

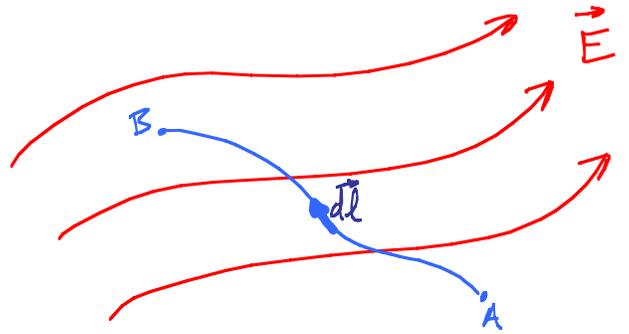
SP212 Blurb 4 2010

1. Intro electrostatic potential difference.
2. Interpretation of electrostatic potential difference.
3. Calculation of ΔV in a constant **E** field.
4. Calculation of ΔV near a point charge.
5. V near a collection of point charges.
6. Energy of a collection of point charges.
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Electrostatic potential: The electrostatic potential difference between two points A and B in a static electric field is defined by the expression,

$$\Delta V = V_B - V_A = - \int_A^B \vec{E} \cdot d\vec{\ell}$$

memorize!



Units: $\frac{Nm}{C} = \frac{J}{C} = \text{Volt}$

What does this mean? Imagine you are moving a charge q from A to B and trying to expend the minimum effort. The work you would be doing can be calculated from,

$$W = \int_A^B \vec{F} \cdot d\vec{\ell}$$

If the work is to be minimized the applied force \vec{F} will be very nearly equal and opposite the electrical force \vec{F}_e , i.e., $\vec{F} = -\vec{F}_e$

$$\Rightarrow W = - \int_A^B \vec{F}_e \cdot d\vec{\ell} \quad \text{and the charge gets to B with zero kinetic energy.}$$

Recall $\vec{F}_e = q\vec{E}$ the electric force $\Rightarrow W = - \int_A^B q\vec{E} \cdot d\vec{\ell}$

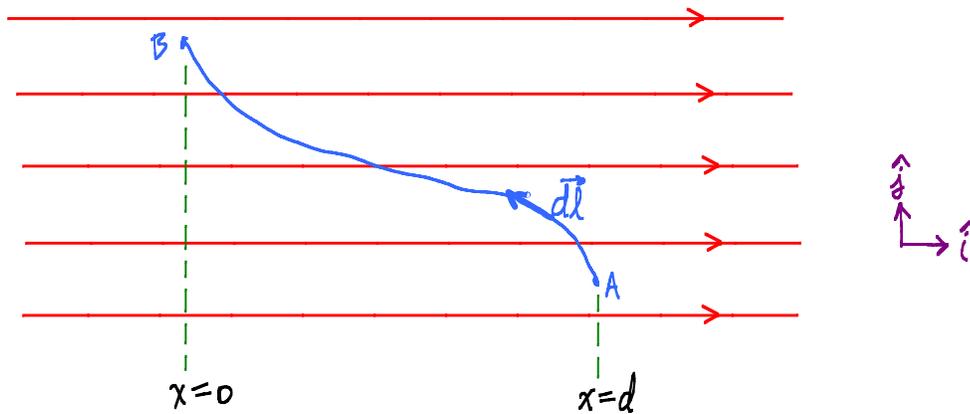
so the work you do depends on the amount of charge q carried, but the work per unit charge W/q , is $-\int_A^B \vec{E} \cdot d\vec{\ell}$

and matches our definition of ΔV , $\Delta V = V_B - V_A = - \int_A^B \vec{E} \cdot d\vec{\ell}$

Units of ΔV are J/C, defined as the Volt. Note also that ΔV is a scalar!

Warning! Electrostatic potential is not the same as potential energy.

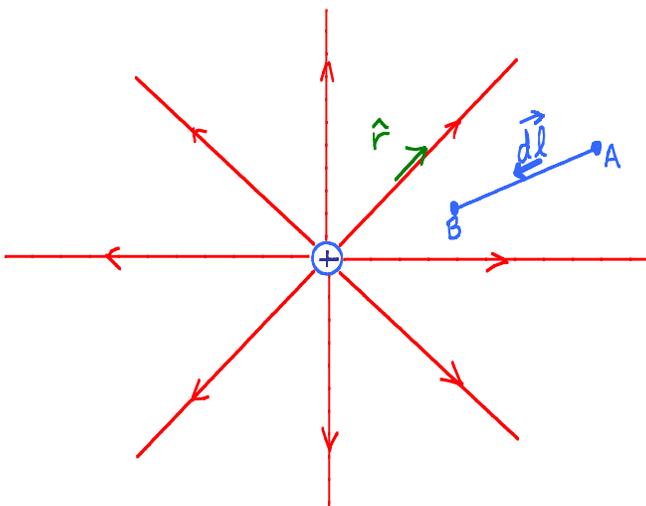
Calculation of ΔV in a constant E field.



$$\begin{aligned} \textcircled{1} \Delta V &= V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{\ell} & \textcircled{2} \vec{E} &= E\hat{i} & \textcircled{3} d\vec{\ell} &= dx\hat{i} + dy\hat{j} \\ \textcircled{4} \Delta V &= -\int_A^B E\hat{i} \cdot (dx\hat{i} + dy\hat{j}) = -\int_{x_A}^{x_B} E dx = -E \int_d^0 dx = -E(-d) = \boxed{Ed} \end{aligned}$$

So, if we move upstream a distance d in a constant electric field, the change in potential is given by $\Delta V = Ed$.

ΔV near a point charge. Let's find the potential difference between two points in a point charge electric field.



$$\begin{aligned} \textcircled{1} \Delta V &= V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{\ell} \\ \textcircled{2} \vec{E} &= \frac{kQ}{r^2} \hat{r} ; d\vec{\ell} = \hat{r} dr \\ \textcircled{3} \Delta V &= -\int_{r_A}^{r_B} \frac{kQ}{r^2} \hat{r} \cdot dr \hat{r} = -kQ \int_{r_A}^{r_B} r^{-2} dr \\ \textcircled{4} \Delta V &= -kQ \frac{r^{-1}}{(-1)} \Big|_{r_A}^{r_B} = kQ \left(\frac{1}{r_B} - \frac{1}{r_A} \right) \end{aligned}$$

Since only potential energy difference is defined, we are able to set electrical potential to zero wherever we wish. In the expression above let the starting point be at infinity.

$$V_B - V_A = kQ \left(\frac{1}{r_B} - \frac{1}{r_A} \right)$$

$$V_B - V_\infty = kQ \left(\frac{1}{r_B} - \frac{1}{\infty} \right) = \frac{kQ}{r_B}$$

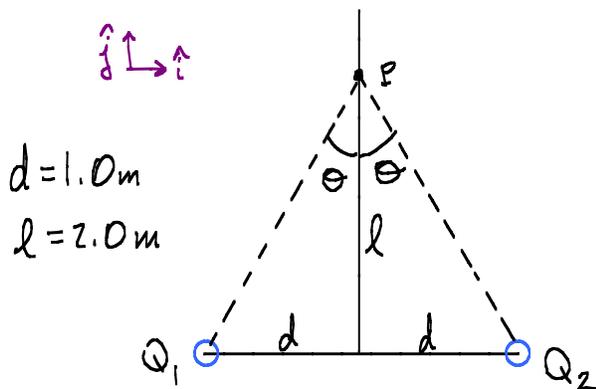
Now, if we set $V_\infty = 0$ we can write,

$$V_B = \frac{kQ}{r_B} \Rightarrow \boxed{V(r) = \frac{kQ}{r}}$$

where the subscript B has been dropped.

The expression developed above is simple, but we must remember when we use it that V at infinity is assumed to be zero!

Example: Calculate the potential at point P due to two point charges in the situation depicted below. Assume each charge is $+ 50.0 \mu\text{C}$.



- ① $V_p = V_1 + V_2$
- ② $V_1 = V_2 = \frac{kQ_1}{r} = \frac{kQ_1}{\sqrt{l^2 + d^2}}$
- ③ $V_p = \frac{2(9 \times 10^9 \frac{\text{Nm}^2}{\text{C}^2})(5.0 \times 10^{-5} \text{C})}{\sqrt{5} \text{ m}}$
- ④ $V_p = 4.02 \times 10^5 \text{ Volts}$

Calculating potential is usually easier than calculating electric field because it's a scalar. Without vector components to consider, extending the calculation of V to N charges is easily done.

$$V_p = \sum_{i=1}^N \frac{kq_i}{r_i}$$

Physical interpretation of ΔV .

A way to understand the physical significance of ΔV is to imagine carrying a small charge as slowly as possible from one point to another with a "work meter" that keeps track of the work you are doing along the way. If, after arriving at the destination, you divide the value shown on the work meter by the charge you carried, you will have calculated the potential difference between the two points.

The reason the charge should be carried as slowly as possible is that an external agent (like you) has the option to do extra work that would impart kinetic energy to the charge. The value of the work meter would thus be too large. To avoid this confusion ΔV is defined as the negative work done by the field divided by the charge carried, since the work done by the field between the two points does not depend on the whims of an external agent. Sometimes, though, it's convenient to think of potential difference as the minimum work, W_{\min} , done by you (the external agent) to carry a charge between the two points and then divide by the charge carried.

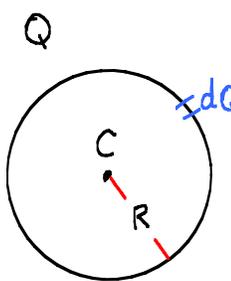
Since the negative of the work done by the field is also the change in potential energy ΔU , a useful expression to remember is,

$$\Delta U = Q\Delta V$$

Example: To illustrate the relationship between potential difference, charge and energy consider a 1.5 volt AAA battery that can deliver 1200mAh of charge. How many joules of energy can this battery deliver?

From the definition of current, $\Delta q = I\Delta t$. So, the battery can deliver $1.2 \text{ A} \cdot 3600 \text{ sec} = 4320 \text{ Coulombs}$ of charge. If 4320 Coulombs of charge moves through a 1.5 Volt potential difference, the work done by the battery is $4320 \text{ Coulombs} \cdot 1.5 \text{ Volts} = 6480 \text{ J}$. This is about the same energy required to raise 660 kg (1450 lbs) one meter near the surface of the earth!

For continuous charge distributions the calculation of V at a point is done by considering the contribution by a differential element of charge dQ , then summing the contributions from all the dQ 's by integrating. As a first example, let's calculate the potential at the center of a ring of charge.

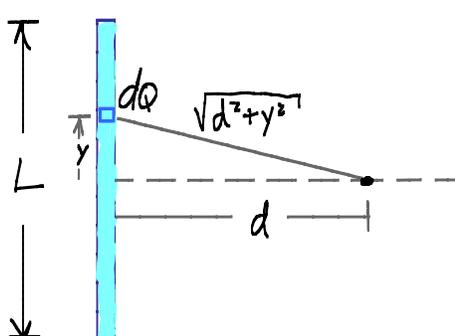


① $dV = \frac{k dQ}{R}$

② $V = \int \frac{k dQ}{R}$

③ $V = \frac{k}{R} \int dQ \Rightarrow V = \frac{kQ}{R}$

Example: Potential on the perpendicular bisector of a uniform line of charge with total charge Q .



① $dV = \frac{k dQ}{r}$ ② $V = \int \frac{k dQ}{r}$

③ $dQ = \frac{Q}{L} dy = \lambda dy$; $r = \sqrt{d^2 + y^2}$

④ $V = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{k \lambda dy}{\sqrt{d^2 + y^2}} = k \lambda \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dy}{\sqrt{d^2 + y^2}}$

⑤ Let $L = 0.10\text{m}$; $Q = 25\text{nC}$; $d = 20.0\text{cm}$

⑥ $\ln[7] = \int_{-.05}^{.05} \frac{1}{\sqrt{.2^2 + x^2}} dx$
 Out[7]= 0.494933

⑦ $V = k \lambda (.495) = (9 \times 10^9) \left(\frac{25 \times 10^{-9}\text{C}}{.1\text{m}} \right) (.495)$

⑧ $V = 1110\text{ Volts}$

Energy of a collection of point charges. Imagine a collection of like point charges held in place by some nonelectrical forces, like rubber bands. If we cut all the rubber bands simultaneously the charges would fly apart because of the net repulsive force on each charge. Thus, kinetic energy develops. The kinetic energy came from the potential energy in the original charge configuration and we are going to use the potential difference concept to calculate the original potential energy of the charge configuration.

We'll start simple with a two charge configuration. Let q_1 and q_2 be two positive charges separated by a distance r_{12} . What is the potential energy of this configuration? To answer this question, consider the work required to build it. Let the two charges start out well separated from each other and an infintedistance away from their final resting place. If we bring in the first charge there is no field exerting force on it and we don't have to do any work to bring it in, i.e., $W_1 = 0$.

But, placing the second charge at its final location is going to require work because the first charge will be exerting force on it. Recall that the potential difference between the starting point and end point for the second charge on its journey is V_2 , the potential at location 2 due to q_1 . This is also the minimum work per unit charge to bring q_2 in from infinity, thus, $W_2 = q_2 * V_2$. If we add the total work to assemble the two charge configuration,

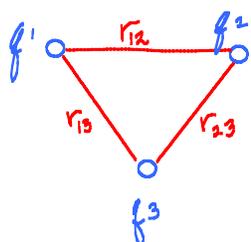
$$W_{\text{total}} = W_1 + W_2$$



$W_{\text{total}} = W_1 + W_2 = 0 + q_2 \frac{kq_1}{r_{12}} = U$ (the potential energy)
 since $\Delta U = Q\Delta V$

$$U = \frac{kq_1q_2}{r_{12}}$$

Following this routine, we can calculate the work to bring in a third charge as q_3V_3 where V_3 is the potential due to the other two charges.



$$V_3 = \frac{kq_1}{r_{13}} + \frac{kq_2}{r_{23}} ; W_3 = q_3 V_3 \Rightarrow W_3 = \frac{kq_1q_3}{r_{13}} + \frac{kq_2q_3}{r_{23}}$$

So, the work to put together the three charge assembly is,

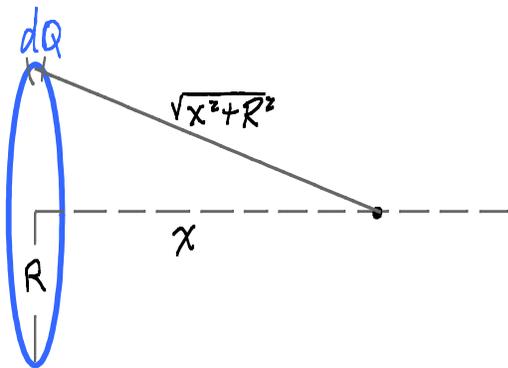
$$W_1 + W_2 + W_3 = 0 + \frac{kq_1q_2}{r_{12}} + \frac{kq_1q_3}{r_{13}} + \frac{kq_2q_3}{r_{23}}$$

$$\Rightarrow U = \frac{kq_1q_2}{r_{12}} + \frac{kq_1q_3}{r_{13}} + \frac{kq_2q_3}{r_{23}}$$

Once we understand this process, it can be extended to any number of charges.

V due to a continuous charge distribution. We can get V for a continuous distribution of charge by integrating the dV contributions from all dQ's that make up the distribution. I'll forgo a grand general statement for a couple examples, which should suffice.

Example: Ring o' charge. Let's find the potential on the axis of a uniformly distributed ring of charge with total charge Q and radius R.



① $dV = \frac{k dQ}{\sqrt{x^2 + R^2}}$ since dQ looks like a point charge

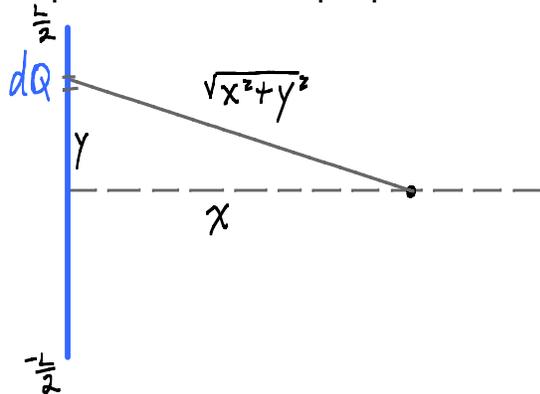
② $V = \int \frac{k dQ}{\sqrt{x^2 + R^2}}$ Now pull constants outside the integral

③ $V = \frac{k}{\sqrt{x^2 + R^2}} \int dQ = \frac{k Q}{\sqrt{x^2 + R^2}}$

Consider the case where $x \gg R$. The ring looks like a point charge and we see the formula in step 3 above gives,

$$V(x) \approx \frac{k Q}{x} \quad \text{when } x \gg R.$$

Example. Finite line of charge. In this development we'll see how to calculate the potential on the perpendicular bisector of a finite line of charge.



① $dV = \frac{k dQ}{\sqrt{x^2 + y^2}}$ since dQ looks like a point charge

② $V = \int \frac{k dQ}{\sqrt{x^2 + y^2}}$

③ Let $\lambda = \frac{Q}{L} \Rightarrow dQ = \lambda dy$

④ $V = k \int \frac{dQ}{\sqrt{x^2 + y^2}} = k \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\lambda dy}{\sqrt{x^2 + y^2}}$ ⑤ $V = k \lambda \ln(y + \sqrt{x^2 + y^2}) \Big|_{-\frac{L}{2}}^{\frac{L}{2}}$

To prove this, take the derivative of step 5 and you will get the integrand in step 4.

⑥ $V = k \lambda \left[\ln\left(\frac{L}{2} + \sqrt{\frac{L^2}{4} + x^2}\right) - \ln\left(-\frac{L}{2} + \sqrt{\frac{L^2}{4} + x^2}\right) \right] = k \lambda \ln\left(\frac{\frac{L}{2} + \sqrt{\frac{L^2}{4} + x^2}}{\sqrt{\frac{L^2}{4} + x^2} - \frac{L}{2}}\right)$

Do we like this result? To answer that let's take the limit as L gets very small compared to x . If we get the result we expect for a point charge, the result is probably true and we can invest our fondness!

First, note that for $L \ll x$, the ratio in parentheses in the last step has to be just slightly larger than 1. Since that is the case we can use $\ln(1+z) \cong z$.

⑦ $V \cong k \lambda \left[\ln\left(\frac{\frac{L}{2} + \sqrt{\frac{L^2}{4} + x^2}}{\sqrt{\frac{L^2}{4} + x^2} - \frac{L}{2}}\right) - \right] = k \lambda \left[\frac{\frac{L}{2} + \sqrt{\frac{L^2}{4} + x^2} - (-\frac{L}{2} + \sqrt{\frac{L^2}{4} + x^2})}{\sqrt{\frac{L^2}{4} + x^2} - \frac{L}{2}} \right]$

⑧ $V = \frac{k \lambda}{\sqrt{\frac{L^2}{4} + x^2} - \frac{L}{2}}$ Now we see that if x is considerably larger than L the denominator becomes nearly x .

⑨ $V \cong \frac{k \lambda L}{x} = \frac{k \frac{Q}{L} L}{x} \Rightarrow V = \frac{kQ}{x}$ the expected result for V a distance x from a point charge.

You might think we can get another limit by letting L get infinitely long, but that won't work here. The reason is that the formula we started with in step 1 assumes the potential at infinity is zero. If we make the line infinitely long, then there is charge at infinity and we can't set potential equal to zero at the location of charge.

Calculating \mathbf{E} from V .

We can calculate ΔV from electric field using the definition of electrostatic potential. Now we'll see that it's possible to calculate electric field if we know electrostatic potential in a region of space. We'll start with the electrostatic potential definition applied to one small differential step,

$$dV = -\vec{E} \cdot d\vec{l}$$

Now let $\vec{E} = E_x \hat{i} + E_y \hat{j} + E_z \hat{k}$ and $d\vec{l} = dx \hat{i} + dy \hat{j} + dz \hat{k}$

$$\Rightarrow dV = -E_x dx - E_y dy - E_z dz$$

Recall from calculus that,

$$dV(x,y,z) = \frac{dV}{dx} dx + \frac{dV}{dy} dy + \frac{dV}{dz} dz$$

Comparing the last two expressions for dV we see,

$$E_x = -\frac{dV}{dx} ; E_y = -\frac{dV}{dy} ; E_z = -\frac{dV}{dz}$$

$$\Rightarrow \vec{E} = -\frac{dV}{dx} \hat{i} - \frac{dV}{dy} \hat{j} - \frac{dV}{dz} \hat{k} \Rightarrow \vec{E} = -\vec{\nabla} V$$

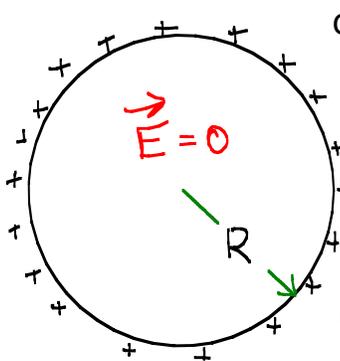
where $\vec{\nabla} = \hat{i} \frac{d}{dx} + \hat{j} \frac{d}{dy} + \hat{k} \frac{d}{dz}$ and is called the gradient operator.

Example: If the potential in a region of space is described by $V(x,y) = 3x^2 + 2y^4$, find the electric field at the point (2,3).

$$\frac{dV}{dx} = 6x = 12 \text{ at } x=2 ; \frac{dV}{dy} = 8y^3 = 8 \cdot 3^3 = 216 \text{ at } y=3$$
$$\Rightarrow \vec{E} = -12\hat{i} - 216\hat{j}$$

The ability to get \mathbf{E} from V means that V gives a complete description of the electric field!

Example: Find the potential inside and outside a charged conducting sphere with total charge Q .



case1: $r > R$

Here we have a spherical charge distribution. Outside the field and potential look like that due to a point charge.

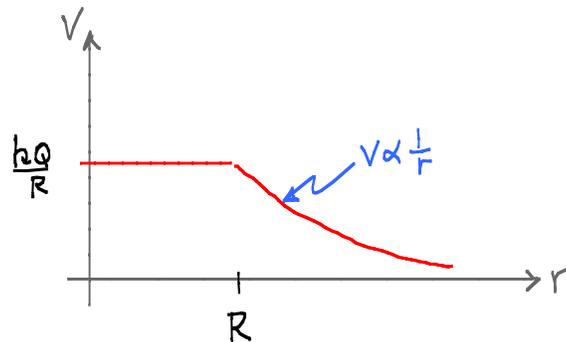
$$V(r) = \frac{kQ}{r}$$

case2: $r < R$

While the most frequent first guess is zero, because the field inside is zero, this is not the case. If we consider the work per unit charge required to move a charge in from infinity, the work invested doesn't drop when we carry it inside the charged sphere. The work done just stays constant and so does the potential. In fact, the potential stays locked at the same value it had at the surface,

$$V = \frac{kQ}{R} \quad r < R$$

A graph of $V(r)$ vs r :



A graph of $E(r)$ vs r :

