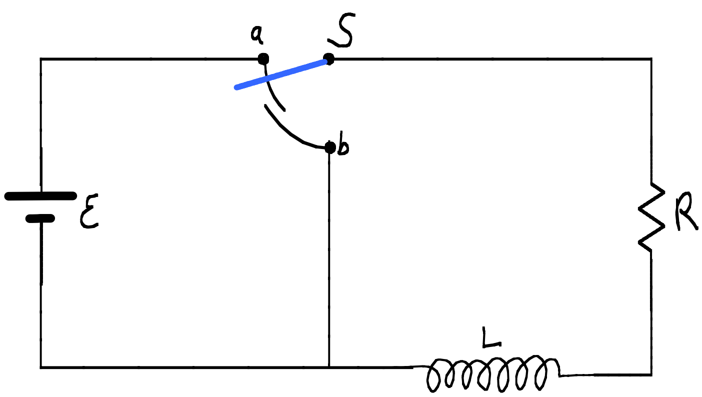
$$\Rightarrow e^{\ln\left(\frac{\mathcal{E}/R - I}{\mathcal{E}/R}\right)} = e^{-t/(L/R)}$$

$$\Rightarrow \frac{\mathcal{E}/R - I}{\mathcal{E}/R} = e^{-t/(L/R)} \Rightarrow \frac{\mathcal{E}}{R} - I = \frac{\mathcal{E}}{R} e^{-t/(L/R)}$$

$$\boxed{I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/(L/R)}\right)} \quad \tau \equiv \frac{L}{R}$$ Time constant for an L-R circuit.

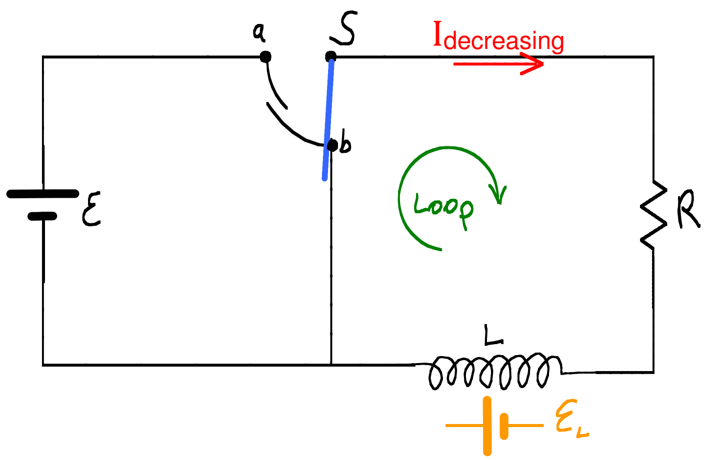


Let's modify our circuit so we can replace the battery with a wire to allow the current to go to zero.



When S is in position a the battery drives the current in the LR circuit around the outer conducting path and after many time constants the current is ε/R.

When the switch is moved to position b the battery branch of the circuit is replaced by a wire so that current can start to decay. Let's apply Kirchhoff's loop rule to the loop on the right.



$$-L\frac{dI}{dt} - IR = 0$$

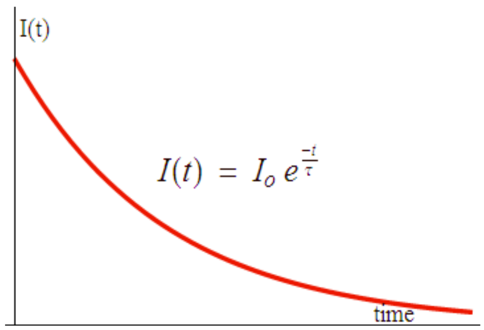
$$L\frac{dI}{dt} = -IR$$

$$\Rightarrow \frac{dI}{I} = \frac{-dt}{(L/R)} \Rightarrow \int_{I_0}^{I}\frac{dI}{I} = \frac{-1}{(L/R)}\int_0^t dt$$

$$\ln\frac{I}{I_0} = -t/(L/R)$$

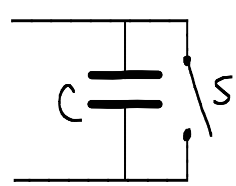
$$\Rightarrow e^{\ln I/I_0} = e^{-t/\tau} \quad ; \ \tau = L/R$$

$$\Rightarrow \boxed{I(t) = I_0 e^{-t/\tau}}$$



So, the inductor prevents instantaneous current change. Since every circuit has self inductance, no circuit can change it's current instantaneously!

If we do something to cause a rapid change in current, like opening a switch, large emfs can be generated that can cause a spark at the switch. To prevent this capacitors can be placed across the switch to accept a charge rather than generating a spark. This sort of switch is often used in volatile surroundings like grain elevators or refineries, maybe even engine rooms on ships.



Energy in a magnetic field

Let's revisit our original LR circuit and recall,

$$\varepsilon = IR + L\frac{dI}{dt}$$

now multiply this equation by I and interpret each term.

$$\underbrace{\mathcal{E}I}_{\text{Power delivered by battery}} = \underbrace{I^2R}_{\text{Power dissipated by resistor}} + \underbrace{LI\frac{dI}{dt}}_{\text{Power delivered to inductor}}$$

Let U represent energy delivered to L. So,

$$\frac{dU}{dt} = LI\frac{dI}{dt} \Rightarrow dU = LI\,dI$$

$$\Rightarrow \int_0^U dU = L\int_0^I I\,dI \Rightarrow \boxed{U = \frac{1}{2}LI^2}$$

The energy stored in the inductor is ½LI². Where is U stored? In the B field! Let's put I in terms of B by assuming our inductor is a solenoid. Recall, $L_{solenoid} = \mu_o n^2 Vol$ and $B = \mu_o nI$.

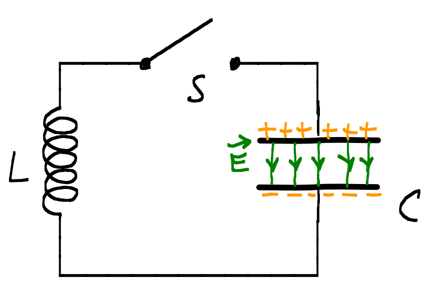
$$\Rightarrow U = \frac{1}{2}LI^2 = \frac{1}{2}(\cancel{\mu_0}\cancel{n^2}Vol)\frac{B^2}{\mu_0^{\cancel{2}}\cancel{n^2}}$$

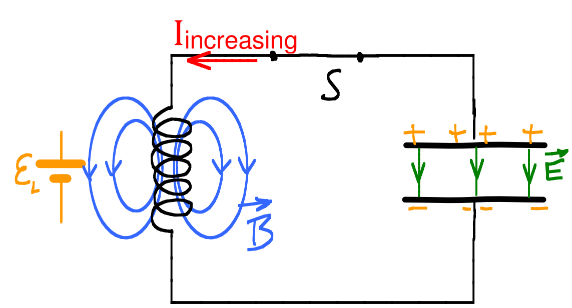
$$\Rightarrow \frac{U}{Vol} = \boxed{\frac{1}{2\mu_0}B^2 = u_B}$$ Energy density of the magnetic field.

Which is valid for any magnetic field, not just inside a solenoid!

The LC Circuit

Suppose we build a circuit with a charged capacitor and an inductor and close switch, S, at time t = 0.





Shortly after the switch, S, is closed.

$-L\dfrac{dI}{dt}+\dfrac{Q}{C}=0$ From the loop rule. Now use $I=-\dfrac{dQ}{dt}$ to get,

$$L\frac{d^2Q}{dt^2}+\frac{Q}{C}=0$$

$$\Rightarrow \frac{d^2Q}{dt^2}+\frac{1}{LC}Q=0$$

Let's try $Q(t)=Q_0\cos(\omega t+\phi)$

$$\Rightarrow \frac{dQ}{dt}=-Q_0\omega\sin(\omega t+\phi)$$

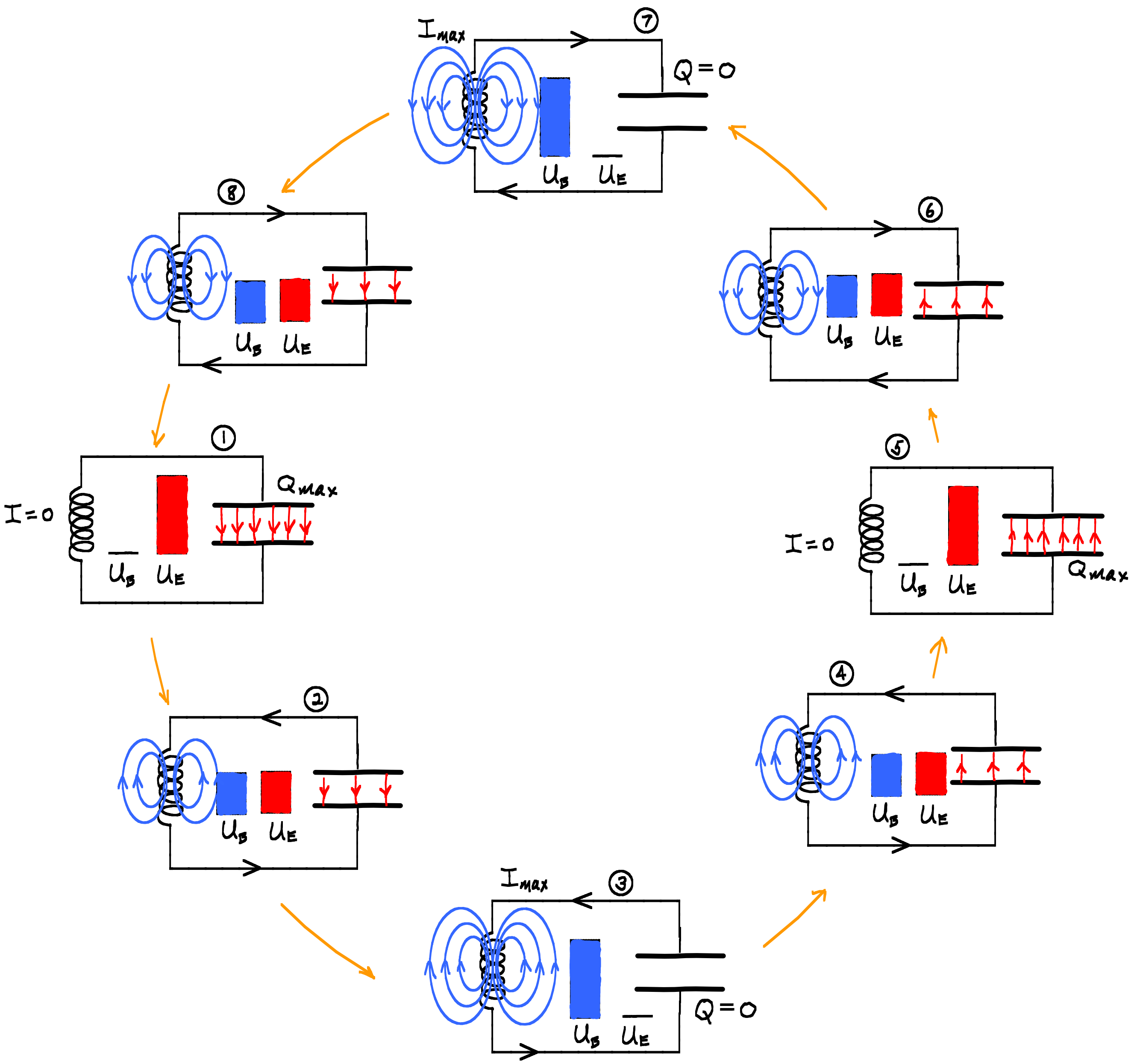
$$\Rightarrow \frac{d^2Q}{dt^2}=-Q_0\omega^2\cos(\omega t+\phi)$$

$$\Rightarrow -\cancel{Q_0}\omega^2\cancel{\cos(\omega t+\phi)}+\frac{1}{LC}\cancel{Q_0}\cancel{\cos(\omega t+\phi)}=0$$

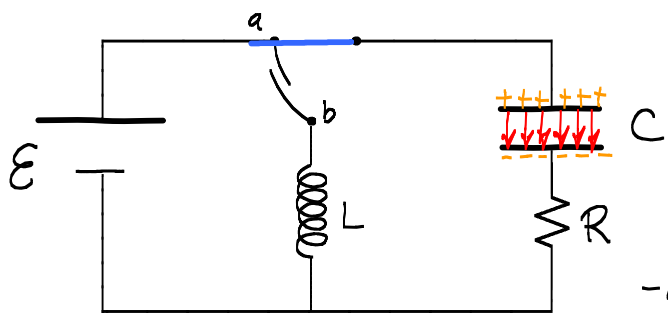
$$\Rightarrow \omega^2=\frac{1}{LC} \Rightarrow \omega=\frac{1}{\sqrt{LC}}=\frac{2\pi}{T}$$

$$T=2\pi\sqrt{LC}$$

So, our analysis tells us that the charge on the capacitor oscillates indefinitely with a period given by the formula above. With no energy loss mechanism in the circuit, zero charge on the capacitor must coincide with maximum current in the inductor to conserve energy. The cyclical exchange of energy between the capacitor and the inductor is characterized in the following diagram. Total energy is the same in every frame, as conservation of energy demands.


⑦
I_{max}
$Q=0$
U_B U_E
⑧
U_B U_E
⑥
U_B U_E
①
$I=0$
Q_{max}
U_B U_E
⑤
$I=0$
Q_{max}
U_B U_E
④
U_B U_E
②
U_B U_E
③
I_{max}
$Q=0$
U_B U_E

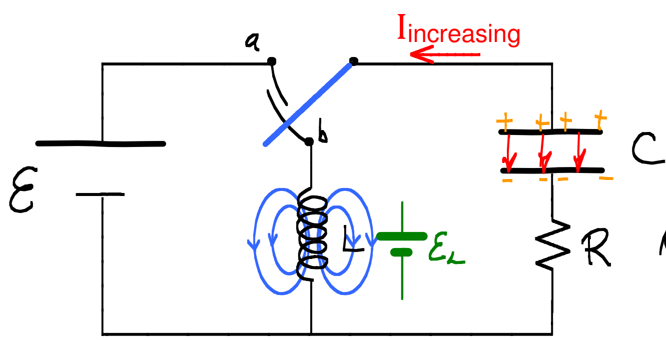
The RLC Circuit - Now put a resistor in our LC circuit.



Place S at a for a long time, then switch to b and start our analysis at t = 0.

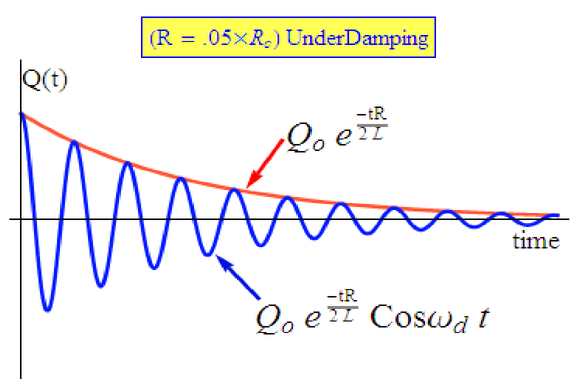
$$-L\frac{dI}{dt} - IR + \frac{Q}{C} = 0 \quad ; \quad I = -\frac{dQ}{dt}$$

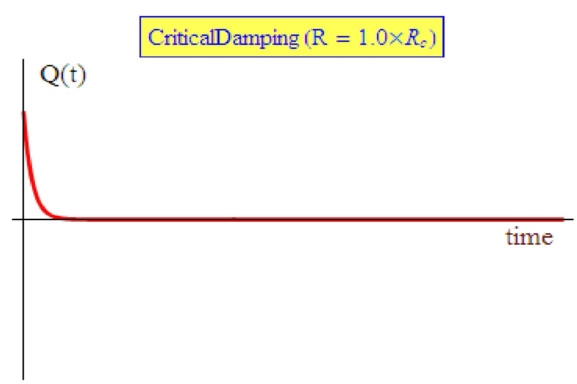
$$\Rightarrow L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

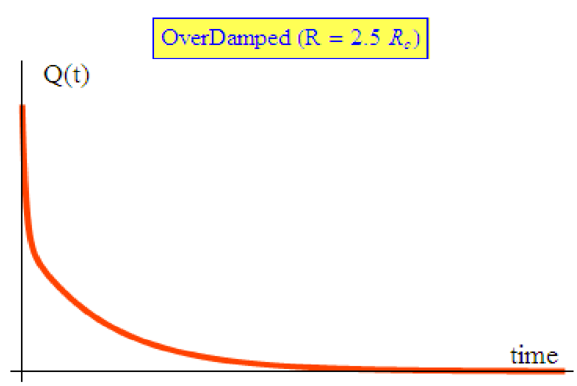


$$Q(t) = Q_{max}\, e^{-t/2(L/R)} \cos(\omega_d t)$$

where $\omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}$

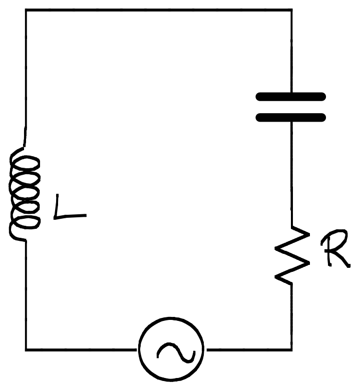






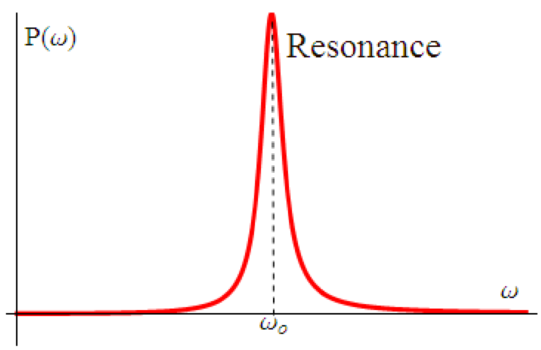
Note that critical damping requires the least time for Q to go to zero.

RLC Application



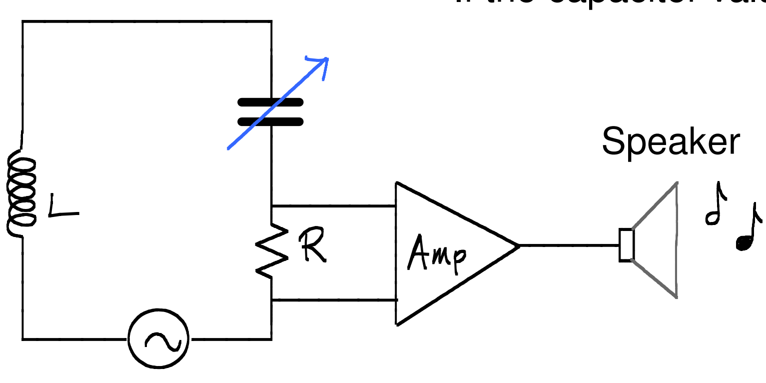
When the frequency of the AC emf source, ω, matches the natural frequency of the circuit, ω_o, large current results. This causes large power dissipation in the resistor.

$$\omega_o = \frac{1}{\sqrt{LC}}$$



The resonance plot at left shows the power dissipated in the resistor as a function of the driving frequency ω_o. When $\omega = \omega_o$, the resonance condition, the power dissipated in the resistor is maximized.

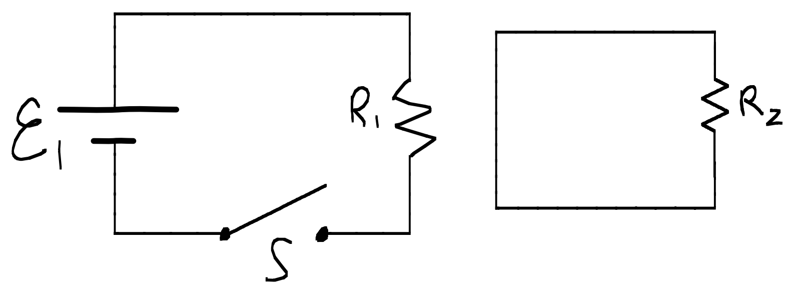
If the capacitor value, C, can be varied then



The LRC circuit is the tuning section of a

Mutual Inductance

Imagine two closely spaced circuits labeled circuit 1 and circuit 2.



If the switch S is closed in circuit 1, B field lines will start to appear in both circuits.The changing magnetic flux will produce a back emf in circuit 1 characterizedby its self inductance, as we have seen. The changing magnetic flux through circuit 2, due to the changing current in circuit 1 also produces an emf and associated current in circuit 2. We summarize this situation by stating,

$$|\mathcal{E}_2| \propto \frac{dI_1}{dt} \Rightarrow \mathcal{E}_2 = -M\frac{dI_1}{dt}$$

where M ≡ mutual inductance of the circuits.

Comparing to $\mathcal{E} = -L\frac{dI}{dt}$ we see M and L have the same units, namely, Henries.

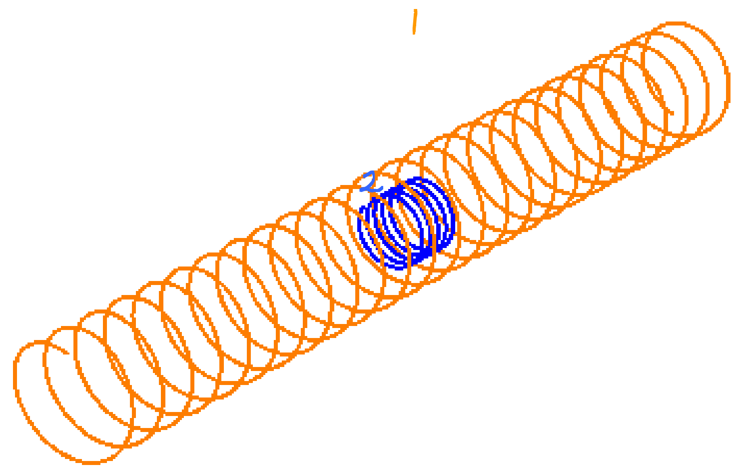
Of course, a changing current in circuit 2 produces a changing flux and associated emf in circuit 1.

$$\Rightarrow |\mathcal{E}_1| \propto \frac{dI_2}{dt}$$

It turns out the mutual inductance in each case is the same, though we won't prove it in this course. So,

$\mathcal{E}_1 = -M\frac{dI_2}{dt}$ uses the same M.

Example: Suppose a short, circular coil with N_c turns and radius r is inside a long solenoid with N turns, length L, and radius R. Find the mutual inductance of the coils.



$$\mathcal{E}_2 = -M\frac{dI_1}{dt}; \qquad |\mathcal{E}_2| = N_2\frac{d\Phi_2}{dt} = N_2\frac{d}{dt}(BA_2)$$

$$\Rightarrow |\mathcal{E}_2| = N_2A_2\frac{dB}{dt} = N_2A_2\frac{d}{dt}\left(\mu_0\frac{N_1}{L}I_1\right)$$

$$\Rightarrow |\mathcal{E}_2| = N_2A_2\mu_0\frac{N_1}{L}\frac{dI_1}{dt} = M\frac{dI_1}{dt}$$

$$\Rightarrow \boxed{M = \frac{\mu_0 N_1N_2A_2}{L}}$$

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