SP212 Blurb 12 2010

1. Maxwell's displacement current

2. Maxwell's Equations

3. Wave equation for E & B

4. Electromagnetic waves

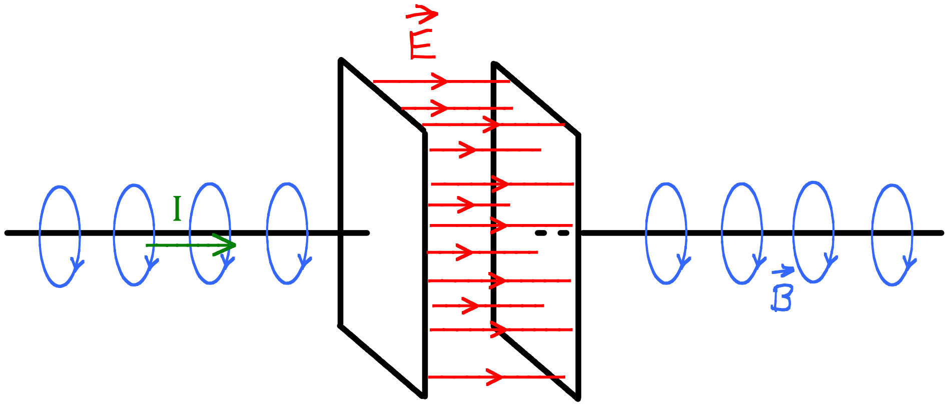
5. Relating E_{max} to B_{max}

6. The Poynting vector

7. Radiation pressure

Maxwell's Displacement Current

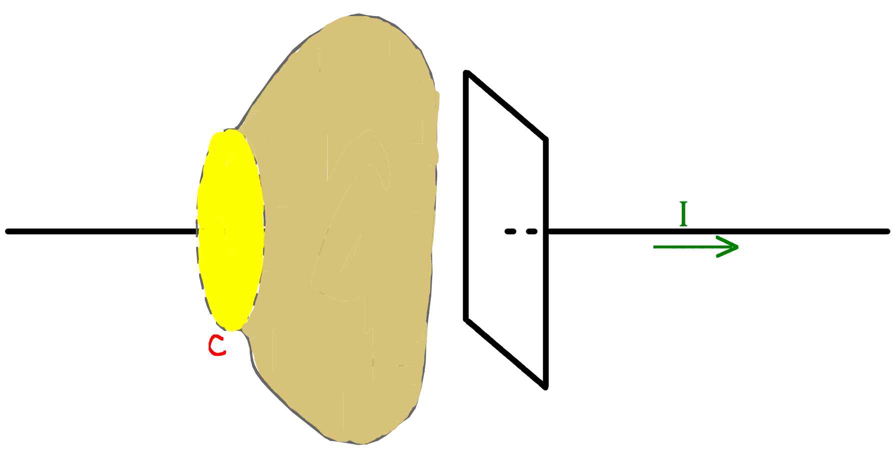
Consider current flowing into a capacitor. As the current flows we know there is a magnetic field around the wire and the electric field between the plates is increasing.



Recall that we can calculate B around the wire from Ampere's law,

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I$$

where I is the current that cuts through the area bounded by the Amperian curve.



In the past we've used the flat yellow area bounded by C to determine I in Ampere's law, but the brown area is also bounded by curve C and seems equally valid. The integral on the left side of Ampere's law should be the same for each area and clearly isn't since no charge crosses the brown surface.

J.C. Maxwell first noticed this problem with Ampere's law and proposed a solution. Namely, Ampere's law, as we've known it, is incomplete and should

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt}$$

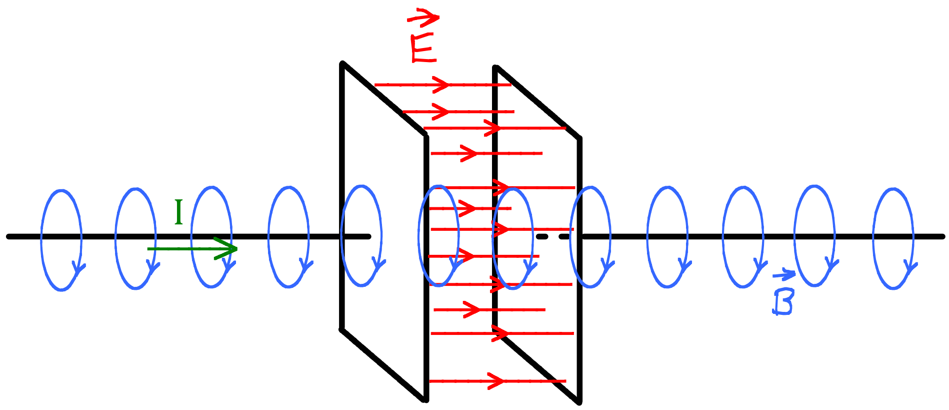
where $\Phi_E = \int \vec{E}\cdot d\vec{a}$ is the electric flux through the area bounded by C.

We define a displacement current I_d, $\quad I_d = \epsilon_0 \frac{d\Phi_E}{dt}$

So the new form of Ampere's Law is,

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 (I + I_d)$$

Outside the capacitor we use I and inside the capacitor, where E is changing, we use I_d.



Now we realize B is the same inside the capacitor as it is outside around the wire. This has profound significance! We already knew that changing magnetic flux produces E field from Faraday's law. Now the corrected form of Ampere's law is telling us that changing electric flux produces magnetic field. You can't change one without producing the other!

Electromagnetic Waves: were first created by electrical means by Heinrich Hertz in1887. From the Maxwell equations with the time derivatives and assuming a medium with no free charge so $I = 0$ we can prove,

$$① \ \frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \quad \text{and} \quad ② \ \frac{\partial B}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E}{\partial t} \qquad \text{Apply } \frac{\partial}{\partial x} \text{ to } ① \text{ and } \frac{\partial}{\partial t} \text{ to } ②$$

$$③ \ \frac{\partial^2 E}{\partial x^2} = \frac{-\partial^2 B}{\partial x \partial t} \quad \text{and} \quad ④ \ \frac{\partial^2 B}{\partial x \partial t} = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \quad \Rightarrow \quad ⑤ \ \boxed{\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}}$$

Maxwell knew about waves on strings, which obey,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

(This is called the wave equation for a wave on a string and can be derived using Newton's laws.)

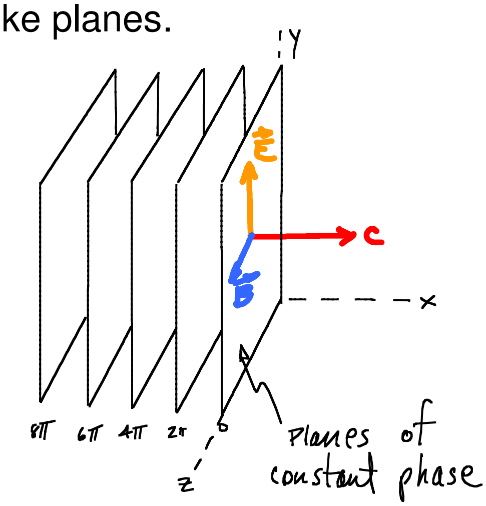
So, Maxwell realized he had a wave equation for electromagnetic waves and by by comparing the last 2 equations he concluded ....

$$⑥ \ \frac{1}{v^2} = \mu_0 \epsilon_0 \quad \Rightarrow \quad v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \simeq 3.0 \times 10^8 \, m/s$$

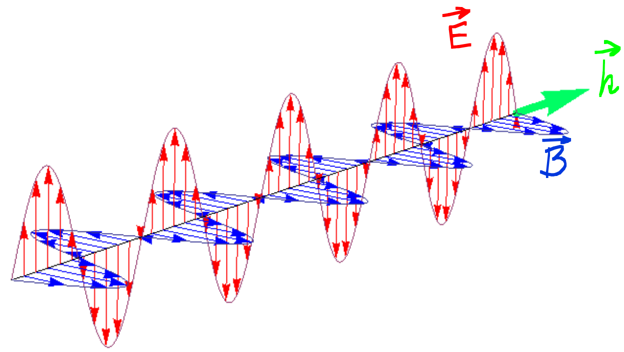
which compared well to the estimated speed of light at the time. So, he postulated that light consists of coupled electric and magnetic waves.

Plane waves

Waves radiating from a point have surfaces of constant phase that are spherical. If we let the spherical waves propagate to infinity then finite sections of the wavefronts look like planes.



From Maxwell's equations we can show that **E**, **B** and **k** form an orthogonal vector triad and that **E** x **B** is in the same direction as **k**.



The simplest solutions to the wave equations for **E** and **B** are,

⑦ $E(x,t) = E_{max} \cos(kx - \omega t)$

⑧ $B(x,t) = B_{max} \cos(kx - \omega t)$

⑨ $\frac{\partial E}{\partial x} = -k E_{max} \sin(kx - \omega t)$

⑩ $\frac{\partial B}{\partial t} = B_{max}\, \omega \sin(kx - \omega t)$

⑪ $-E_{max}\, k \sin(kx - \omega t) = -B_{max}\, \omega \sin(kx - \omega t)$

⑫ $E_{max}\, k = B_{max}\, \omega \Rightarrow$ ⑬ $\frac{E_{max}}{B_{max}} = \frac{\omega}{k} = c$

So if we know E_{max} we can get B_{max} from, $\boxed{B_{max} = E_{max}/c}$

Energy Density in E-M waves

For an electric field, $u_E = \frac{1}{2}\varepsilon_o E^2$

For a magnetic field, $u_B = (1/(2\mu_o))B^2$

Since an electromagnetic field contains both,

$$u_{EM} = u_E + u_B = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2$$

$$= \frac{1}{2}\epsilon_0 E_{max}^2 \cos^2(kx-\omega t) + \frac{1}{2\mu_0} B_{max}^2 \cos^2(kx-\omega t)$$

$$= \frac{1}{2}\epsilon_0 E_{max}^2 \cos^2(kx-\omega t) + \frac{1}{2\mu_0} \frac{E_{max}^2}{c^2} \cos^2(kx-\omega t)$$

Now use $\frac{1}{c^2} = \mu_0 \epsilon_0$ in the second term above

$$\Rightarrow u_{EM} = \frac{1}{2}\epsilon_0 E_{max}^2 \cos^2(kx-\omega t) + \frac{1}{2\cancel{\mu_0}} \cancel{\mu_0} \epsilon_0 E_{max}^2 \cos^2(kx-\omega t)$$

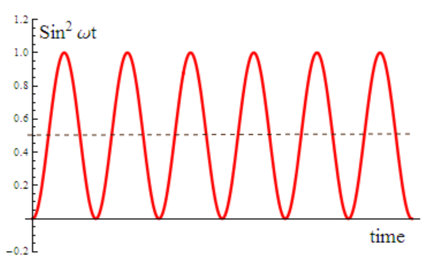
Since the second term above equals the first, the energy in the **B** part of the e-m wave is the same as the energy in the **E** part of the wave.

$$u_{EM} = \epsilon_0 E_{max}^2 \cos^2(kx-\omega t)$$

For visible light waves there are no detectors fast enough to detect the individual waves. Thus, we're usually interested in time averages of energy related quantities. (Imagine yourself in sunlight. You don't really care that sunlight is oscillating at around 10^{14} Hz. What you care about is the average rate energy is being deposited on your skin.)

$$\langle u_{em} \rangle = \langle \epsilon_0 E_{max}^2 \cos^2(kx-\omega t) \rangle = \epsilon_0 E_{max}^2 \langle \cos^2(kx-\omega t) \rangle$$

Now let $x = 0 \Rightarrow \langle u_{em} \rangle = \epsilon_0 E_{max}^2 \langle \cos^2 \omega t \rangle$ < > denotes time average



It's easy to see $\langle \sin^2\omega t \rangle = ½$ from the graph.

If you need more convincing,
$\sin^2\omega t + \cos^2\omega t = 1 \Rightarrow \langle \sin^2\omega t \rangle + \langle \cos^2\omega t \rangle = 1$,
but $\langle \sin^2\omega t \rangle = \langle \cos^2\omega t \rangle$, so each must equal ½ .

$$\Rightarrow \boxed{\langle u_{EM} \rangle = \frac{1}{2}\epsilon_0 E_{max}^2}$$

Poynting vector

Note that energy is carried in the direction of **E** x **B**. Mr. Poynting noticed this too!

Definition of Poynting Vector, **S** $\vec{S} \equiv \frac{1}{\mu_0}\vec{E}\times\vec{B}$

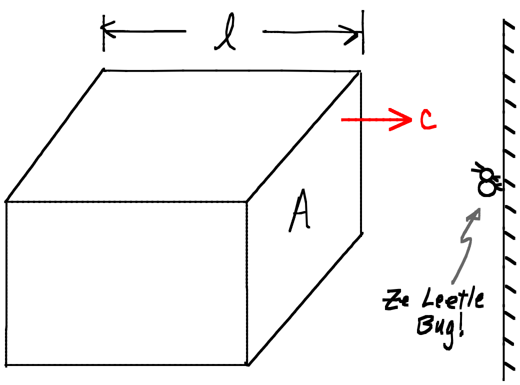
Assuming our wave has the form of the plane wave above,

$$\vec{S} = \frac{1}{\mu_0} E_{max} B_{max} \cos^2(kx-\omega t)\,\hat{i}$$

$$\langle|\vec{S}|\rangle = \frac{1}{2\mu_0} E_{max} \frac{E_{max}}{c} = \frac{1}{2}\epsilon_0 \frac{1}{\mu_0\epsilon_0 c} E_{max}^2 = \frac{1}{2}\epsilon_0 \frac{c^2}{c} E_{max}^2 = \left(\frac{1}{2}\epsilon_0 E_{max}^2\right) c$$

$$\Rightarrow \langle|\vec{S}|\rangle = \langle u_{EM}\rangle c \equiv I \text{ (intensity)}$$

Imagine a little bug clinging to a wall with a box of electromagnetic energy about to slam into him!



If the energy is spread out over a large area or if the box is long and the energy gets deposited over a long time, the bug doesn't get burned. So, the bug is very interested in energy/(time*area).

$$\text{Energy} = \langle u_{EM}\rangle \text{Vol} = \langle u_{EM}\rangle A l \quad \text{but } l = c\Delta t$$

$$\Rightarrow \frac{\text{Energy}}{\text{time} * \text{Area}} = \frac{\langle u_{EM}\rangle \cancel{A} c \cancel{\Delta t}}{\cancel{\Delta t}\,\cancel{A}} = \langle u_{EM}\rangle c = I = \langle|\vec{S}|\rangle$$

Radiation Pressure

When E-M energy, U gets absorbed it transfers momentum p. U and p are related by,

$$p = \frac{U}{c} \Rightarrow \Delta p = \frac{\Delta U}{c} \Rightarrow \frac{\Delta p}{\Delta t} = \frac{1}{c}\frac{\Delta U}{\Delta t}$$

$$\Rightarrow F = \frac{\text{Power}}{c}$$

$$\Rightarrow \frac{F}{A} = \frac{\text{Power}}{Ac} \Rightarrow \boxed{P \text{ (pressure)} = \frac{I}{c}}$$

$$\boxed{P = \frac{2I}{c}} \text{ for perfect reflector.}$$

Radiation pressure force on earth is about 100,000 tons (approx. weight of a large aircraft carrier)
Applications: Comet tails, space probes,....