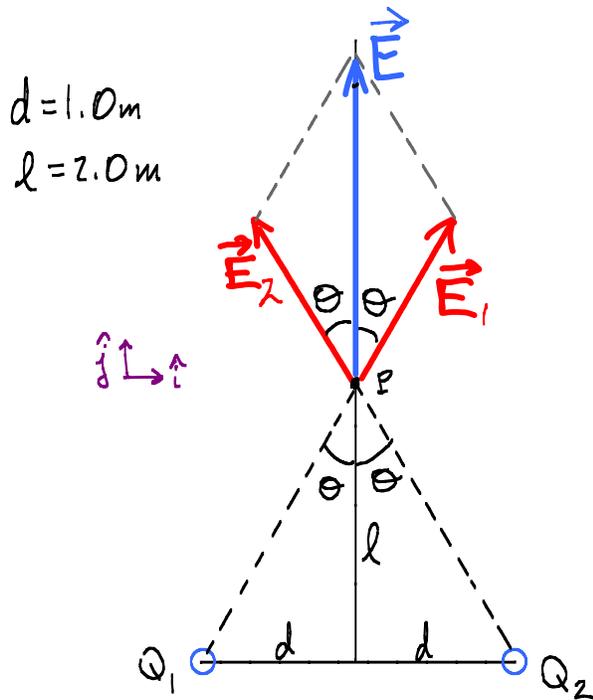


## SP212 Blurb 2 2010

1. Example: Net **E** field from 2 point charges.
2. **E** field from continuous charge distributions.
  - a. Line o' charge
  - b. Ring o' charge
  - c. Disc o' charge (not dance related!)
3. Electric Field Lines
4. Charge Motion in a Uniform Field

Example: + 50.0  $\mu\text{C}$  charges lie on the x-axis of a coordinate system at  $x = -1.0\text{ m}$  and  $x = +1.0\text{ m}$ . Find the electric field on the y axis at  $y = 2.0\text{ m}$ .



- ①  $\vec{E} = \vec{E}_1 + \vec{E}_2$
- ②  $\vec{E}_1 = E_1 \sin\theta \hat{i} + E_1 \cos\theta \hat{j}$
- ③  $\vec{E}_2 = -E_2 \sin\theta \hat{i} + E_2 \cos\theta \hat{j}$
- ④  $E_1 = E_2 = \frac{kQ_1}{l^2 + d^2}$
- ⑤  $\vec{E} = 2E_1 \cos\theta \hat{j}$
- ⑥  $\vec{E} = \frac{2kQ_1}{l^2 + d^2} \frac{l}{\sqrt{l^2 + d^2}} \hat{j}$
- ⑦  $\vec{E} = \frac{2 \times 9 \times 10^9 \frac{\text{Nm}^2}{\text{C}^2} (5.0 \times 10^{-5} \text{C}) (2.0\text{m})}{(2^2 + 1^2)^{3/2}}$

- ⑧  $\vec{E} = 1.61 \times 10^5 \frac{\text{N}}{\text{C}} \hat{j}$

Note that I took care of the vector stuff before dragging out the electric field formula. This is usually good practice since it allows you to know something about the solution before you calculate it, and can keep you from doing unnecessary calculations.

If a 3.0  $\mu\text{C}$  charge is placed at point P above, what is the force on it? One approach would be to reformulate the problem above using forces and Coulomb's law. A faster approach is to multiply the electric field we found at P by the charge we're going to place there.

$$\vec{F} = q \vec{E} = (3.0 \times 10^{-6} \text{C}) (1.61 \times 10^5 \frac{\text{N}}{\text{C}} \hat{j}) = 0.483 \text{N} \hat{j}$$

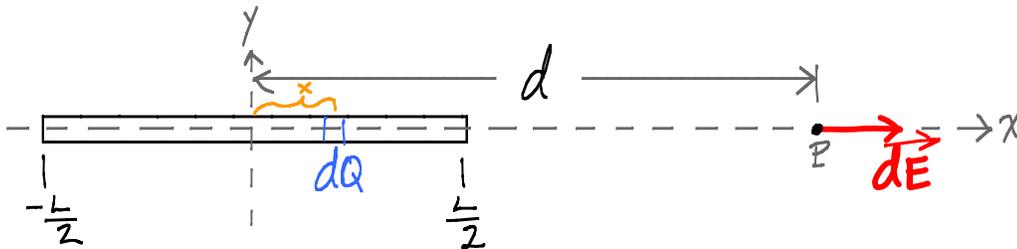
Now do the problem above with the left charge negative!

**Conceptual Question:** What can we say about the electric field inside a conductor in electrostatics?

### Continuous Charge Distributions

Since charges come in more geometrical configurations than simply points or spheres, we need a way to develop E field expressions for continuous charge distributions. Using the E field for a point charge and calculus allows us to get the job done. We break the charge distribution up into differential elements of charge and integrate to get the net E field. We'll illustrate this technique first with a uniform line of charge.

**Example:** A 15.0  $\mu\text{C}$  charge is uniformly distributed on a 25.0 cm rod that lies along the x-axis with its center at the origin. Find the electric field at  $x = +0.50$  cm.



$$\textcircled{1} \vec{E} = \int d\vec{E} = \int dE \hat{i} = \hat{i} \int dE$$

$$\textcircled{2} dE = \frac{k dQ}{(d-x)^2}$$

$$\textcircled{3} \text{ Define } \lambda \equiv \frac{Q}{L} \Rightarrow dQ = \lambda dx$$

$$\textcircled{4} \vec{E} = \hat{i} \int_{-L/2}^{L/2} \frac{k \lambda dx}{(d-x)^2} = \hat{i} k \lambda \int_{-L/2}^{L/2} \frac{dx}{(d-x)^2} \quad \leftarrow \text{use substitution}$$

$$\text{Let } \alpha = d-x \Rightarrow d\alpha = -dx \quad \int_{-L/2}^{L/2} \rightarrow \int_{d+L/2}^{d-L/2}$$

$$\Rightarrow \int_{-L/2}^{L/2} \frac{dx}{(d-x)^2} = \int_{d+L/2}^{d-L/2} \frac{-d\alpha}{\alpha^2} = - \left[ \frac{\alpha^{-1}}{-1} \right]_{d+L/2}^{d-L/2} = \left[ \frac{1}{d-L/2} - \frac{1}{d+L/2} \right]; \quad d = 0.5 \text{ m}$$

$$= 1.067 \text{ m}^{-1} \quad L = .25 \text{ m}$$

$$\textcircled{5} \quad \vec{E} = \hat{i} \left( 9 \times 10^9 \frac{\text{Nm}^2}{\text{C}^2} \right) \left( \frac{15.0 \times 10^{-6} \text{C}}{.25 \text{m}} \right) (1.067 \text{m}^{-1}) = \boxed{5.76 \times 10^{-5} \frac{\text{N}}{\text{C}} \hat{i}}$$

Is the result above reasonable? One way to tell is to look at an extreme value. If the detection point moves very far from the line of charge, the E field should begin to look like that from a point charge.

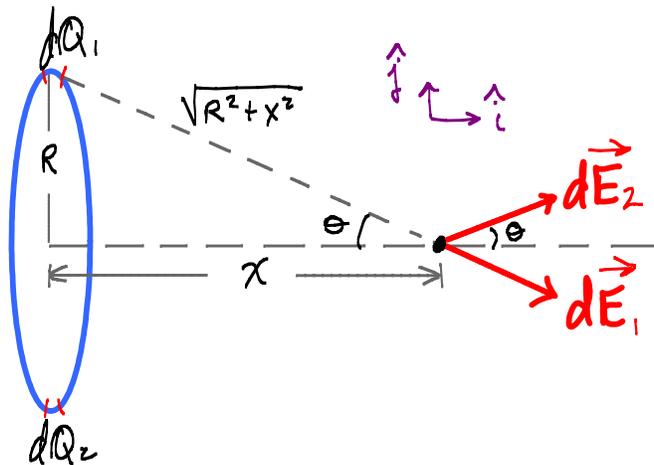
$$\begin{aligned} \vec{E} &= \hat{i} k \lambda \left[ \frac{1}{d - \frac{L}{2}} - \frac{1}{d + \frac{L}{2}} \right] = \hat{i} k \lambda \left[ \frac{d + \frac{L}{2} - (d - \frac{L}{2})}{(d - \frac{L}{2})(d + \frac{L}{2})} \right] = \hat{i} k \lambda \left[ \frac{L}{(d - \frac{L}{2})(d + \frac{L}{2})} \right] \\ &= \hat{i} k \frac{Q}{(d - \frac{L}{2})(d + \frac{L}{2})} \end{aligned}$$

Now let d get so large that L/2 can be ignored.

$$\Rightarrow \quad \vec{E} = \hat{i} \frac{kQ}{d^2} \quad \text{as we'd expect for } \mathbf{E} \text{ from a point charge.}$$

### Ring of Charge (with apologies to Johnny Cash!)

Now let's calculate the E field on the axis of a uniformly charged ring. Let the radius of the ring be R and the distance of the point from the center of the ring be x. Total charge on the ring is Q.



We see from the vector diagram that components perpendicular to the axis cancel.

$$\begin{aligned} \textcircled{1} \quad \vec{E} &= \int d\vec{E} = \int dE \cos \theta \hat{i} \\ \textcircled{2} \quad dE &= \frac{k dQ}{R^2 + x^2} \end{aligned}$$

$$\textcircled{3} \cos\theta = \frac{x}{\sqrt{x^2+R^2}}$$

$$\textcircled{4} \vec{E} = \hat{i} \int \frac{k dQ}{R^2+x^2} \frac{x}{\sqrt{R^2+x^2}} = \hat{i} k \int \frac{x dQ}{(R^2+x^2)^{3/2}}$$

$$\textcircled{5} \vec{E} = \frac{\hat{i} k x}{(R^2+x^2)^{3/2}} \int dQ$$

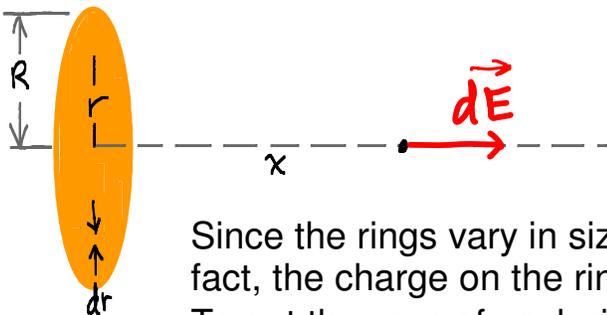
$$\textcircled{6} \vec{E} = \frac{\hat{i} k x Q}{(R^2+x^2)^{3/2}}$$

Again, let's see if this result holds for the extremes. If  $x$  goes to 0 we are in the center of the ring. Symmetry demands  $E = 0$  and when  $x = 0$  in the equation above we get zero.

Also, when  $x$  gets very large compared to  $R$ , the field from the ring should start looking like that due to a point charge. You should be able to convince yourself that the equation in step 6 above agrees in this limit.

### Disc o' Charge (It's so 80's!)

Now let's calculate the  $E$  field on the axis of a uniformly charged disc. Let the radius of the disc be  $R$  and the distance of the point from the center of the disc be  $x$ . Total charge on the disc is  $Q$ .



We can imagine the disc is made of an infinite set of rings, each with charge  $dQ$ , and making its own contribution  $d\vec{E}$ . We can then integrate the  $d\vec{E}$ 's to get net  $\vec{E}$ .

Since the rings vary in size, the charge  $dQ$  on each one varies. In fact, the charge on the ring will be proportional to the area of the ring. To get the area of each ring,  $dA$ , multiply its circumference,  $2\pi r$ , by its thickness,  $dr$ . If the disc is uniformly charged then its charge per unit area  $\sigma$ , is  $Q/(\pi R^2)$ . We can then write  $dQ = \sigma dA = \sigma 2\pi r dr$ . Using the ring result developed previously, we can proceed.

$$\textcircled{1} \vec{E} = \int d\vec{E} = \int dE \hat{i} \quad \textcircled{2} dE = \frac{k x dQ}{(r^2+x^2)^{3/2}}$$

$$\textcircled{3} \vec{E} = \hat{i} \int_0^R \frac{kx dQ}{(r^2+x^2)^{3/2}} = \hat{i} kx \int_0^R \frac{2\pi r dr}{(r^2+x^2)^{3/2}} = \hat{i} kx \sigma 2\pi \int_0^R \frac{r dr}{(r^2+x^2)^{3/2}}$$

$$\text{Let } u = r^2 + x^2 \Rightarrow du = 2r dr$$

$$\textcircled{4} \vec{E} = \hat{i} kx \sigma 2\pi \int_{x^2}^{R^2+x^2} u^{-3/2} \frac{du}{2} = \hat{i} kx \sigma \pi \frac{u^{-1/2}}{-1/2} \Big|_{x^2}^{R^2+x^2}$$

$$= \hat{i} kx \sigma 2\pi \left[ \frac{1}{x} - \frac{1}{\sqrt{R^2+x^2}} \right] \Rightarrow \vec{E} = \hat{i} k \frac{Q}{R^2} \cdot 2\pi \left[ 1 - \frac{x}{\sqrt{R^2+x^2}} \right]$$

$$\vec{E} = \frac{\hat{i} k 2Q}{R^2} \left[ 1 - \frac{x}{\sqrt{R^2+x^2}} \right]$$

Limiting Case 1: Let  $x$  get very large compared to  $R$ . Rewriting the expression,

$$\vec{E} = \frac{\hat{i} k 2Q}{R^2} \left[ 1 - \frac{1}{\sqrt{1+R^2/x^2}} \right] = \frac{\hat{i} k 2Q}{R^2} \left[ 1 - \left(1 + \frac{R^2}{x^2}\right)^{-1/2} \right]$$

but  $\left(1 + \frac{R^2}{x^2}\right)^{-1/2} \approx 1 - \frac{R^2}{2x^2} \Rightarrow \vec{E} = \frac{\hat{i} k 2Q}{R^2} \left[ x - \left(x - \frac{R^2}{2x^2}\right) \right] = \frac{\hat{i} k 2Q R^2}{R^2 2x^2}$

$$\Rightarrow \vec{E} = \frac{\hat{i} k Q}{x^2} \text{ as expected when one is so far away that the disc looks like a point charge!}$$

Limiting Case 2: Let  $R$  get very large compared to  $x$ . Reexamining the expression,

$$\vec{E} = \frac{\hat{i} k 2Q}{R^2} \left[ 1 - \frac{x}{\sqrt{R^2+x^2}} \right] \text{ we see that the second term in the square brackets goes to zero.}$$

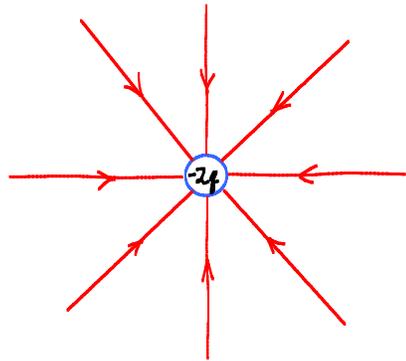
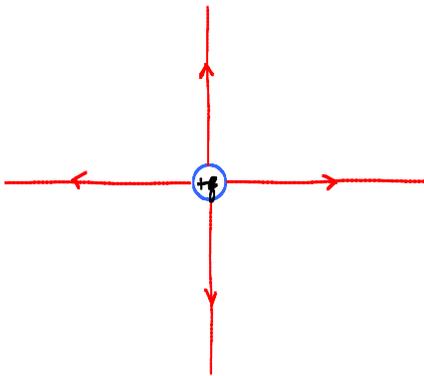
$$\vec{E} = \frac{\hat{i} k 2Q}{R^2} = \frac{\hat{i} k 2\sigma \pi R^2}{R^2} \Rightarrow \vec{E} = \hat{i} k 2\pi \sigma \text{ for an infinite plane of charge with charge per unit area } \sigma.$$

Note that  $x$  dependence has disappeared and  $\vec{E}$  is constant!

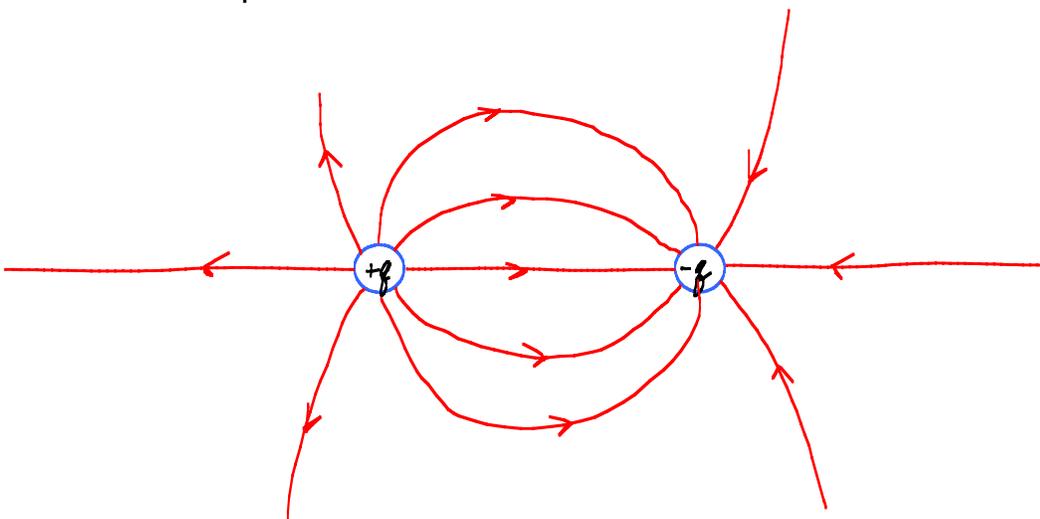
**Electric Field Lines** provide a convenient method for visualizing the electric field and obey the following rules:

1. They originate on positive charges and terminate on negative charges.
2. The number of lines originating or terminating on a charge is proportional to the charge magnitude.
3. Their density is proportional to the strength of the field while the direction of the field at a point is tangent to the **E** field line.
4. They never cross.

Separate point charges:



The electric dipole:

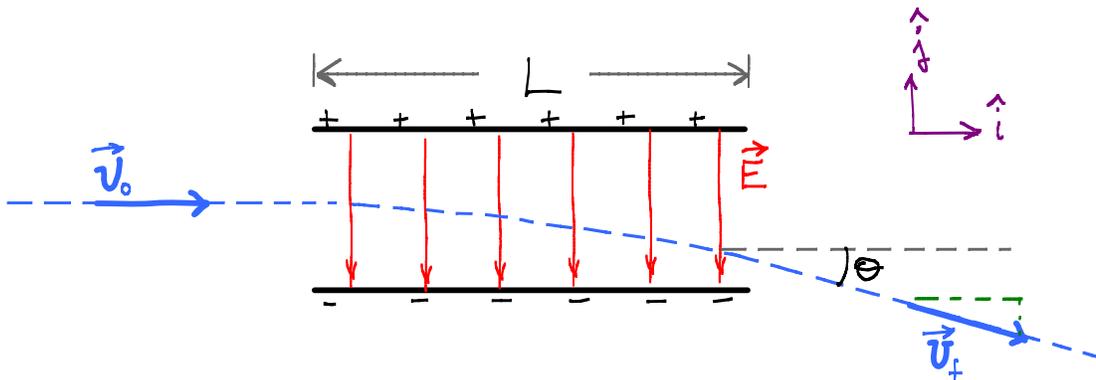


## Charge Motion in a Uniform Field.

Since the force on a charge in an electric field is given by  $\vec{F} = q\vec{E}$ , a constant  $\vec{E}$  produces a constant force on the charge and, according to Newton's second law, constant acceleration.

$$\vec{F} = q\vec{E} ; \vec{F} = m\vec{a} \Rightarrow \vec{a} = \frac{q\vec{E}}{m}$$

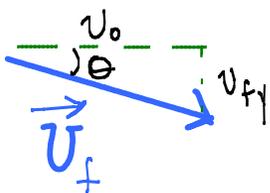
Example: Oppositely charged plates with a nearly constant electric field between them are often used to steer charged particles. In the system shown below, protons enter from the left with a speed of  $4.5 \times 10^5$  m/s. If the plates are 2.0 cm long and the field strength is 3700 N/C, what is the angle  $\theta$  for the emerging protons?



How does the x component of  $\vec{v}_f$  compare to  $\vec{v}_0$ ?

$$\Rightarrow \text{time between plates is } \frac{L}{v_0} = \frac{0.02 \text{ m}}{4.5 \times 10^5 \text{ m/s}} = 4.4 \times 10^{-8} \text{ sec}$$

$$\Rightarrow v_{fy} = at = \frac{qE}{m} t = \frac{(1.6 \times 10^{-19} \text{ C})(3700 \text{ N/C})(4.4 \times 10^{-8} \text{ s})}{1.67 \times 10^{-27} \text{ kg}} = 1.58 \times 10^4 \text{ m/s}$$



$$\theta = \tan^{-1} \left( \frac{v_{fy}}{v_0} \right) = \tan^{-1} \left( \frac{1.58 \times 10^4}{4.5 \times 10^5} \right) = 2.0^\circ$$

