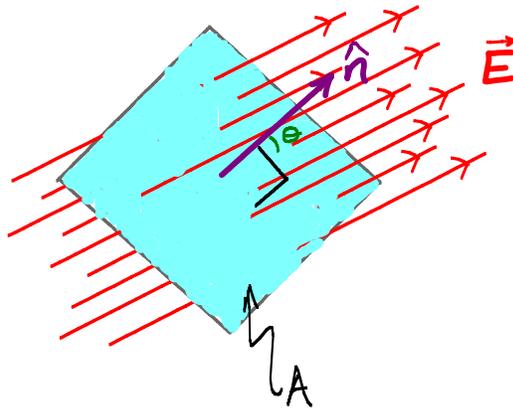


SP212 Blurb 3 2010

1. Definition of electric flux
2. Statement of Gauss' Law
3. Calculating with Gauss' Law.
 - a) E from a point charge
 - b) E inside and outside a uniformly charged sphere
 - c) E from a line of charge
 - d) E from a large uniform sheet of charge
4. Electrostatic conductors and E fields
5. Example problem

We're going to introduce a fundamental statement about the relationship between charges and electric field called Gauss' Law. As an added bonus, it makes calculations of \vec{E} easier in a few important cases. To be able to state Gauss' law we have to develop the concept of **electric flux**. To understand electric flux, imagine electric field lines in a region of space that cut through an area, assumed rectangular for now and depicted below.



Electric flux Φ through a surface is proportional to the number of field lines that cut through the surface. This number depends on the orientation of the surface, so we characterize the orientation of the surface with a normal vector. We see that the number of lines through the surface is proportional to the quantity,

$$\Phi = \vec{E} \cdot \hat{n} A = EA \cos \theta$$

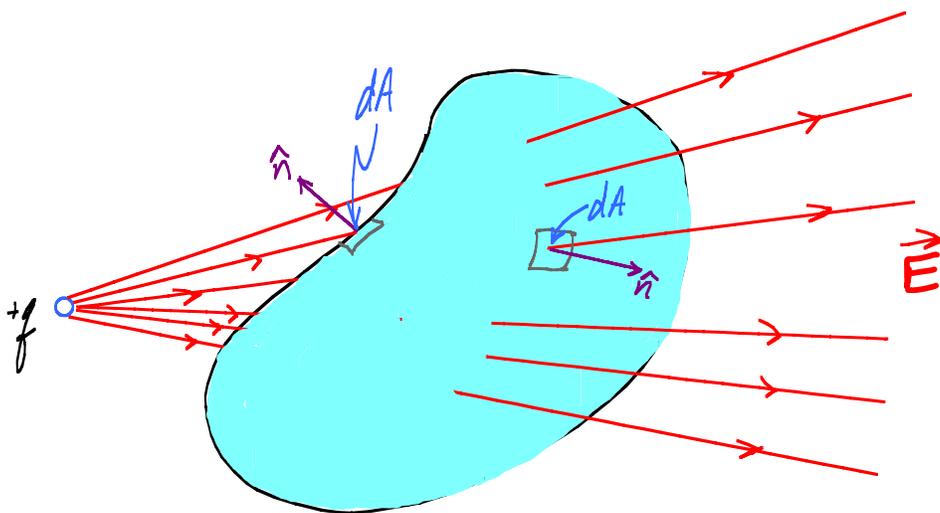
In the diagram above we see the number of lines cutting through the surface maximizes when the unit vector aligns with the field and goes to zero when the unit vector is perpendicular to the field, which supports our defining equation above.

Another simple idea we need to introduce to understand Gauss' law is the concept of a closed surface which is simply a surface with a definite inside and outside. Examples include a sphere and a cylinder with end caps. Unclosed surfaces include coffee cups, cylinders without end caps, planes, ... Now I can finally state Gauss' Law.

Gauss' Law: The net electric flux through a closed surface is proportional to the net charge enclosed.

$$\oint_S \vec{E} \cdot \hat{n} dA = \frac{q_{\text{enclosed, net}}}{\epsilon_0}$$

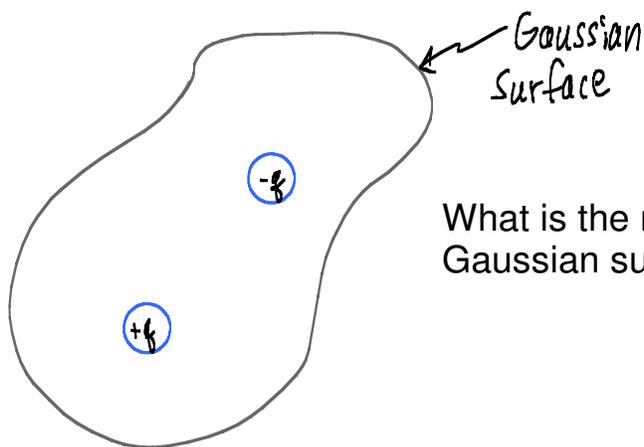
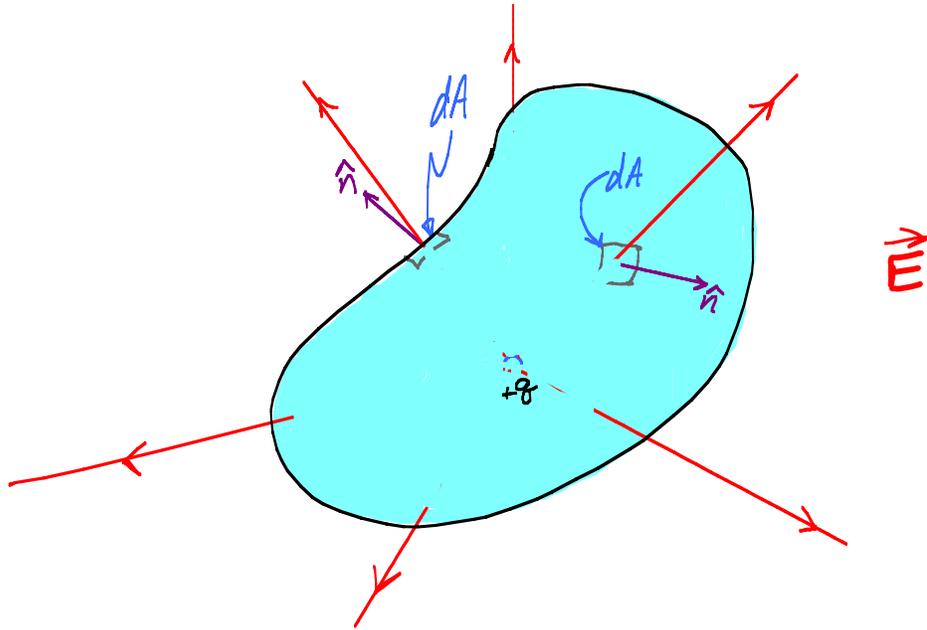
Memorize!



The Gaussian surface is an imaginary construct, not a material object!

In the diagram above, the right side of Gauss' Law tells us that the net flux through the closed surface is zero because the charge resides outside the surface. To understand why the left side of Gauss' law adds to zero, two small dA 's are shown. One is near an E line emerging from the closed surface and one is near an E line entering the closed surface. We see from the definition of flux that the emerging E line gives positive flux while the E line entering the surface gives a negative contribution. The summation of all these flux contributions is zero because there are as many lines leaving the closed surface as entering it.

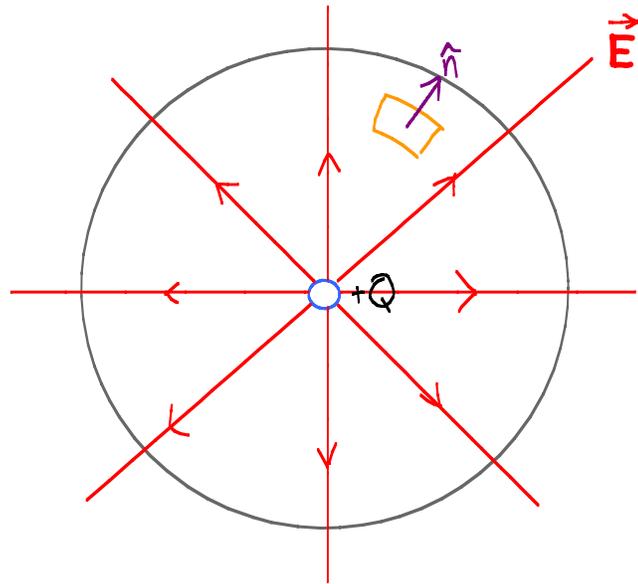
In the following diagram the right side of Gauss' Law says the net flux through the closed surface is positive because a positive net charge is enclosed. The left side is positive because the angles between the normal vectors and the E field lines are small and their dot product gives positive values.



What is the net flux through the Gaussian surface at left?

Calculating E field with Gauss' Law: In a few situations simple geometry allows Gauss' Law to be a powerful calculational tool. There are only about four, but if you learn them well, you can easily handle the variations that you will be expected to handle later. First, we'll look at E field due to a point charge, which we already know, to see how to use Gauss' Law in this known case.

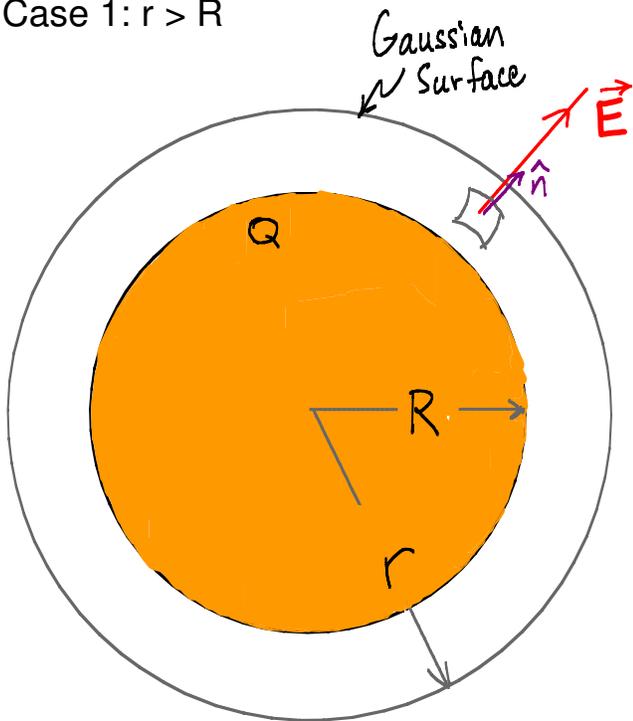
E due to a point charge



- ① $\oint_s \vec{E} \cdot \hat{n} da = \frac{q_{\text{net, enclosed}}}{\epsilon_0}$
- ② $\oint_s E da$ since $\hat{n} \parallel \vec{E}$
- ③ $\oint_s E da = E \oint_s da$ since E is constant
- ④ $\oint_s da = A_{\text{sphere}} = 4\pi r^2$
- ⑤ $E 4\pi r^2 = Q/\epsilon_0$
- ⑥ $E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \Rightarrow E = \frac{kQ}{r^2}$

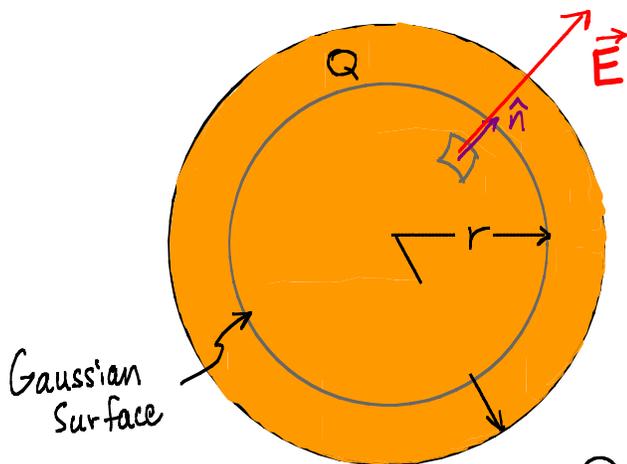
E due to a uniformly charged nonconducting sphere. Let a sphere of radius R have uniform charge Q throughout. Find E for all distances r from the center.

Case 1: $r > R$



- ① $\oint_s \vec{E} \cdot \hat{n} da = \frac{q_{\text{net, enclosed}}}{\epsilon_0}$
- ② $\oint_s E da$ since $\hat{n} \parallel \vec{E}$
- ③ $\oint_s E da = E \oint_s da$ since E is constant
- ④ $\oint_s da = A_{\text{sphere}} = 4\pi r^2$
- ⑤ $E 4\pi r^2 = Q/\epsilon_0$
- ⑥ $E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \Rightarrow E = \frac{kQ}{r^2}$

Case 2: $r < R$



Since only a fraction of the total charge Q is within the Gaussian surface we define charge per unit volume,

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}$$

With this definition the charge within the Gaussian surface can be written as,

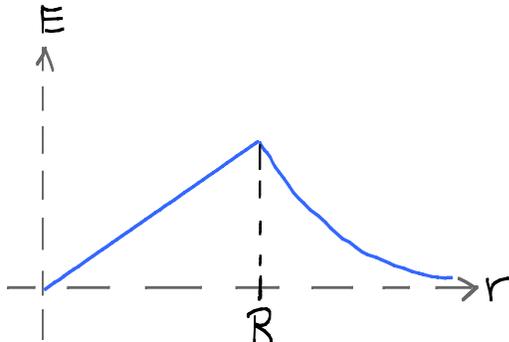
$$q_{\text{enclosed}} = \rho V_{\text{Gaussian sphere}} = \rho \frac{4}{3}\pi r^3$$

- ① $\oint \vec{E} \cdot \hat{n} da = q_{\text{enclosed}}/\epsilon_0$
- ② $\oint \vec{E} \cdot \hat{n} da = \oint E da$ since $\vec{E} \parallel \hat{n}$
- ③ $\oint E da = E \oint da = E 4\pi r^2$

$$\textcircled{4} \quad E \cdot 4\pi r^2 = \rho \frac{4}{3}\pi r^3 / \epsilon_0$$

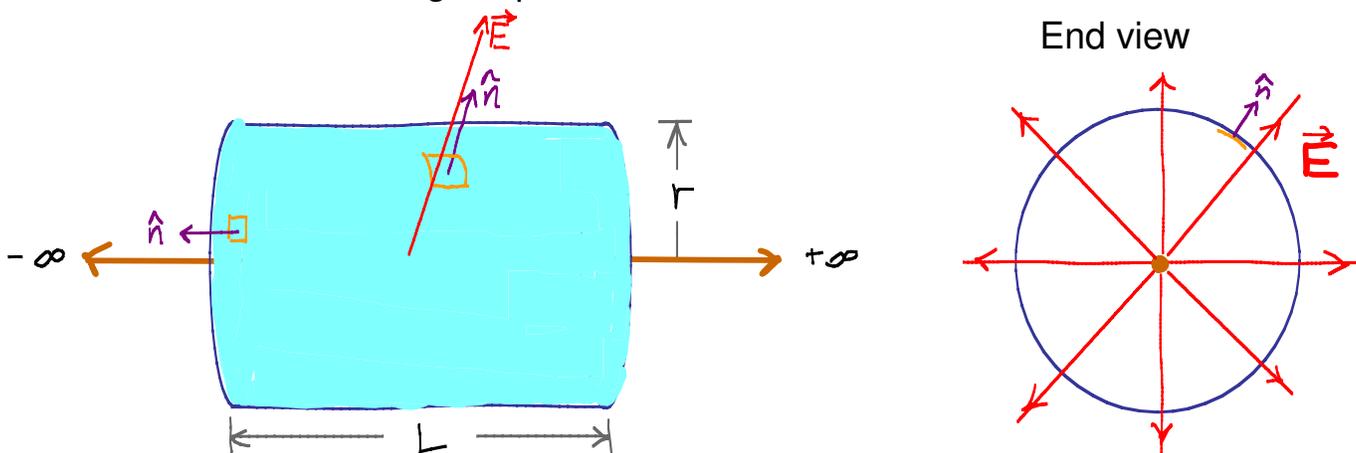
$$\textcircled{5} \quad E = \frac{\rho}{3\epsilon_0} r \Rightarrow E = \left(\frac{\rho}{4\pi\epsilon_0 R^3} \right) r$$

Check: The formulas for E inside and outside the charged sphere should be the same when $r = R$ and you can see they are.



Our results are summarized in the graph at left. The E field increases linearly from the center of the sphere to the surface and outside falls off as $1/r^2$.

E due to a uniformly charged line of charge. Let an infinite line of charge have uniform charge per unit length λ throughout. Find E for all distances r from the center. Assume the charge is positive.



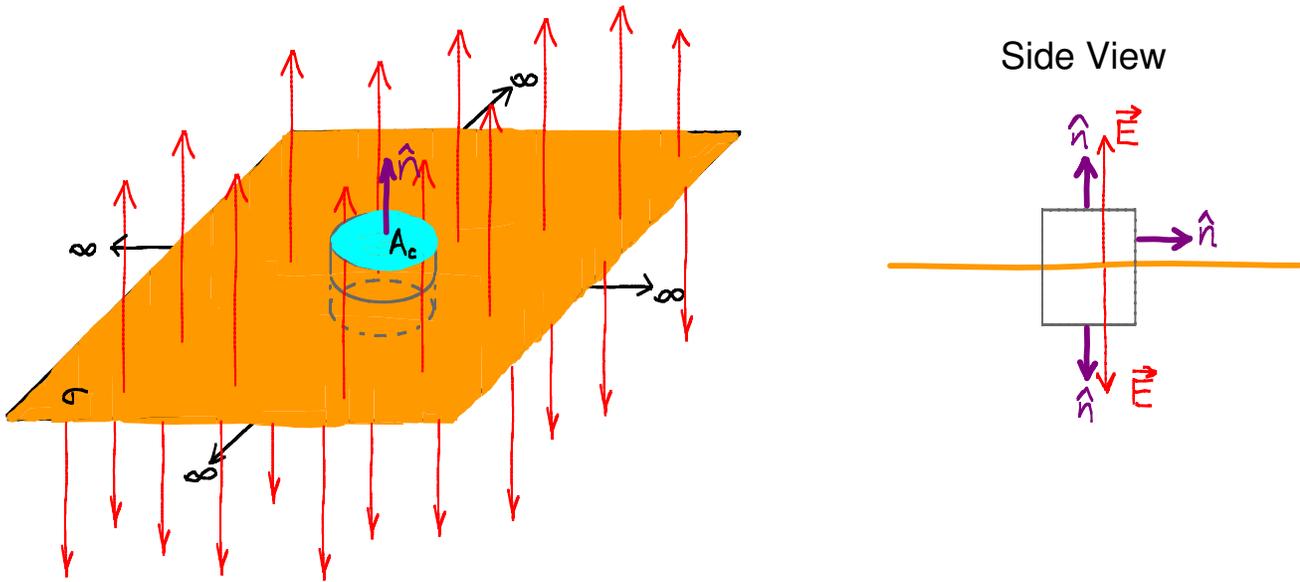
$$\textcircled{1} \quad \oint_S \vec{E} \cdot \hat{n} da = \frac{q_{\text{net, enclosed}}}{\epsilon_0} \quad \textcircled{2} \quad \oint_S \vec{E} \cdot \hat{n} da = \int_{\text{End Caps}} \vec{E} \cdot \hat{n} da + \int_{\text{curved surface}} \vec{E} \cdot \hat{n} da$$

$$\textcircled{3} \quad \int_{\text{End Caps}} \vec{E} \cdot \hat{n} da = 0 \quad \text{Since } \vec{E} \perp \text{ to } \hat{n} \text{ on end caps}$$

$$\textcircled{4} \quad \int_{\text{curved surface}} \vec{E} \cdot \hat{n} da = \int_{\text{curved surface}} E da = E \int da = E 2\pi r L \quad \textcircled{5} \quad q_{\text{enc}} = \lambda L$$

⑥ $E 2\pi r \cancel{L} = \frac{\lambda L}{\epsilon_0}$ ⑦ $E = \frac{\lambda}{2\pi\epsilon_0 r}$ Note the $\frac{1}{r}$ dependence

E due to a uniformly charged infinite plane of charge. Let a plane have uniform charge per unit area σ . Find E for all distances from the plane.



From symmetry we can argue that the E field is perpendicular to the plane. This leads us to use a pill shaped Gaussian surface as shown in the diagram. As you can see, there is a nonzero flux contribution through the flat end caps, but zero contribution from the curved surface.

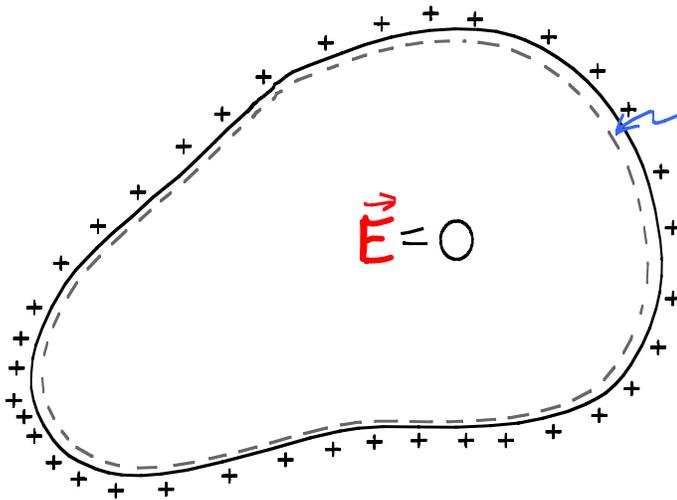
① $\oint_S \vec{E} \cdot d\vec{a} = \frac{q_{net, enclosed}}{\epsilon_0}$ ② $\oint_S \vec{E} \cdot d\vec{a} = \int_{End\ Caps} \vec{E} \cdot \hat{n} da + \int_{curved\ surface} \vec{E} \cdot \hat{n} da$

③ $\int_{curved\ surface} \vec{E} \cdot \hat{n} da = 0$ since $\hat{n} \perp$ to E ④ $\int_{End\ Caps} \vec{E} \cdot \hat{n} da = \int_{End\ Caps} E da$ since $\hat{n} \parallel \vec{E}$

⑤ $\int_{End\ Caps} E da = E \int_{End\ Caps} da = 2EA$ where A is the area of a single end cap.

⑥ $q_{net, enclosed} = \sigma A$ ⑦ $2EA = \frac{\sigma A}{\epsilon_0}$ ⑧ $E = \frac{\sigma}{2\epsilon_0}$

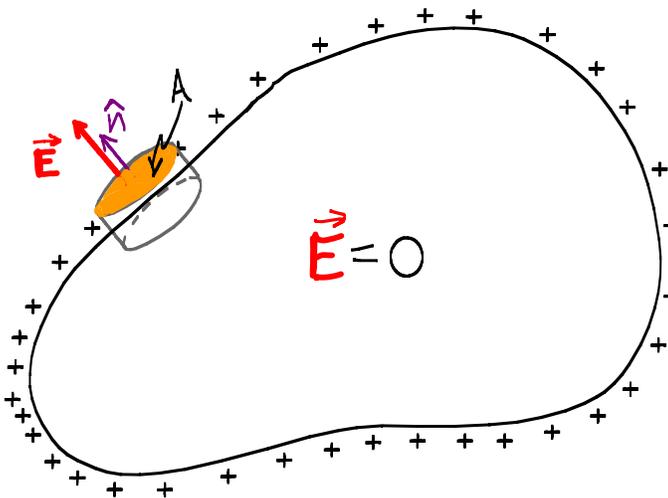
Gauss' Law applied to electrostatically charged conductors. We've already argued that the E field inside an electrostatically charged conductor is zero since a nonzero field would have charges moving. With this in mind let's consider a Gaussian surface just barely below the outer surface of a charged conductor.



Gaussian surface

$$\oint \vec{E} \cdot \hat{n} da = \frac{q_{net, enclosed}}{\epsilon_0}$$

The left side of Gauss' law above is clearly zero since E is zero everywhere on the Gaussian surface. So, if net charge resides on the conductor, it must be outside the Gaussian surface, therefore, on the outer skin of the conductor.

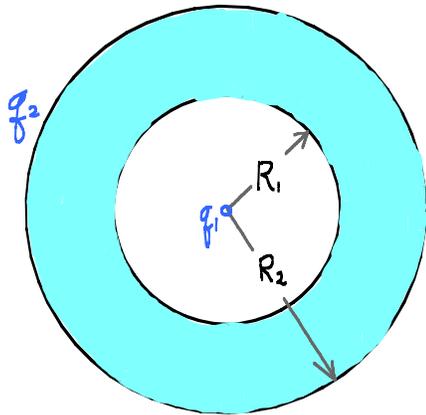


At left the conductor considered previously is redrawn with a pillbox Gaussian surface drawn so its end caps are parallel to the conducting surface with one end cap inside the conductor while the other is just outside. The electric field has to be perpendicular to the surface in the electrostatic case. If there were a tangential component to E, charges would be moved along the surface contradicting the static assumption.

The charge within the Gaussian surface is σA where σ is the charge per unit area at the location of the Gaussian surface. The net flux through the Gaussian surface is EA since only the top cap contributes. Thus, Gauss' law says,

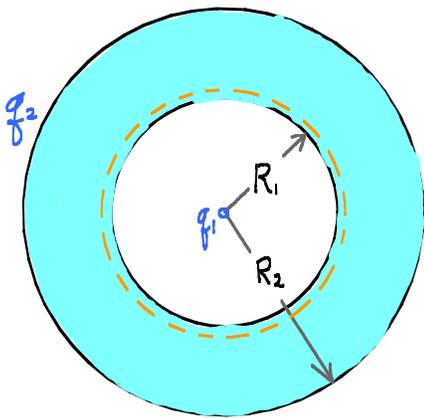
$$EA = \frac{\sigma A}{\epsilon_0} \Rightarrow E = \frac{\sigma}{\epsilon_0}$$

So, we can now relate the electric field just outside a charged conductor to the surface charge density at that location.

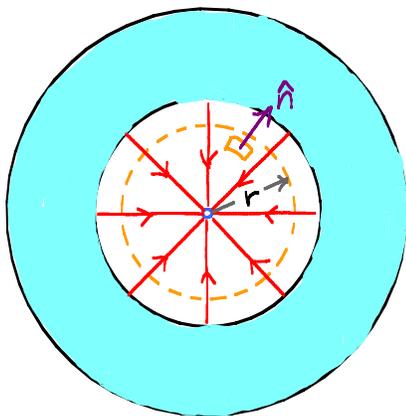


Example: The diagram shows a charged hollow spherical conducting shell with inner radius R_1 and outer radius R_2 with a point charge q_1 at its center. If the charge q_2 on the spherical conductor is $+65.0 \text{ nC}$ and $q_1 = -45.0 \text{ nC}$, find $E(r)$. Assume $R_1 = 12.0 \text{ cm}$ and $R_2 = 28.0 \text{ cm}$.

The first step in the solution is to figure out the charge distribution. We'll use the fact that $E = 0$ in a conductor, Gauss' Law, and conservation of charge to answer this question.



At left a Gaussian surface is drawn just beneath the surface of the spherical cavity. Since the static electric field is zero inside the conductor, Gauss' law says there must be no net charge within the Gaussian surface. So, there must be a charge $-q_1$ on the inner surface. Also, for conservation of charge to hold, the conducting sphere must have a net charge of q_2 , which requires the outermost layer of the conductor to have a charge of $q_1 + q_2$.



Next, we'll find the E field in the spherical cavity by constructing a spherical Gaussian surface with $r < R_1$. In our example negative charge resides at the center, so the electric field in this region points toward the center.

$$\textcircled{1} \oint_s \vec{E} \cdot d\vec{a} = \frac{q_{\text{net, enclosed}}}{\epsilon_0}$$

$$\textcircled{2} \oint \vec{E} \cdot \hat{n} da = -\oint E da \text{ since } \vec{E} \text{ is antiparallel to } \hat{n}$$

$$\textcircled{3} -\oint E da = -E \oint da = -E 4\pi r^2 = \frac{q_1}{\epsilon_0}$$

$$\textcircled{4} E = \frac{-1}{4\pi\epsilon_0} \frac{|q_1|}{r^2} = \frac{(-9.0 \times 10^9 \frac{\text{Nm}^2}{\text{C}^2})(45.0 \times 10^{-9} \text{C})}{r^2} = \frac{-405}{r^2}$$

$$\text{When } r = .12 \text{ m} \Rightarrow E = \frac{-405}{.12^2} \frac{\text{N}}{\text{C}} = -2.81 \times 10^4 \text{ N/C}$$

Outside, as we've seen in previous examples, the E field will look like that of a point charge. Since the net charge is positive,

$$\textcircled{5} \quad E = \frac{1}{4\pi\epsilon_0} \frac{q_{\text{net}}}{r^2} = \frac{(9 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2})(20.0 \times 10^{-9} \text{C})}{r^2} = \frac{+180}{r^2}$$

when $r = 0.28 \text{ m}$ $E = \frac{180}{.28^2} \frac{\text{N}}{\text{C}} = 2.30 \times 10^3 \frac{\text{N}}{\text{C}}$

