

## SP212 Blurb 5 2010

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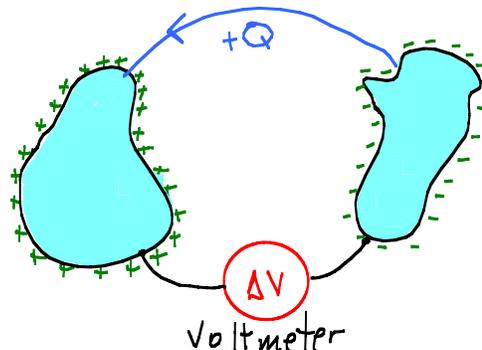
## Definition of capacitance

If charge  $Q$  is moved from one conductor to another, an electric field develops between them, resulting in a potential difference  $\Delta V$ . If a small  $Q$  results in a large  $\Delta V$  we say the system has a small capacity for holding charge. If it takes a large transfer of charge to get appreciable  $\Delta V$  we say the system has a large capacity for holding charge. This forms the basis for our definition of capacitance  $C$ , defined as follows,

$$C = \frac{Q}{\Delta V}$$

The unit of capacitance is Coulombs per Volt

$$1 \frac{C}{V} \equiv 1 \text{ Farad}$$



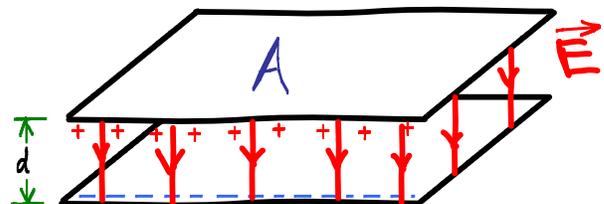
The ratio we have defined as capacitance depends on geometry and, as we will explore later, the material between the two conductors. Capacitors are used to store electrical energy and as filters in electrical circuits. They are especially useful in applications where electrical energy needs to be delivered quickly. Examples include flashbulbs and the new railgun technology where batteries and generators cannot meet this demand.

**Parallel Plate Capacitors:** The first example is also one of the most practical as examples of it are found in nearly every piece of electrical equipment ever made! Imagine two uncharged parallel conducting plates that are arranged with their large areas parallel and held a distance  $d$  apart as shown in the diagram. Charge  $Q$  is transferred from the bottom plate to the top resulting in a potential difference  $\Delta V$ . Assuming  $d$  is very small compared to the dimensions of the plate allows us to assume a constant electric field between the plates and potential difference,

$$\Delta V = Ed$$

Now recall that the electric field on the surface of a conductor is given by,

$$E = \sigma / \epsilon_0$$



Writing  $Q = \sigma A$ , we now have all the pieces to compute the capacitance.

$$C = \frac{Q}{\Delta V} = \frac{\sigma A}{Ed} = \frac{\sigma A}{\frac{\sigma}{\epsilon_0} d}$$

$$C_{||} = \frac{\epsilon_0 A}{d}$$

As a numerical example let's calculate the plate area of a 1 Farad parallel plate capacitor with 1 mm plate separation.

$$C = \frac{\epsilon_0 A}{d} \Rightarrow A = \frac{Cd}{\epsilon_0} = \frac{(1F)(.001m)}{8.85 \times 10^{-12} \frac{C^2}{F \cdot m}} = 1.13 \times 10^8 m^2$$

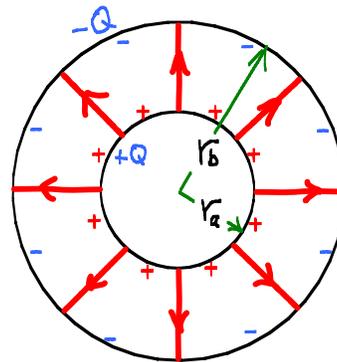
Assuming square plates, how long is one side of this plate?

$$A = L^2 \Rightarrow L = \sqrt{A} \Rightarrow L = \sqrt{1.13 \times 10^8 m^2} \Rightarrow L \cong 10,600m$$

Since 10.6 km equals 6.6 miles, you might conclude that 1 Farad capacitors cannot be made. This is not true as modern materials and clever designs have, just within the past several years, made 1 Farad capacitors small enough to swallow, though not recommended by this author.

Spherical capacitor: While not as practically useful, this case is interesting and can be easily be computed.

Depicted in the diagram are two concentric spherical shells. If positive charge,  $Q$  is moved from the inner sphere to the outer sphere an E field and potential difference develops between them where,



$$\Delta V = kQ \left( \frac{1}{r_a} - \frac{1}{r_b} \right) = \frac{Q}{4\pi\epsilon_0} \left( \frac{1}{r_a} - \frac{1}{r_b} \right)$$

$$\Rightarrow \frac{Q}{\Delta V} = \frac{4\pi\epsilon_0}{\left( \frac{1}{r_a} - \frac{1}{r_b} \right)} = C$$

An interesting special case: Let  $r_b$  go to infinity and then we are dealing with an isolated charged sphere with radius  $r_a$ . In the expression above the  $1/r_b$  term goes to zero. This leaves,

$$C = 4\pi\epsilon_0 r_a$$

and can be interpreted as the capacitance of an isolated sphere. To get a numerical example let's again compute the size of a 1 Farad capacitor

$$r_a = \frac{C}{4\pi\epsilon_0} = \frac{1 \text{ Farad}}{4\pi(8.85 \times 10^{-12} \text{ F/m})} \cong 9 \times 10^9 \text{ m}$$

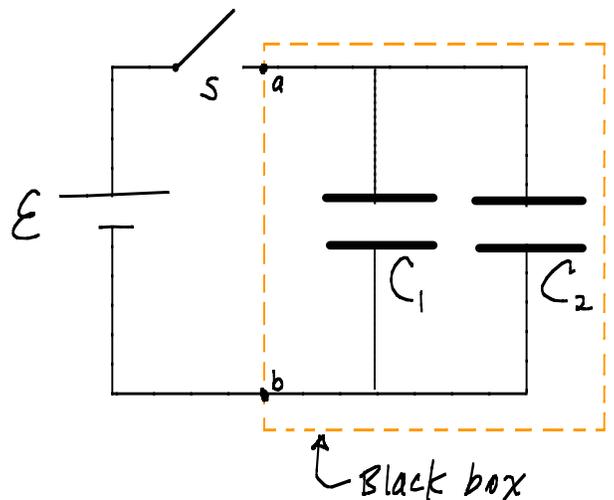
The radius of the earth is only  $6.4 \times 10^6 \text{ m}$ , so a 1 Farad sphere would be humongous and is not seen in captivity!

**Capacitor combos.** Capacitors as circuit devices are often needed with values that are not manufactured. It is usually possible to combine capacitors to get the required design. Two important rules are developed.

Capacitors in parallel are connected so that they have the same potential difference. If we imagine two capacitors connected in parallel, what's the equivalent capacitance of the pair?

If we close the switch S, charge flows out of the battery and stores in the two capacitors. Charge only flows a short time in this circuit because the gaps in the two capacitors prevent charge flow across them. The capacitors get full.

If we imagined a black box with these two capacitors inside, we'd measure the capacitance of the box as the charge  $Q$  extracted from the battery divided by  $\Delta V_{ab}$  between the terminals of the box, which equals the battery emf,  $\epsilon$ .



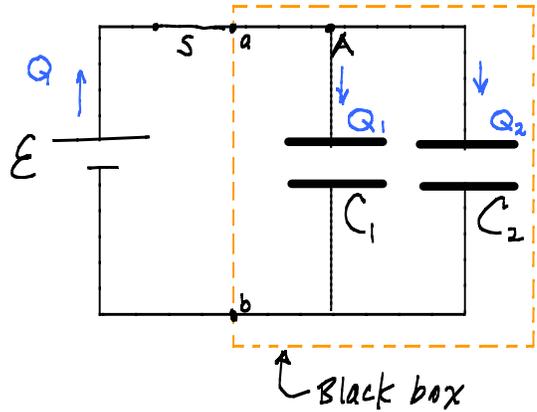
When charges get to the branch point A, in the drawing below, some go to  $C_1$  and some go to  $C_2$ . From conservation of charge we can say,

$$\textcircled{1} Q = Q_1 + Q_2$$

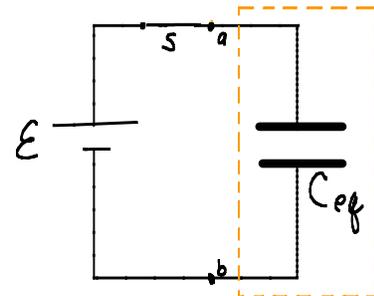
$$\textcircled{2} C_{\text{eff}} = \frac{Q}{E}; C_1 = \frac{Q_1}{E}; C_2 = \frac{Q_2}{E}$$

$$\textcircled{3} C_{\text{eff}} E = EC_1 + EC_2$$

$$\textcircled{4} C_{\text{eff}} = C_1 + C_2$$



So the equivalent capacitance in the black box is simply the sum of the two capacitors in parallel.



Capacitors in series are connected so that they have the same charge. If we imagine two capacitors connected in series, what's the equivalent capacitance of the pair? A charge  $q$  going clockwise around the circuit picks up energy  $qE$  going through the battery and loses energy  $qE_1d_1$  through  $C_1$  and  $qE_2d_2$  through  $C_2$ . From conservation of energy,

$$\textcircled{1} qE = qE_1d_1 + qE_2d_2$$

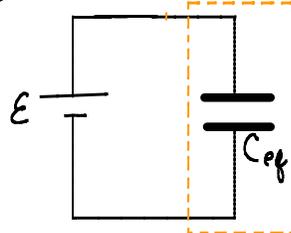
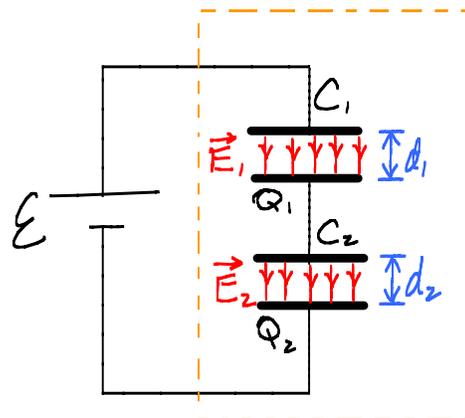
$$\textcircled{2} \Delta V_1 = E_1d_1; \Delta V_2 = E_2d_2$$

$$\textcircled{3} E = \Delta V_1 + \Delta V_2$$

$$\textcircled{4} \Delta V_1 = \frac{Q}{C_1}; \Delta V_2 = \frac{Q}{C_2}; E = \frac{Q}{C_{\text{eff}}}$$

$$\textcircled{5} \frac{Q}{C_{\text{eff}}} = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$\textcircled{6} \frac{1}{C_{\text{eff}}} = \frac{1}{C_1} + \frac{1}{C_2}$$



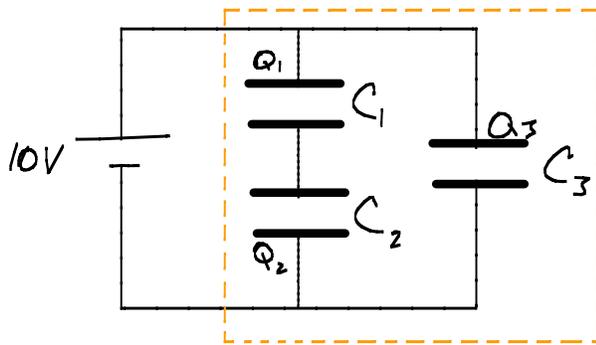
Note: Two capacitors in series have an equivalent resistance smaller than either of the constituent capacitors. Proof,

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \Rightarrow \frac{1}{C_{eq}} = \frac{C_1 + C_2}{C_1 C_2} \Rightarrow C_{eq} = \frac{C_1 C_2}{C_1 + C_2} = \left( \frac{C_1}{C_1 + C_2} \right) C_2$$

Since capacitance is always positive the quantity in parentheses above is always less than one. Therefore,

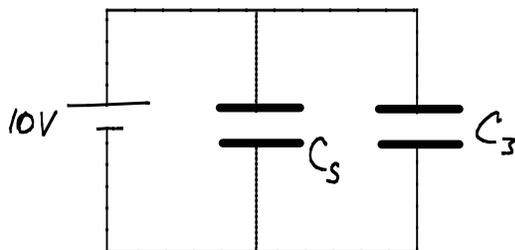
$$C_{eq} < C_2 \quad \left( \text{Same argument for } C_{eq} < C_1 \right)$$

Capacitor network analysis. Find the equivalent capacitance of the three capacitor network.



$$\begin{aligned} C_1 &= 1.0 \mu F \\ C_2 &= 3.0 \mu F \\ C_3 &= 5.0 \mu F \end{aligned}$$

Let's find the charge and voltage on each cap. Rewrite the circuit as,



$$\begin{aligned} \frac{1}{C_s} &= \frac{1}{C_1} + \frac{1}{C_2} \Rightarrow C_s = \frac{C_1 C_2}{C_1 + C_2} \\ &= \frac{(1.0 \mu F)(3.0 \mu F)}{4.0 \mu F} = 0.75 \mu F \end{aligned}$$

where  $C_s$  is the series equivalent of  $C_1$  and  $C_2$ .

$$C_{eq} = C_s + C_3 = 0.75 \mu F + 5.0 \mu F = 5.75 \mu F$$

Now find the charge and voltage on each capacitor in this example.

$C_3$  and  $C_5$  each have 10 V across them, so use the definition of capacitance to get the charge on each.

$$C_3 = \frac{Q_3}{\Delta V} \Rightarrow Q_3 = (5 \mu\text{F})(10\text{V}) = 50 \mu\text{C}$$

$$C_5 = \frac{Q_5}{\Delta V} \Rightarrow Q_5 = (0.75 \mu\text{F})(10\text{V}) = 7.5 \mu\text{C}$$

Now use the fact that caps in series have the same charge to find  $V_1$  and  $V_2$ ;

$$Q_5 = Q_1 = Q_2 \Rightarrow C = Q/\Delta V \Rightarrow \Delta V_1 = \frac{Q_1}{C_1} = \frac{7.5 \mu\text{C}}{1 \mu\text{F}}$$

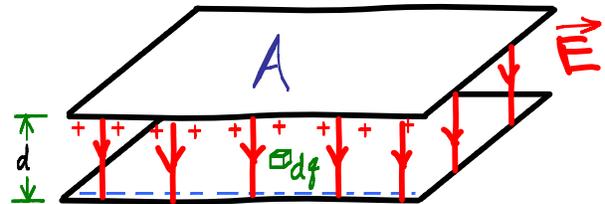
$$\Rightarrow \Delta V_1 = 7.5 \text{ Volts}$$

$$\Delta V_2 = \frac{Q_2}{C_2} = \frac{7.5 \mu\text{C}}{3.0 \mu\text{F}} = 2.5 \text{ Volts}$$

We didn't have to do the last calculation. Why?

## Energy storage in capacitors.

To obtain the energy stored in a capacitor we will calculate the work required to move charge from one plate to another in a parallel plate capacitor.



Consider a small amount of charge  $dq$ , moved from the bottom plate in the depicted capacitor to the top plate. The bit of work required to do this is,

$$dW = dq \Delta V$$

Now use the definition of capacitance to write,  $\Delta V = \frac{q}{C}$

and integrate to get the total work to charge the capacitor. This is the energy stored.

$$dW = \frac{q}{C} dq \Rightarrow W = \int_0^Q \frac{q}{C} dq = \frac{1}{C} \int_0^Q q dq = \frac{1}{2C} Q^2$$

$$\therefore \boxed{U = \frac{1}{2C} Q^2} \Rightarrow U = \frac{1}{2C} (C\Delta V)^2 \Rightarrow \boxed{U = \frac{1}{2} C\Delta V^2}$$

**Energy in electric fields.** Using the result above and our expression for C in a parallel plate capacitor, we can make a statement about the energy contained in the electric field.

$$U = \frac{1}{2} C\Delta V^2 = \frac{1}{2} \frac{\epsilon_0 A}{d} (Ed)^2 = \frac{1}{2} \epsilon_0 (Ad) E^2$$

but  $(Ad)$  = volume between plates

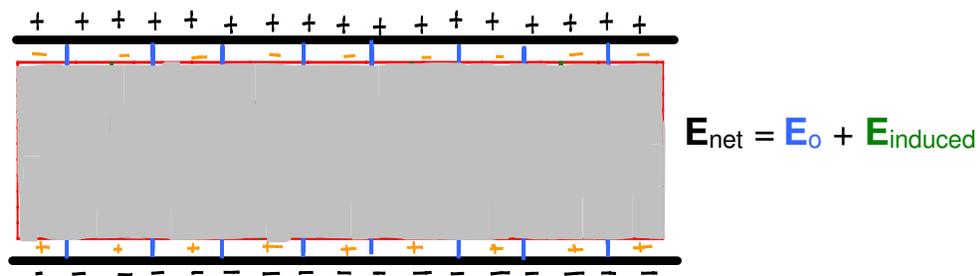
$$\Rightarrow \frac{U}{\text{vol}} = \frac{1}{2} \epsilon_0 E^2 \quad \text{which we define as energy density, } u_E$$

$$\boxed{u_E = \frac{1}{2} \epsilon_0 E^2}$$

This is a pretty fundamental statement! It means this thing we made up to explain how one charge feels the force of another, the electric field, actually contains energy. Who knew!?

## Capacitors with Dielectrics

Now let's consider what happens when there is a dielectric (non-conducting) material separating capacitor electrodes.



We see that the original  $\mathbf{E}$  field between the plates is reduced by the induced charges in the dielectric since  $\mathbf{E}_0$  and  $\mathbf{E}_{\text{induced}}$  point in opposite directions. We use a dimensionless constant called the dielectric constant  $\kappa$  to relate the new  $\mathbf{E}$  field to the original field  $\mathbf{E}_0$ .

$$\vec{E} = \frac{\vec{E}_0}{\kappa} \text{ where } \kappa \gg 1$$

$$\Rightarrow \Delta V = \frac{\Delta V_0}{\kappa} \text{ since } \Delta V = Ed$$

There is a table of dielectric constants in your book. Some values are,

Material	$\kappa$	$V_{\text{breakdown}} (10^6 \text{ Volts})$
Air	1.00059	3
Nylon	3.4	14
Pyrex glass	5.6	14
Teflon	2.1	60
Water	80	—

What happens to capacitance when we place the dielectric between the plates?  
We'll resort to the definition,

$$C = \frac{Q}{\Delta V} = \frac{Q}{\Delta V_0 / \kappa} = \kappa \frac{Q}{\Delta V_0} = \kappa C_0$$

$\Rightarrow$  Capacitance gets bigger by  $\kappa$  when a dielectric fills the region between the plates. We make an effectively bigger capacitor.

What happens to energy stored in the capacitor when a dielectric is placed between the plates? Recall,

$$U = \frac{1}{2C} Q^2 \text{ and use the fact that } Q \text{ doesn't change}$$

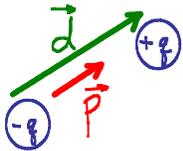
$$U_0 = \frac{1}{2C_0} Q^2 \text{ and } U = \frac{1}{2C} Q^2$$

where  $U_0$  is the energy in the capacitor without dielectric.  $U$  is the energy in the capacitor after a dielectric is placed between the plates

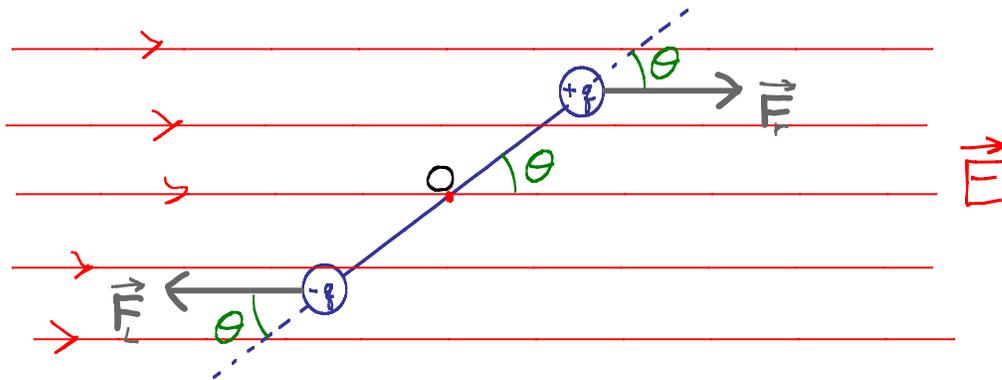
$$\Rightarrow \frac{U}{U_0} = \frac{C_0}{C} = \frac{C_0}{C_0 \kappa} \Rightarrow U = \frac{U_0}{\kappa}$$

So, the energy stored in the cap goes down.

**Dipoles:** We can interpret the above by looking on the microscopic scale at the response of the dielectric material to the electric field in the space between plates. We defined a dipole in an earlier blurb as equal and opposite charges separated by some distance. An electric dipole moment  $\mathbf{p}$  is a vector defined by  $\mathbf{p} = q\mathbf{d}$  where  $\mathbf{d}$  is the displacement vector from negative to positive charge as represented in the following diagram.



Now, suppose we put this dipole in a constant electric field and note the force on each of the charges.



We can see clearly that the net force is zero but the dipole will experience a net torque that tends to align the dipole with the electric field. We see the torque about point O caused by the each charge is  $\frac{1}{2}qEd \sin\theta$ , producing a net torque about O of,  $qEd \sin\theta$ . Since the rotation for the depicted dipole tends clockwise, the torque is into the page, which also happens to be the same direction as  $\mathbf{p} \times \mathbf{E}$ . Comparing the result above for the magnitude of the torque on the dipole we see that it exactly matches the magnitude of this cross product. So, in general, we can write the torque on a dipole in vector form

Potential energy of an electric dipole in an electric field. If we now rotate the electric dipole depicted above counterclockwise we can get the work invested back by releasing the dipole and collecting the kinetic energy. Thus, the energy invested corresponds to potential energy and we can write,

$$\Delta U = U(\theta) - U(\theta_0) = \int_{\theta_0}^{\theta} \tau d\theta = \int_{\theta_0}^{\theta} pE \sin\theta d\theta$$

$$\Rightarrow U(\theta) - U(\theta_0) = pE \int_{\theta_0}^{\theta} \sin\theta d\theta$$

$$U(\theta) - U(\theta_0) = -pE \cos\theta \Big|_{\theta_0}^{\theta}$$

$$U(\theta) - U(\theta_0) = -pE(\cos\theta - \cos\theta_0)$$

Now let  $\theta_0 = 90^\circ$  and set  $U(90^\circ) = 0$

$$\Rightarrow U(\theta) = -pE \cos\theta$$

$$\Rightarrow U(\theta) = -\vec{p} \cdot \vec{E}$$

So  $U$  represents the potential energy of an electric dipole in an electric field where it is understood that  $U = 0$  at  $\theta = 90^\circ$ .

