SP212 Blurb 7 2010

1. RC circuits (charging & discharging)

2. Resistors in series

3. Resistors in parallel

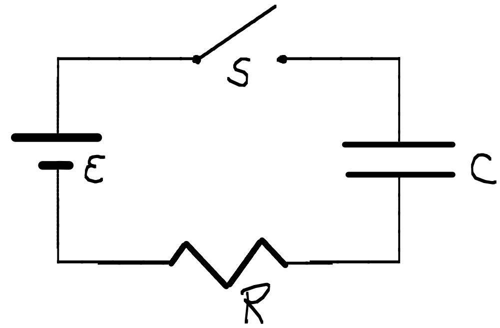
4. Kirchhoff's Rules. (2 loop example)

5. Kirchhoff Supplement

RC Circuits

i) Charging case:

Let's consider what happens when the switch is closed in the circuit at right at time t = 0.



Charges flow and the capacitor gradually fills. How long this takes depends on how large the charge reservoir (capacitor) is and how large the "flow control valve" (resistor) is. If we apply Kirckhoff's loop rule while current is flowing,

$$\mathcal{E} - \frac{Q}{C} - IR = 0 \text{ and } I = \frac{dQ}{dt} \Rightarrow R\frac{dQ}{dt} = \mathcal{E} - \frac{Q}{C}$$

$$\Rightarrow \frac{dQ}{dt} = \frac{\mathcal{E}}{R} - \frac{Q}{RC} \Rightarrow \frac{dQ}{Q/RC - \mathcal{E}/R} = -dt \Rightarrow \frac{dQ}{Q - \mathcal{E}C} = \frac{-dt}{RC}$$

$$\Rightarrow \int_0^Q \frac{dQ}{Q - \mathcal{E}C} = \frac{-1}{RC}\int_0^t dt \Rightarrow \ln(Q - \mathcal{E}C)\Big|_0^Q = \frac{-t}{RC}$$

$$\Rightarrow \ln(Q - \mathcal{E}C) - \ln(-\mathcal{E}C) = -t/RC \Rightarrow \ln\left(\frac{Q - \mathcal{E}C}{-\mathcal{E}C}\right) = -t/RC$$

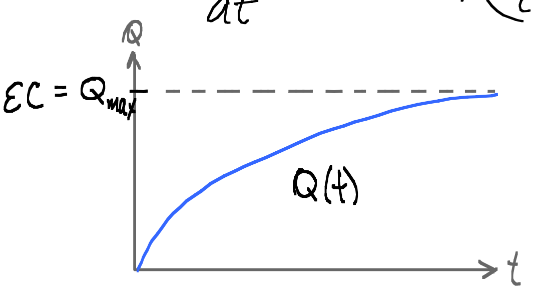
$$\Rightarrow \ln\left(\frac{\mathcal{E}C - Q}{\mathcal{E}C}\right) = -t/RC \Rightarrow \frac{\mathcal{E}C - Q}{\mathcal{E}C} = e^{-t/RC}$$

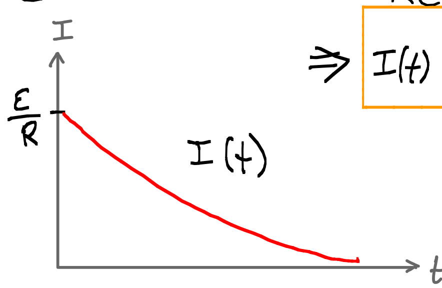
$$\Rightarrow \mathcal{E}C - Q = \mathcal{E}C e^{-t/RC} \Rightarrow \mathcal{E}C(1 - e^{-t/RC}) = Q(t)$$

$$\mathcal{E}C = Q_{max}\,; \boxed{RC \equiv \tau \text{ (time constant)}} \quad \boxed{Q(t) = Q_{max}(1 - e^{-t/\tau})}$$

$$I(t) = \frac{dQ}{dt} = -Q_{max}\left(\frac{-1}{\tau}\right)e^{-t/\tau} \qquad I(t) = \frac{Q_{max}}{RC}e^{-t/\tau} = I_{max}e^{-t/\tau}$$

$$\Rightarrow \boxed{I(t) = I_{max}e^{-t/\tau}}$$





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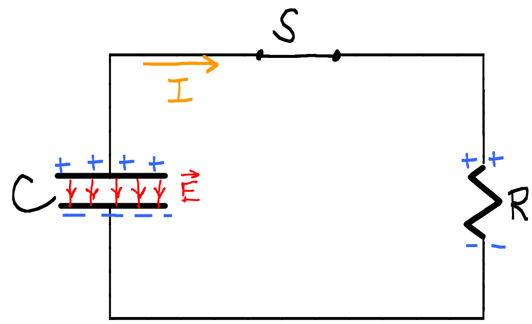
ii) <u>Discharging case</u>:

Switch is closed at t=0.
Applying loop rule:



$$① \quad \frac{Q}{C} - IR = 0$$

Since Q represents charge on the capacitor and current is due to charge lost from the capacitor,

$$② \; I = -\frac{dQ}{dt} \Rightarrow ③ \; \frac{Q}{C} = -\frac{dQ}{dt}R \Rightarrow ④ \; \frac{dQ}{Q} = -\frac{dt}{RC}$$

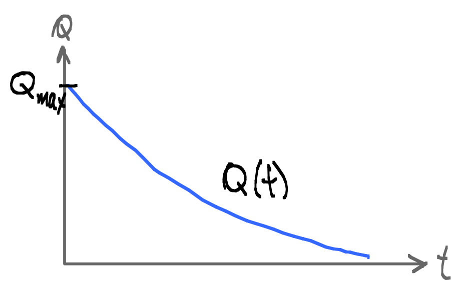
$$\Rightarrow ⑤ \; \int_{Q_{max}}^{Q} \frac{dQ}{Q} = -\frac{1}{RC}\int_0^t dt \Rightarrow ⑥ \; \ln Q \Big|_{Q_{max}}^{Q} = -t/RC$$

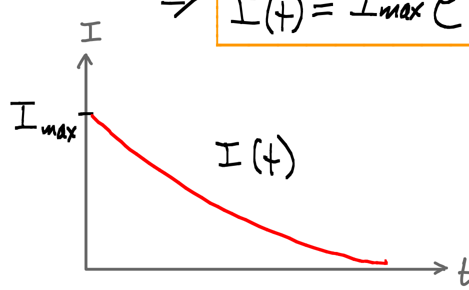
$$\Rightarrow ⑦ \; \ln Q - \ln Q_{max} = \ln Q/Q_{max} = -t/RC$$

$$\Rightarrow ⑧ \; e^{\ln Q/Q_{max}} = e^{-t/RC} \qquad ⑨ \; \frac{Q}{Q_{max}} = e^{-t/RC}$$

$$\Rightarrow \boxed{Q(t) = Q_{max} e^{-t/\tau}} \quad \text{Since } I(t) = -\frac{dQ}{dt} \quad I(t) = -\frac{Q_{max}}{-RC} e^{-t/RC}$$

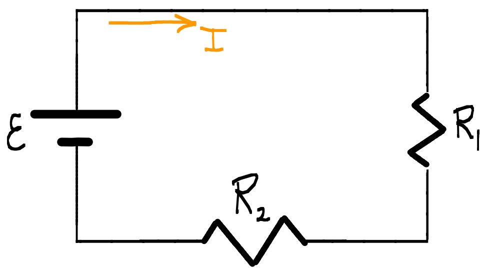
$$\Rightarrow \boxed{I(t) = I_{max} e^{-t/\tau}}$$





Resistors in Series

If we connect resistors in series, as in the following diagram, we know they must have the same current.



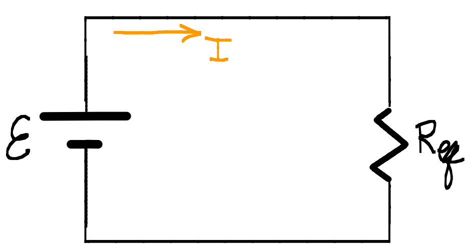
(Ignore battery internal resistance for now.)

Since the voltage drop across the two resistors adds up to the battery voltage we can say,

$$① \ \mathcal{E} = \Delta V_1 + \Delta V_2 \qquad ② \ \Delta V_1 = I R_1 \text{ and } \Delta V_2 = I R_2$$

$$\Rightarrow ③ \ \mathcal{E} = I R_1 + I R_2 = I(R_1 + R_2)$$

Considering the two resistors to be a single resistive load on the battery we could draw the circuit as,

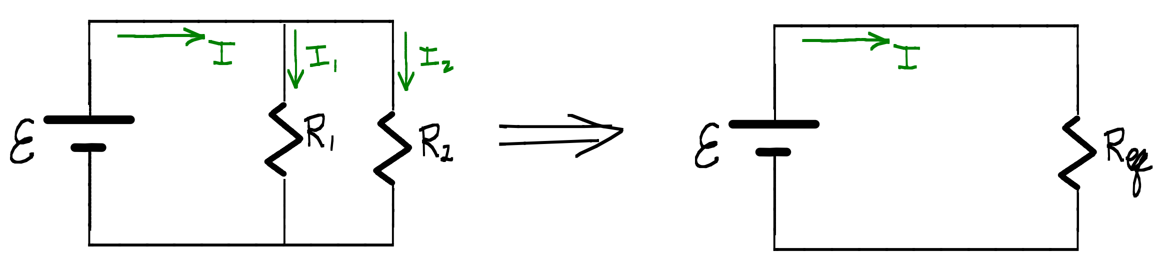


$$\mathcal{E} = I R_{eq} \Rightarrow \boxed{R_{eq} = R_1 + R_2}$$

where $R_{eq} \equiv$ the equivalent resistance.

So, the rule for adding resistances in series is $R_{eq} = R_1 + R_2 + R_3 + \ldots$ for however many there are.

Resistors in parallel



New we want to know the relationship between R_1, R_2 and R_{eq}.We can answer this question by noting that R_1 and R_2 have the same voltage across them.

$$\Rightarrow \mathcal{E} = I_1 R_1 \text{ and } \mathcal{E} = I_2 R_2$$

We also know that $I = I_1 + I_2$ and, from the equivalent circuit, $\varepsilon = I\,R_{eq}$

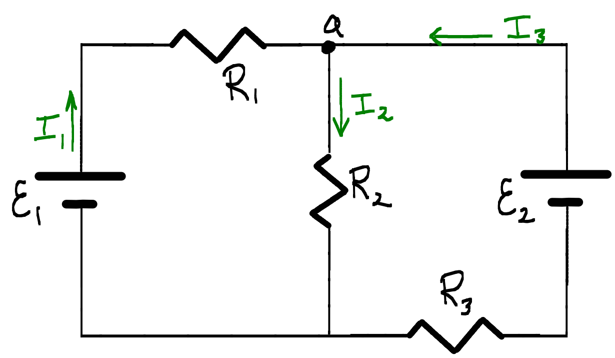
$$\Rightarrow \frac{\mathcal{E}}{R_{eq}} = \frac{\mathcal{E}}{R_1} + \frac{\mathcal{E}}{R_2} \Rightarrow \boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}}$$

Note: $\frac{1}{R_{eq}} = \frac{R_1+R_2}{R_1R_2} \Rightarrow R_{eq} = \frac{R_1R_2}{R_1+R_2} = R_1\left(\frac{R_2}{R_1+R_2}\right) = R_2\left(\frac{R_1}{R_1+R_2}\right)$

$\therefore R_{eq} < R_1$ or R_2 in the parallel case.

Kirchhoff's Rules

Suppose we want to analyze a circuit that is more complicated than the previous 2. Consider the circuit in the diagram.



Can we simplify this circuit by combining series or parallel resistors. i.e., do any two resistors have to have the same current or potential differences?

Answer: NO!

So we need a way to analyze this circuit. If we connect three resistors and two batteries as shown we'd like to be able to predict I_1, I_2 and I_3.

Charge is conserved so, looking at junction a, we see $I_1 + I_2 = I_3$ because all the charge going into a has to leave a. We also know that a charge going around any loop of a circuit has to end up with the same energy it started with. If this weren't true a charge could gain energy each time around a loop and the energy of the charge could become infinite.

Since a charge going around a loop maintains the same charge, and energy change through each circuit element is the charge times the potential difference across the element, the sum of the potential differences around any loop must add to zero.

Kirchhoff's Rules.

Junction rule: The current into a junction equals the current out of the junction.

Loop rule: The potential differences across all components in a circuit loop add to zero.

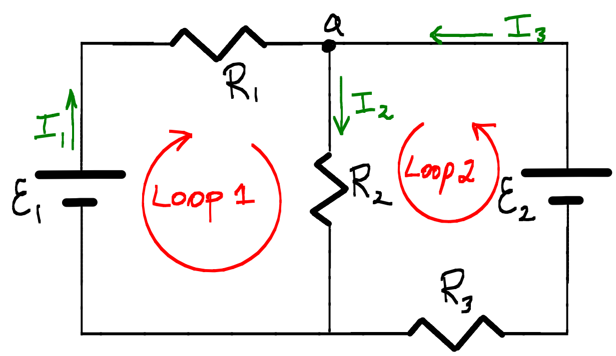
memorize!

Let's find all three currents in the circuit depicted above with the numerical values: ε_1 = 10.0 V; ε_2 = 8.0 V; R_1 = 2.0 Ω ; R_2 = 5.0 Ω ; R_3 = 3.0 Ω

Junction at a: $I_1 + I_3 = I_2$

Loop 1: $\mathcal{E}_1 - I_1 R_1 - I_2 R_2 = 0$

Loop 2: $\mathcal{E}_2 - I_2 R_2 - I_3 R_3 = 0$



Rewriting:

$$I_1 - I_2 + I_3 = 0 \qquad\qquad I_1 - I_2 + I_3 = 0$$
$$10 - 2I_1 - 5I_2 = 0 \quad \Rightarrow \quad 2I_1 + 5I_2 + 0 = 10$$
$$8 - 5I_2 - 3I_3 = 0 \qquad\qquad 0 + 5I_2 + 3I_3 = 8$$

Check:

```
In[2]:= Solve[{I1 - I2 + I3 == 0, 10 - 2 * I1 - 5 * I2 == 0,
          8 - 5 * I2 - 3 * I3 == 0}, {I1, I2, I3}]
```

Out[2]= $\left\{\left\{I1 \to \frac{40}{31},\ I2 \to \frac{46}{31},\ I3 \to \frac{6}{31}\right\}\right\}$

Cramer's Rule:

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 5 & 0 \\ 0 & 5 & 3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 8 \end{pmatrix} \quad ; \quad \mathbb{D} = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 5 & 0 \\ 0 & 5 & 3 \end{vmatrix}$$

$$\mathbb{D} = 1(5\cdot3 - 0\cdot5) - (-1)(2\cdot3 - 0\cdot0) + 1(2\cdot5 - 0\cdot5)$$

$$= 15 + 6 + 10 = 31$$

$$I_1 = \frac{\begin{vmatrix} 0 & -1 & 1 \\ 10 & 5 & 0 \\ 8 & 5 & 3 \end{vmatrix}}{\mathbb{D}} = \frac{0 + 1(30) + 1(50 - 40)}{31} = \frac{40}{31}$$

$$I_2 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 2 & 10 & 0 \\ 0 & 8 & 3 \end{vmatrix}}{\mathbb{D}} = \frac{1(30) - 0 + 1(16)}{31} = \frac{46}{31}$$

From the junction rule:

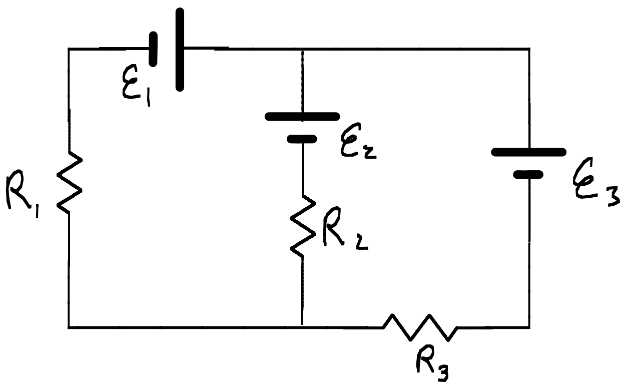
$$I_3 = I_2 - I_1 = \frac{46}{31} - \frac{40}{31} = \frac{6}{31}$$

In decimals:

$$I_1 = 1.290\,A$$
$$I_2 = 1.484\,A$$
$$I_3 = 0.193\,A$$

Kirchhoff Supplement:

Goal: Find currents in the three branches of the following circuit,

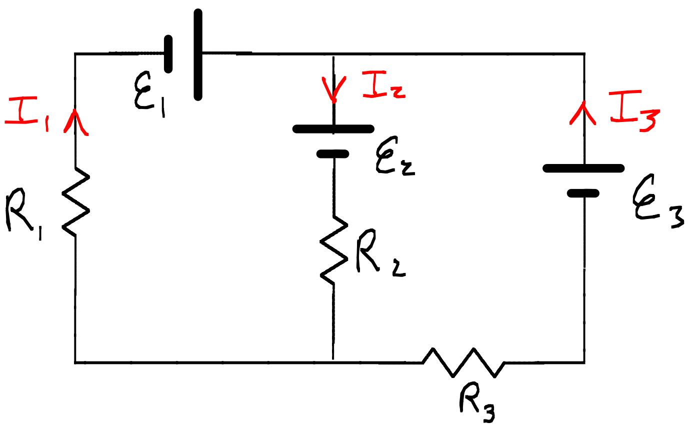


$\mathcal{E}_1 = 5V$ $\quad R_1 = 10\Omega$

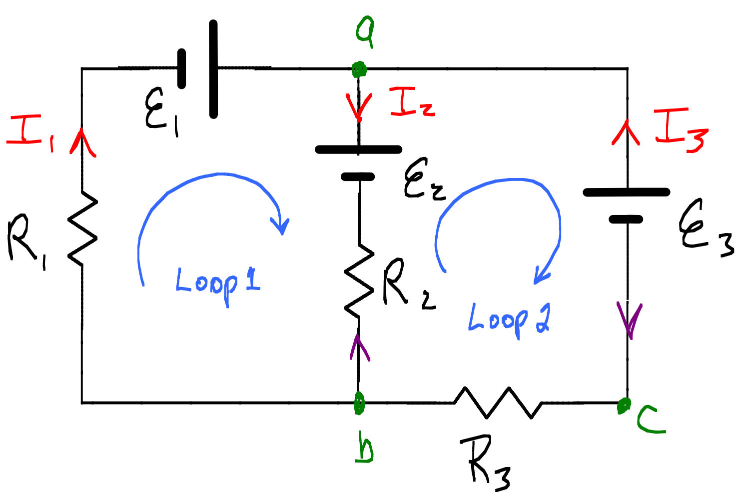
$\mathcal{E}_2 = 3V$ $\quad R_2 = 6\Omega$

$\mathcal{E}_3 = 2V$ $\quad R_3 = 1\Omega$

Step 1: Label each current and assume



Step 2: Define and label loops. Label junctions.



Step 3: Use junction rule. $\quad I_1 + I_3 = I_2$

$\mathcal{E}_1 = 5V$ $\quad R_1 = 10\Omega$

$\mathcal{E}_2 = 3V$ $\quad R_2 = 6\Omega$

$\mathcal{E}_3 = 2V$ $\quad R_3 = 1\Omega$

Step 4: Use loop rule to get solvable system.

From loop 1: $\quad -I_1 R_1 + \mathcal{E}_1 - \mathcal{E}_2 - I_2 R_2 = 0$

From loop 2: $\quad \mathcal{E}_2 - \mathcal{E}_3 + I_3 R_3 + I_2 R_2 = 0$

Step 5: Put in numbers and rewrite.

$$I_1 - I_2 + I_3 = 0 \qquad\qquad I_1 - I_2 + I_3 = 0$$
$$5 - 3 - 10I_1 - 6I_2 = 0 \quad\Rightarrow\quad 2 - 10I_1 - 6I_2 = 0$$
$$3 - 2 + I_3 + 6I_2 = 0 \qquad\qquad 1 + 6I_2 + I_3 = 0$$

Step 6: Solve the system.

```
In[1]:= Solve[{I1 - I2 + I3 == 0, 2 - 10 * I1 - 6 * I2 == 0,
           1 + 6 * I2 + 1 * I3 == 0}, {I1, I2, I3}]
```

$$\text{Out[1]=}\ \left\{\left\{I1 \to \frac{5}{19},\ I2 \to -\frac{2}{19},\ I3 \to -\frac{7}{19}\right\}\right\}$$

Step 7: Examine results: Note I_2 and I_3 are negative, which means the actual direction of flow in the associated branches is opposite the assumed directions.

Now let's get the potential difference between points a and c.

$$V_a - \mathcal{E}_2 + I_2R_2 + I_3R_3 = V_c$$
$$\Rightarrow V_c - V_a = I_2R_2 + I_3R_3 - \mathcal{E}_2$$
$$= \left(\frac{2}{19}\right)6 + \frac{7}{19}(1) - 3$$
$$= \frac{12}{19} + \frac{7}{19} - 3 = -2\,\text{Volts}$$

$\mathcal{E}_1 = 5V$ $\quad R_1 = 10\Omega$

$\mathcal{E}_2 = 3V$ $\quad R_2 = 6\Omega$

$\mathcal{E}_3 = 2V$ $\quad R_3 = 1\Omega$

Uses actual
current directions!

Check!-Let's take a different path:

$$V_a - \mathcal{E}_1 + I_1R_1 + I_3R_3 = V_c$$
$$\Rightarrow V_c - V_a = I_1R_1 + I_3R_3 - \mathcal{E}_1$$
$$= \frac{5}{19}*10 + \frac{7*1}{19} - 5V = \frac{50}{19} + \frac{7}{19} - 5$$
$$= \frac{57}{19} - 5 = 3 - 5 = -2\,\text{Volts}\ !!$$

(How reassuring!)