Fourier Series: An Example of Characteristic Behaviors

Concepts of primary interest:

- **Independent behavior**
- **Completeness**
- **Inner product, a gauge of common character/behavior**
- **Orthogonality**
- **Projection - projecting out a behavior coefficient**
- **Fundamental frequency - relation to the period**
- **Convergence**
- **Rapid changes $\Rightarrow$ greater high frequency content**
- **Parseval's identity**
- **Relating series with different periods**
- **Power series expansions are good – why choose Fourier?**
- **FS and the Vector Space of periodic functions**

Sample Calculations and Applications:

- **Anti-symmetric square wave**
- **Anti-symmetric triangular wave**
- **Oscillator with a periodic driving force**
- **Summing series**

Tools of the trade:

- **The Unit Circle for evaluating trig functions**
- **Converting sums to integrals**

Discovery Exercise:

- **Summing terms to match a specified function**
- **Give f(x) = the sum of a few terms with integer …**

See problem 41 on page 79+ **********

Contact: tank@usna.edu

The value c_o as defined in the section is equivalent to ½ a_o as defined everywhere else in the known universe. Note that this 'distinct' notation also leads to a change in the form of Parseval's equality. The rule will be to replace c_o by ½ a_o throughout to compare with other Fourier references.

Introduction to Fourier Series: More haphazard history ⇒ It's just a story with no basis in fact used to introduce a topic.

Fourier studied the problem of the vibrating string fixed at both ends discovering that it has a set of characteristic vibration patterns in which the string oscillates in the shape of any multiple of half cycles of a sinusoid with an amplitude that oscillates sinusoidally in time at frequencies that are proportional to that multiple of half cycles ($\omega_m = m\ \omega_o$). The result was so appealing that Fourier declared that these were the only shapes in which one could expect to find the string. A musician objected immediately claiming that his violin string had a triangular shape when he plucked it. Stung by the prospect of being proven wrong, Fourier boldly proposed that all shapes assumed by the violin string could be represented as a sum of his sinusoids. Indeed any function can be represented as a sum of sinusoids with just a few restrictions. The allowed sine and cosine functions are periodic, and each has zero average value. If we restrict ourselves to periodic functions or functions defined over a finite interval which can be assumed to be extended periodically beyond that interval, and we toss in an additive constant to handle the average value, and it works. It works as long as the correct definition for working is adopted.

Functions are also restricted in that they must be finite and continuous except for perhaps a finite number of finite discontinuities. Some more exotic behaviors can be tolerated, but here the behavior assumed is

finite, piecewise continuous and possessing only a finite number of discontinuities in a period. See the Dirichlet conditions presented later for a more precise statement of the requirements.

In the time-frequency realm, Fourier's conjecture takes the form: Any function $f(t)$ which is periodic in time with period T can be represented as a sum of a constant plus sinusoidal terms with frequencies $\omega_m = m(2\pi/T)$. The sinusoids are the sine and cosine. The set of frequencies ω_m ensures that each sinusoid completes an integer (m) number of cycles in the time T, the period of $f(t)$. The base frequency for the expansion is $\omega_0 = \omega_1 = (2\pi/T)$ which is called the **fundamental frequency**. This frequency is set by the **period of the function,** and it is the same for all functions with that period. The frequency of every term in the expansion is an integer multiple (harmonic) of this fundamental frequency.

To be periodic with period T, each sinusoid must complete an integer number of cycles in the time T. This condition sets the allowed frequencies to the multiples of the fundamental $2\pi/T$ independent of any feature of the function other than its period.

$$f(t) \quad = \quad c_0\,(1) \quad + \quad \sum_{n=1}^{\infty} a_m \cos[\omega_n t] \quad + \quad \sum_{m=1}^{\infty} b_m \sin[\omega_m t] \quad \text{where } \omega_m = m(2\pi/T). \qquad \text{[FS.1]}$$

Each term in the expansion represents a distinct behavior of a function with period T. The first term c_0 (1), a fancy way to write c_0, is the average value of the function with the (1) indicating that it represents the constant behavior of the function. The terms $a_m \cos[\omega_m t]$ represent the independent or distinct ways in which an even function ***of period T*** can wiggle, and the terms $b_m \sin[\omega_m t]$ represent the independent or distinct ways in which an odd function of period T can wiggle. It is crucial to recognize than each term represents an **independent behavior**.

(*The constant c_0 is replaced by ½ a_0 in most treatments of Fourier series.*)

Recall the Cartesian representation of a particle's displacement:

$$\Delta\vec{r} = \Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}$$

Each term in the representation of $\Delta\vec{r}$ is an independent contribution to the displacement. No amount of $\Delta x\,\hat{i}$ can change the *y* or *z* coordinate of the particle. The statement of this independence or orthogonality is in the relations:

$$\hat{i}\cdot\hat{j} = 0;\quad \hat{k}\cdot\hat{i} = 0;\quad \hat{j}\cdot\hat{k} = 0;\quad \hat{i}\cdot\hat{i} = 1;\ \hat{j}\cdot\hat{j} = 1;\ \hat{k}\cdot\hat{k} = 1$$

The first three equations are orthogonality relations, and the last three normalization relations. A displacement in the *x* direction is exactly like itself and nothing like either of the other two possibilities. These (dot) inner product relations provide the method to find the components of $\Delta\vec{r}$.

$$\left(\Delta\vec{r}\right)_x = \hat{i}\cdot\Delta\vec{r} = \hat{i}\cdot\left(\Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}\right) = \Delta x$$

One evaluates the inner product of the *x*-direction and $\Delta\vec{r}$ to find the *x*-component of $\Delta\vec{r}$. This process is called projection. The orthogonality of the coordinate directions insures that the projected components are measures of independent (displacement) behaviors. The choice of unity normalization ($\hat{i}\cdot\hat{i} = 1;\ \hat{j}\cdot\hat{j} = 1;$ $\hat{k}\cdot\hat{k} = 1$) for the three coordinate directions leads to the familiar result that:

$$\left|\Delta\vec{r}\right|^2 = \left(\Delta x\right)^2 + \left(\Delta y\right)^2 + \left(\Delta z\right)^2$$

There is a relation between the norm of the object represented and the sum of the squares of the expansion coefficients (components). This paragraph may seem a pointless now. Review it after completing this chapter. It reveals the tip of a megacept. Track the terms: inner product, projection, orthogonality.

Fourier representations work the same way. Orthogonality is the statement that each term represents an independent wiggling of the function. The inner product takes the form of an integral.

$$\left\langle g(t)\middle| f(t)\right\rangle = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\,f(t)\,dt$$

The strange notation $\left\langle g(t)\middle| f(t)\right\rangle$ is chosen to represent a general inner product. The

symbolic representation is chosen to be unfamiliar forcing you to regard it as a new process. The integral is sometimes designated as the overlap of $g(t)$ and $f(t)$ over the interval. Each of the terms: $\{c_0 1;\{a_m \cos[\omega_m t]\};\{b_p \sin[\omega_p t]\}\}$ is an independent wiggling or behavior that has zero overlap with any other member of the set. Therefore, the coefficient in the expansion for a member of the set $\{1;\{\cos[\omega_m t]\};\{\sin[\omega_p t]\}\}$ can be **projected out** by multiplying the sum by the member (behavior) associated with that coefficient and integrating over the range $[-T/2, T/2]$.

Fuzzyspeak; not a definition: An **inner product** gauges degree to which one entity embodies the behavior of another entity of the same class. To what degree do two vectors point in the same direction? To what degree do $g(t)$ and $f(t)$ wiggle together ? More about this choice for the inner product for wiggling functions is to be presented in a meandering mind section below.

Projection is important as a concept and as a procedure.

PROJECTION: A process to isolate a coefficient or behavior in an entity. Multiply the entity and the expansion that represents it by a behavior and execute the inner product operation. Use orthogonality relations. To find the coefficient a_m for $\cos[\omega_m t]$, multiply by $\cos[\omega_m t]$ and compute the inner product with the function *f(t)* and with its Fourier expansion.

Begin with 1. Find out how much constant behavior is in the function.

$$\frac{1}{T}\int_{-T/2}^{T/2}(1)\, f(t)\, dt = \frac{1}{T}\int_{-T/2}^{T/2}(1)\, c_0\, dt + \sum_{m=1}^{\infty} a_m \frac{1}{T}\int_{-T/2}^{T/2}(1)\cos[\omega_m t]\, dt + \sum_{m=1}^{\infty} b_m \frac{1}{T}\int_{-T/2}^{T/2}(1)\sin[\omega_m t]\, dt$$

The integrals of the sine and cosines are over an integral number of cycles; hence they vanish. Wiggling like a sine or cosine for an integral number of cycles is nothing like being constant. So:

$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2}(1)\, f(t)\, dt \quad = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, dt = \langle f(t)\rangle \qquad \text{[FS.2]}$$

The constant part of the function $f(t)$ is just its average value over the interval $\left[-T/2, T/2\right]$.

To identify the distinct wiggle contributions, the coefficients for each distinct wiggle behavior are projected out of the sum by invoking the following orthogonality relations:

$$\begin{aligned} \frac{1}{T}\int_{-T/2}^{T/2} \sin\left[\omega_p t\right]\cos\left[\omega_m t\right] dt &= 0 \\ \frac{1}{T}\int_{-T/2}^{T/2} \sin\left[\omega_p t\right]\sin\left[\omega_m t\right] dt &= \tfrac{1}{2}\,\delta_{pm} \\ \frac{1}{T}\int_{-T/2}^{T/2} \cos\left[\omega_p t\right]\cos\left[\omega_m t\right] dt &= \tfrac{1}{2}\,\delta_{pm} \end{aligned} \qquad \text{[FS.3]}$$

The first relation is trivial as the product of sine and cosine is an odd function. The next two relations are proven below, and a second set of proofs is to be developed in a homework problem. That is: proof by exhaustion follows.

The coefficients for the independent wiggle modes become:

$$a_p = \frac{2}{T}\int_{-T/2}^{T/2} \cos\left[\omega_p t\right]\ f(t)\,dt\,;\ b_m = \frac{2}{T}\int_{-T/2}^{T/2} \sin\left[\omega_m t\right]\ f(t)\,dt \text{ for } p,\, m \geq 1 \qquad \text{[FS.4]}$$

The factor of two appear because the average values of $\cos^2\left[\omega_p t\right]$ and of $\sin^2\left[\omega_p t\right]$ are one half (while the average value of 1 is 1).

Do not focus on the equations to compute the expansion coefficients. Focus on the orthogonality relations and the projection process to isolate the coefficients. That is: use the orthogonality relations to essentially repeat the derivation of the equations above each time you use them. It sounds extreme, but the reward is that you command a powerful technique that applies far beyond the sine and cosine series expansions.

Exercise: Find the coefficient b_p by multiplying the series representation by

$\sin\left[\omega_p t\right]$ and integrating over the range $\left[-T/2, T/2\right]$. Use the orthogonality relations.

The proofs of the orthogonality relations above follow from the identities for products of sines and of cosines. Recalling that $\omega_m = m\left(2\pi/T\right) = m\omega_0$:

$$\sin(x)\sin(y) = \tfrac{1}{2}\left[\cos(x-y) - \cos(x+y)\right] \Rightarrow$$
$$\sin\left[\omega_p t\right]\sin\left[\omega_m t\right] = \tfrac{1}{2}\left[\cos\left[(p-m)\omega_0 t\right] - \cos\left[(p+m)\omega_0 t\right]\right]$$

* *Treat the cases p = m and* $p \neq m$ *separately*. Always start with $p \neq m$.

$$p \neq m: \quad \frac{1}{T}\int_{-T/2}^{T/2} \sin\left[\omega_p t\right]\sin\left[\omega_m t\right] dt = \frac{1}{T}\int_{-T/2}^{T/2} \tfrac{1}{2}\left[\cos\left[(p-m)\omega_0 t\right] - \cos\left[(p+m)\omega_0 t\right]\right] dt$$

$$= \frac{1}{2T}\left[\frac{\sin\left[(p-m)\omega_0 t\right]}{(p-m)\omega_0} - \frac{\sin\left[(p+m)\omega_0 t\right]}{(p+m)\omega_0}\right]_{-T/2}^{+T/2} = 0 \quad if \quad p \neq m$$

The sines are periodic with period T so $\sin[(p - m)\,\omega_o T/2] = \sin[-(p - m)\,\omega_o T/2]$. Note that the result above is ill defined for $p = m$. This divergence requires that the $p = m$ case be treated separately.

$$p = m: \quad \frac{1}{T}\int_{-T/2}^{T/2} \sin\left[\omega_m t\right]\sin\left[\omega_m t\right] dt = \frac{1}{T}\int_{-T/2}^{T/2} \tfrac{1}{2}\left[\cos\left[0\right] - \cos\left[(2m)\omega_0 t\right]\right] dt$$

$$= \frac{1}{2T}\left[t - \frac{\sin\left[(2m)\omega_0 t\right]}{(2m)\omega_0}\right]_{-T/2}^{+T/2} = \frac{1}{2} \quad if \quad p = m$$

as $\sin\left[\omega_m\left(T/2\right)\right] = \sin\left[m\left(2\pi/T\right)\left(T/2\right)\right] = \sin\left[m\pi\right] = 0$ for all integers m is. Hence:

$$\left(1/T\right)\int_{-T/2}^{T/2} \sin\left[\omega_p t\right]\sin\left[\omega_m t\right] dt = \tfrac{1}{2}\delta_{pm}$$

Exercise: Examine: $\frac{1}{2T}\left[\frac{\sin\left[(p-m)\omega_0 t\right]}{(p-m)\omega_0} - \frac{\sin\left[(p+m)\omega_0 t\right]}{(p+m)\omega_0}\right]_{-T/2}^{+T/2} = 0$. What is the problem that arises for the $p = m$ case? What is $\cos[(p - m)\omega_o t)$ in the $p = m$ case?

Exercise: Use the change of variable $u = \omega_0 t$ and the trigonometric identity:

$$\cos(x)\cos(y) = \tfrac{1}{2}\left[\cos(x+y)+\cos(x-y)\right] \rightarrow$$
$$\cos\left[\omega_p t\right]\cos\left[\omega_m t\right] = \tfrac{1}{2}\left[\cos\left[(p+m)\omega_0 t\right]+\cos\left[(p-m)\omega_0 t\right]\right]$$

to show that: $\left(\tfrac{1}{T}\right)\int_{-T/2}^{T/2}\cos\left[\omega_p t\right]\cos\left[\omega_m t\right]dt = \tfrac{1}{2}\delta_{pm}$.

Exercise: Repeat the proof that: $\left(\tfrac{1}{T}\right)\int_{-T/2}^{T/2}\sin\left[\omega_p t\right]\sin\left[\omega_m t\right]dt = \tfrac{1}{2}\delta_{pm}$ using the change of variable $u = \omega_0 t$.

The Orthogonality Relations for the Fourier Expansion Set

Orthogonality relations are crucial to the application of Fourier series methods. They are used in 70% of all Fourier problems and applications. **Highlight your every application of these relations. Try to prepare a prose summary of the manner in which the relations were used to solve each problem.** The Fourier expansion set has members

$$\{1;\{\ldots,\cos\left[\omega_m t\right],\ldots\};\{\ldots,\sin\left[\omega_p t\right],\ldots\}\}.$$

This set of functions is mutually orthogonal [with a typical member $\boldsymbol{F_i(t)}$] which means that the inner product of any two distinct members of the set vanishes:

$$\frac{1}{T}\int_{-T/2}^{T/2}\left[F_i(t)\right]^* F_j(t)\,dt = \begin{cases} 0 \; for\; i \neq j \\ N_j \; for\; i = j \end{cases} \qquad \text{General Orthogonality Relations}$$

The inner product is zero unless the two functions are the same in which case the result is a non-zero value. This equation is shorthand for the following relations: [FS.5]

$$\frac{1}{T}\int_{-T/2}^{T/2}[1]\sin\left[\omega_m t\right]dt = 0; \quad \frac{1}{T}\int_{-T/2}^{T/2}\sin\left[\omega_p t\right]\cos\left[\omega_m t\right]dt = 0; \quad \frac{1}{T}\int_{-T/2}^{T/2}[1]\cos\left[\omega_m t\right]dt = 0$$

$$\frac{1}{T}\int_{-T/2}^{T/2}[1]\ [1]\,dt = 1;\ \frac{1}{T}\int_{-T/2}^{T/2}\sin[\omega_p t]\sin[\omega_m t]\,dt = \tfrac{1}{2}\delta_{pm};\ \frac{1}{T}\int_{-T/2}^{T/2}\cos[\omega_p t]\cos[\omega_m t]\,dt = \tfrac{1}{2}\delta_{pm}$$

The relations state that a member of the Fourier expansion set wiggles like itself and not at all like any other member of the set over the domain [- ½ T, ½ T].

The **Kronecker** delta δ_{ij} is a shorthand for 0 if $i \neq j$ *and* 1 if $i = j$.

Inner Products and the Meandering Mind: (This section is not strictly required to understand the topic of expansions and othogonality, but the ideas presented may provide the key to a more global appreciation of the topic.)

The goal is to motivate the definition of the inner product as:

$$\langle g(t)|f(t)\rangle = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\,f(t)\,dt \qquad \text{[FS.6]}$$

Backtracking to 3D vectors, $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$ which becomes more positive for each component pair that points in the same direction (has the same sign). As vectors point, this inner (dot) product grows for common pointing (behavior). The inner product gauges the common behavior of two entities of the same class. How can this concept be generalized to the inner product of two functions?

Start with a simple case and work up to the real problem. A function assumes a value at each point in the interval. To begin, represent the function by its values at N evenly spaced values of its argument. Let $t_m = -\left(T/2\right) - \left(T/2N\right) + m\left(T/N\right)$ for m = $\{1,2,\ldots,N\}$ which selects the midpoints of the N equal width bins covering the range [- ½ T, ½ T]. The functions $g(t)$ and $f(t)$ are each represented by N values.

$$g(t) \to \{g(t_1), g(t_2), \ldots, g(t_N)\};\quad f(t) \to \{f(t_1), f(t_2), \ldots, f(t_N)\}$$

Following the expression for the dot product of two displacement vectors, the inner product of the two functions proposed is:

$$\langle g(t)|f(t)\rangle = g(t_1)f(t_1) + g(t_2)f(t_2) + \ldots + g(t_N)f(t_N) = \sum_m g(t_m)f(t_m)$$

The sum of the products of the corresponding **values at positions behaviors**. Not a bad start, but to get a better picture, the limit $N \to \infty$ is desired which requires division by N to keep the expression finite.

$$\langle g(t)|f(t)\rangle = \left(1/N\right)\left[g(t_1)f(t_1) + g(t_2)f(t_2) + \ldots + g(t_N)f(t_N)\right] = \sum_m g(t_m)f(t_m)\left(1/N\right)$$

The integral of a function is the limit of the sum of terms of that function times an infinitesimal step of the integration variable. A step from one bin to the next is $\Delta t = T/N$ so multiplication and division by T is in order. (See Converting Sums to Integrals in the Tools of the Trade section. Our function to integrate is the product of $f(t)$ and $g(t)$.)

$$\langle g(t)|f(t)\rangle = \left(1/T\right)\sum_m g(t_m)f(t_m)\left(T/N\right) = \left(1/T\right)\sum_m g(t_m)f(t_m)\left(\Delta t\right) = \left(1/T\right)\int g(t)f(t)dt$$

This prescription is almost great! The inner product of something with itself should always be positive. Something should positively behave like itself. To ensure this feature, a final tweak is made to the definition.

$$\boxed{\langle g(t)|f(t)\rangle = \tfrac{1}{T}\int_{-T/2}^{T/2}[g(t)]^* f(t)\,dt \qquad \text{[FS.7]}}$$

The * represents complex conjugation. This definition is correct for real valued functions and for complex valued ones. ***Correct*** is a slippery concept. Many definitions of the inner product can be ***correct*** as long as each properly gauges the common behavior of the functions. For example, the scale factor is arbitrary. As long as the rule is consistently applied, the definition could be modified to be $\left(1/2T\right)\int_{-T/2}^{T/2}\left(g(t)\right)^* f(t)dt$ or even just $\int_{-T/2}^{+T/2}\left(g(t)\right)^* f(t)dt$. Be alert to changes in notation and normalizations as you review different references.

The goal has been to motivate the choice of the overlap integral as the definition of the inner product. If you have been paying attention, you should not be buying it! The inner product is about gauging common behavior. The behaviors under discussion are being constant (having an average value) and wiggling periodically which are represented by the Fourier coefficients $\left\{c_0,\{\ldots,a_m,\ldots\},\{\ldots,b_p,\ldots\}\right\}$ which would *naturally* lead to the inner product definition (which is similar to the sum of the products of the corresponding components):

$$\langle g|f\rangle = \left(\tfrac{1}{T}\right)\int g(t)^* f(t)dt = c_{g0}^* c_{f0} + \tfrac{1}{2}\sum_m a_{gm}^* a_{fm} + \tfrac{1}{2}\sum_p b_{gp}^* b_{fp} \qquad \text{[FS.8]}$$

The factors of one-half are an annoying penalty for choosing the expansion set $\{1;\{\cos[\omega_m t]\};\{\sin[\omega_p t]\}\}$ with the uneven mean square value (normalization).

$$\langle 1^2\rangle = 1;\quad \left\langle\left(\cos[\omega_m t]\right)^2\right\rangle = \tfrac{1}{2};\quad \left\langle\left(\sin[\omega_m t]\right)^2\right\rangle = \tfrac{1}{2}$$

Other than that, life is wonderful! The two definitions agree. It takes a little perspiration to establish the equality, but they do agree. The name of some long-dead mathematician should be tied to this equality. Failing to find a clearly culpable individual, Parseval is to be given credit for the result. The integral definition of the inner product is in the **coordinate representation** in which the behavior of a function is to take on a value at a point. The equivalent expression just above is in the **Fourier or frequency-space representation** in which the set of wiggle mode behaviors provides a basis for representing functions. As long as the functions $f(t)$ and $g(t)$ are periodic with period T, these modes provide a **complete set** of the possible behaviors for well-behaved functions with period T. In either of the representations, the inner product gauges the degree to which the functions share behavior. The coordinate and frequency representations both provide complete descriptions of the problem. An engineer studying string clearances in the design of a violin might choose the coordinate space

representation for the violin string. A musician playing the instrument might be more directly interested in the tone and might choose the frequency representation.

Parseval's Equality Notation: in other references expect:

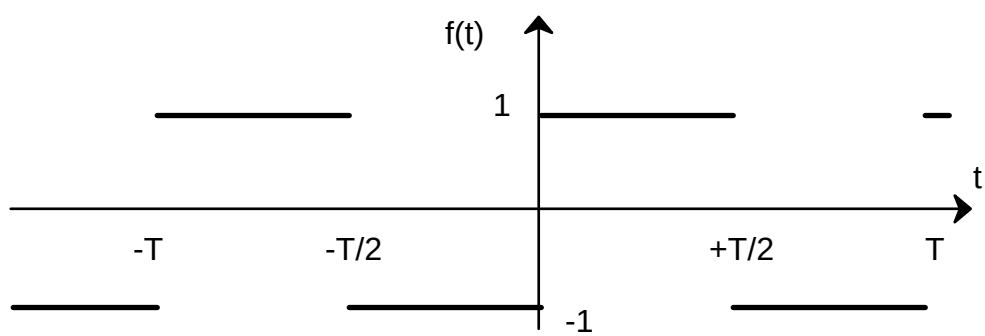
$$\langle g|f\rangle = \left(\tfrac{1}{T}\right)\int_{-\frac{1}{2}T}^{\frac{1}{2}T} g(t)^* f(t)dt = \tfrac{1}{4}a_{g0}^* a_{f0} + \tfrac{1}{2}\sum_{m=1}^{\infty} a_{gm}^* a_{fm} + \tfrac{1}{2}\sum_{p=1}^{\infty} b_{gp}^* b_{fp}$$

Recall the average value of the function c_o is to be replaced by ½ a_o to agree with the notation more commonly used.

$$c_0 = \left(\tfrac{1}{2}\right) a_0 = \frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} (1) f(t)dt \quad = \frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} f(t)dt = \langle f(t)\rangle \qquad \text{[FS.9]}$$

Sample Calculations: Fourier series for square and triangular waves

Example 1: Square waves come in several forms. As an example, an odd square wave with period T stepping from -1 to +1 is presented.



The square wave $f(t)$ has a period of T and is extended beyond the base period to display the discontinuities at each multiple of $^T/_2$. The function is represented as:

$$f(t) = \begin{cases} +1 & for \quad 0 < t < T/2 \\ -1 & for \quad -T/2 < t < 0 \end{cases} \qquad \text{[FS.10]}$$

Clearly, the function is odd and has zero average value. The Fourier series only includes terms with the behaviors intrinsic to the function. Only the sine terms

match the zero average value and odd function behaviors. Evaluating the integrals to find c_0 and the $\{\ldots a_p \ldots\}$ yields zero contributions. The expansion has the form:

$$f(t) \;=\; c_0\,(1) \;+\; \sum_{m=1}^{\infty} a_m \cos[\omega_m t] \;+\; \sum_{m=1}^{\infty} b_m \sin[\omega_m t] \;\rightarrow\; \sum_{m=1}^{\infty} b_m \sin[\omega_m t] \qquad \text{[FS.11]}$$

Using the symmetry, the expression for b_P reduces to:

$$b_p = \frac{2}{T}\int_{-T/2}^{T/2} \sin\left[\omega_p t\right] f(t)\,dt = \frac{2}{T}\left[\int_{-T/2}^{0} \sin\left[\omega_p t\right](-1)\,dt + \int_{0}^{T/2} \sin\left[\omega_p t\right](+1)\,dt\right]$$

$$b_p = \frac{4}{T}\int_{0}^{T/2} \sin\left[p\left(2\pi/T\right)t\right] f(t)\,dt$$

Invoking standard practice, the change of variable $u = p\left(2\pi/T\right)t$ is elected which yields $dt = {}^{T}/_{(2\pi p)}\,du$. [The standard choice is the argument of the most complicated function in the integrand. In this case, either $u = p\left(2\pi/T\right)t$ or $v = \left(2\pi/T\right)t$ is a good choice. In all cases, a dimensionless variable should be chosen.]

$$b_p = \frac{4}{T}\left(\frac{T}{2\pi\, p}\right)\int_{0}^{p\pi} \sin[u]\;du = \left(\frac{2}{\pi\, p}\right)\Big[-\cos(u)\Big]\Big|_{0}^{p\pi} = \left(\frac{2}{\pi\, p}\right)\left[1-\cos(p\pi)\right]$$

Cosine of even multiples of π is one; of odd multiples minus one. Hence:

FOURIER SERIES FOR THE ANTI-SYMMETRIC SQUARE WAVE:

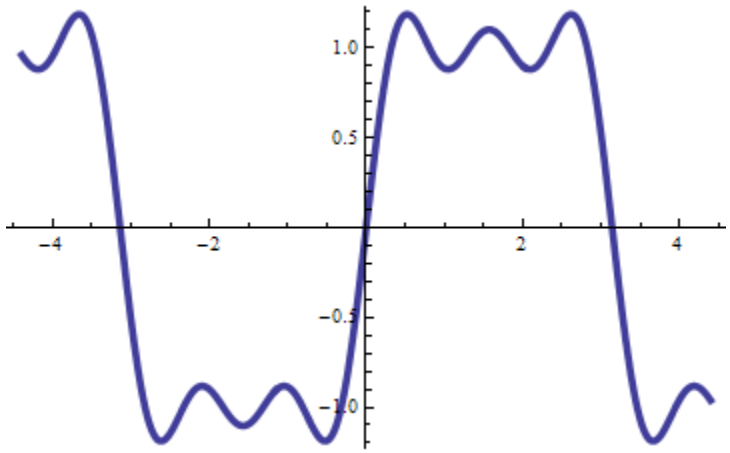
$$b_p = \begin{cases} \left({}^{4}\!/_{\pi\, p}\right) & \textit{for p odd} \\ 0 & \textit{for p even} \end{cases} \qquad \text{or} \qquad f(t) \;= \frac{4}{\pi}\sum_{m=0}^{\infty}\left(\frac{1}{2m+1}\right)\sin\left[\left(2m+1\right)\frac{2\pi\, t}{T}\right]$$

The inverse p dependence means that the coefficients vanish only slowly in the high frequency (large p) limit. High frequencies (fast wiggles) are necessary to reproduce the very rapid changes at the discontinuities. The factor $(2\,m+1)$ is a standard representation for the odd integers.

Exercise: Sketch the sines on a plot of the square wave. Use the sketches to explain the absence of the even p terms and the $^1/_p$ dependence of the coefficients for the odd p terms.

Plotting the first three terms:

```
Plot[(4/Pi)*(Sin[x]+Sin[3x]/3+Sin[5 x]/5),{x,-1.4
Pi,1.4 Pi}, PlotStyle->Thickness[0.01]]
```

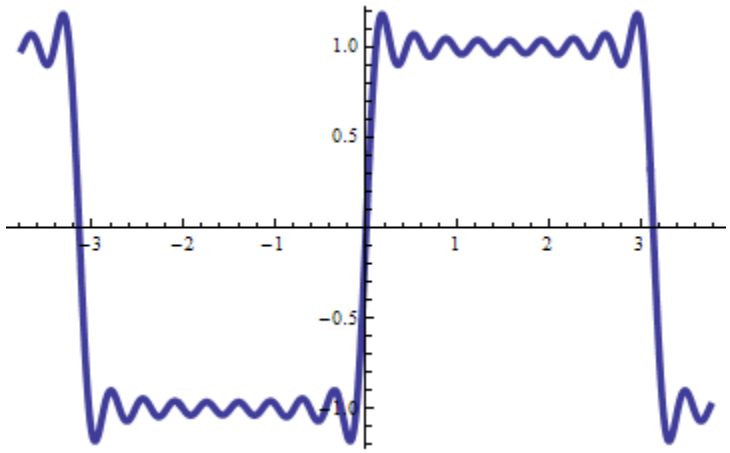


The square wave is evident with only the first three terms. The few term expansion does is not a good representation in the region around the discontinuity, but that is expected. A discontinuity is a rapid change that can only be represented by rapidly changing functions, high-frequency sinusoids. At the discontinuity, the series sums to zero, ***the average of the limits of the function as the discontinuity is approached from smaller values of the argument and from larger values***. Examining the first nine terms of the expansion does not alter these observations. The inclusion of the higher frequencies reproduces the jump more faithfully. One potentially disturbing point is noted. While the miss region near the discontinuity is shrinking in extent, the magnitude of the overshoot just before it is not. It remains at *about* **9%** of the discontinuity for sums of any finite number of terms. This behavior is the *Gibbs Overshoot Phenomenon* (after the American physicist J.W. Gibbs who studied the problem after Albert Michelson brought it to his attention).

The overshoot is discussed further when convergence properties are covered in a later section.

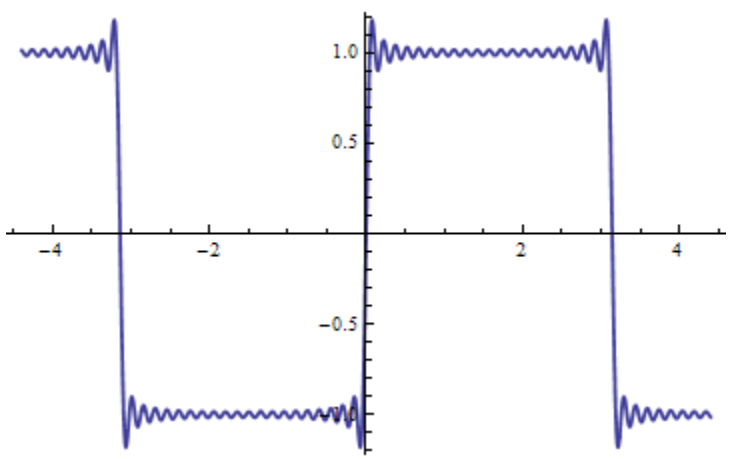
Plot with first nine non-zero terms ($m_{\max} = 17$):

```
Plot[(4/Pi) * ( Sin[x] + Sin[3x]/3 + Sin[5 x]/5 +Sin[7 x]/7 + Sin[9 x]/9 +
    Sin[11 x]/11 +Sin[13 x]/13+Sin[15 x]/15+Sin[17 x]/17), {x,-1.2  Pi,1.2 Pi}]
```



plot through $m = 17$

```
Plot[(4/Pi) * Sum[ Sin[m x]/m, {m,1,41,2}],{x,-1.4 Pi,1.4 Pi},  =
PlotStyle-> Thickness[0.005]]
```



plot through $m = 41$

Why were the coefficients of even index integer terms zero? Draw a few pictures.

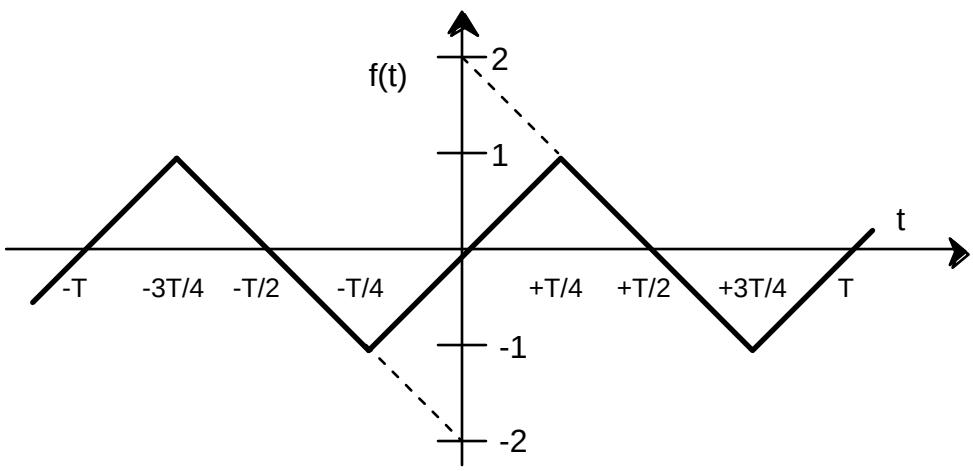
Draw one period of the square wave; add a sketch of $\sin\left[p\left(2\pi/T\right)\right]$ for $p = 1$.

Sketch the product of the functions. Repeat using successive values of p. For p even, $\sin\left[p\left(2\pi/T\right)\right]$ is odd about the points $-{}^{T}/_{4}$ and ${}^{T}/_{4}$ while the square wave is even about those points which leads to zero overlap or inner product. Being even or odd about a value of the argument is a distinct characteristic behavior.

Functions lacking distinct even or odd behavior can be expressed as a sum of an even function and an odd function. Choosing even and odd about $x = a$:

$$f(x) = \frac{f(a+[x-a]) + f(a-[x-a])}{2} + \frac{f(a+[x-a]) - f(a-[x-a])}{2} = f_{even} + f_{odd}$$

Example 2: The second example chosen is a triangular wave. This function is smoother than the square wave as it has no discontinuities. It does have periodic discontinuities in its first derivative. *Be alert*; *note the differences associated with expanding a smoother function.*



base period: $-{}^{T}//_{2} < t < {}^{T}//_{2}$

$$f(t) = \begin{cases} 2 - {}^{4t}/_{T} & for \quad {}^{T}/_{4} < t < {}^{T}/_{2} \\ {}^{4t}/_{T} & for \quad -{}^{T}/_{4} < t < {}^{T}/_{4} \\ -2 - {}^{4t}/_{T} & for \quad -{}^{T}/_{2} < t < -{}^{T}/_{4} \end{cases} \qquad \text{[FS.12]}$$

Once again, the function is odd with average value zero.

$$f(t) \;=\; \sum_{m=1}^{\infty} b_m \sin\left[\omega_m t\right] = \sum_{m=1}^{\infty} b_m \sin\left[m\,\frac{2\pi\, t}{T}\right] \text{ where}$$

$$b_p = \frac{2}{T}\int_{-T/2}^{T/2} \sin\left[\omega_p t\right] f(t)\, dt = \frac{2}{T}\int_{-T/2}^{-T/4} \sin\left[p\left(2\pi/T\right)t\right]\left(-2-4t/T\right) dt +$$

$$\frac{2}{T}\int_{-T/4}^{T/4} \sin\left[p\left(2\pi/T\right)t\right]\left(4t/T\right) dt + \frac{2}{T}\int_{T/4}^{T/2} \sin\left[p\left(2\pi/T\right)t\right]\left(2-4t/T\right) dt$$

The integrations necessary to compute the Fourier coefficients are not extraordinarily difficult, but the process does become tedious. The tools required to complete the problem are ***integration by parts*** and ***perseverance***.

Integration by parts is based on the identity for the differential of a product: $d(uv) = u\,dv + v\,du$ which can be applied to $d(u^n \cos[u]) = u^n\left(-\sin[u]\right) + n\,u^{n-1}\cos[u]$ or $u^n \sin[u] = -d(u^n \cos[u]) + n\,u^{n-1}\cos[u]$.

This result provides the tool to compute the integrals.

$$\int_{-T/4}^{T/4} t^n \sin\left[p\left(2\pi/T\right)t\right]\, dt = \left(T/2\pi\, p\right)^{n+1} \int_{-p\pi/2}^{p\pi/2} u^n \sin\left[u\right]\, du$$

$$= \left(T/2\pi\, p\right)^{n+1}\left[\left[-u^n \cos\left[u\right]\right]\Big|_{-p\pi/2}^{p\pi/2} + \int_{-p\pi/2}^{p\pi/2} n\,u^{n-1}\cos\left[u\right]\, du\right]$$

For our case of interest:

$$\int_{-T/4}^{T/4} t\, \sin\left[p\left(2\pi/T\right)t\right]\, dt = \left(T/2\pi\, p\right)^{2} \int_{-p\pi/2}^{p\pi/2} u\, \sin\left[u\right]\, du$$

$$= \left(T/2\pi\, p\right)^{2}\left[\left[-u\, \cos\left[u\right]\right]\Big|_{-p\pi/2}^{p\pi/2} + \int_{-p\pi/2}^{p\pi/2} \cos\left[u\right]\, du\right]$$

$$= \left(T/2\pi\, p\right)^{2}\left[-p\pi\cos\left(p\pi/2\right) + \sin\left(p\pi/2\right) - \sin\left(-p\pi/2\right)\right]$$

$$= \left(T/2\pi\, p\right)^{2} * \begin{cases} 2\,(-1)^{\left[p-1/2\right]} & \textit{for } p \textit{ odd} \\ -p\pi\,(-1)^{p/2} & \textit{for } p \textit{ even} \end{cases}$$

A miracle occurs, and the answer appears. Some of the steps toward that miracle include recognizing that the triangle wave is symmetric about $t = {}^{T}/_{4}$. The functions $\sin\left[\frac{m2\pi t}{T}\right]$ share this property if m is odd, but not if m is even. The Fourier expansion of the anti-symmetric triangular wave under study contains only sine terms with frequencies that are odd multiples of the fundamental.

Exercise: Sketch the functions $\sin\left[\frac{m2\pi t}{T}\right]$ form m = 1, 2, 3 and 4 on a sketch of the triangular wave for the region $0 < t < {}^{T}/_{2}$. For which m values is the integral of the product of the sine and triangular wave non-zero for the range: $0 < t < {}^{T}/_{2}$?

$$b_p = \frac{2}{T}\int_{-T/2}^{T/2} \sin\left[\omega_p t\right] f(t)\, dt \quad \rightarrow \frac{2}{T}(4)\int_{0}^{T/4} \sin\left[\omega_p t\right] f(t)\, dt \quad for\ p\ odd$$

$$b_p = \frac{2}{T}(4)\int_{0}^{T/4} \sin\left[\omega_p t\right] \frac{4t}{T}\, dt = \frac{32}{T^2}\int_{0}^{T/4} t\sin\left[\omega_p t\right]\, dt \quad where\ \omega_p = \frac{p2\pi}{T}$$

$$b_p = \frac{8}{\pi^2 p^2}\int_{0}^{p\pi/2} u\sin\left[u\right]\, du = \frac{8}{\pi^2 p^2}\left[-u\cos u + \sin u\right]\Big|_{0}^{p\pi/2} \quad where\ u = \frac{p2\pi t}{T}$$

$$b_p = \frac{8}{\pi^2 p^2}(-1)^{\left[(p-1)/2\right]} \quad for\ p\ odd;\ b_p = 0 \quad for\ p\ even \qquad \text{[FS.13]}$$

Exercise: Complete the detailed evaluation of the b_p suggested in the four lines above this exercise.

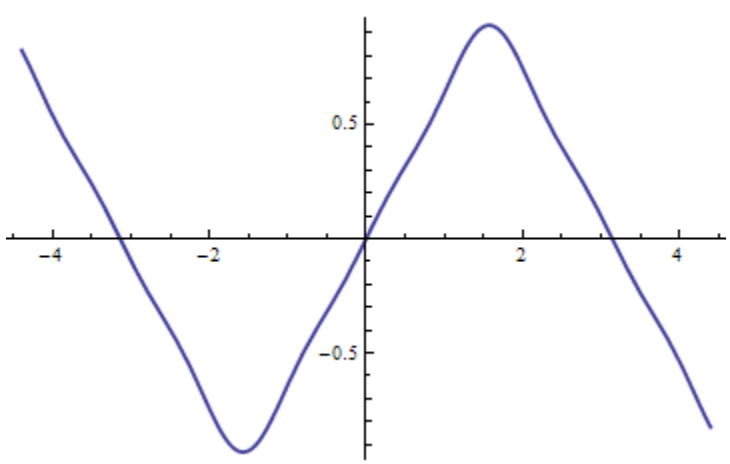
FOURIER SERIES FOR THE ANTI-SYMMETRIC TRIANGULAR WAVE:

$$f(t) = \frac{8}{\pi^2}\left[\sin\left[m\frac{2\pi\, t}{T}\right] - \frac{1}{3^2}\sin\left[m\frac{2\pi\, t}{T}\right] + \frac{1}{5^2}\sin\left[m\frac{2\pi\, t}{T}\right] + \ldots\right]$$
$$= \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\left[(2m+1)\frac{2\pi\, t}{T}\right] \qquad \text{[FS.14]}$$

Again, only the odd index terms appear. As before, for p even, $\sin\left[p(2\pi/T)\right]$ is odd about the points $-T/4$ and $T/4$ while the triangular wave is even about those points which leads to zero overlap or inner product. For large *m*, the expansion coefficients vanish as $1/m^2$ which is rather more rapid than the $1/m$ for the square wave expansion coefficients. High frequencies are necessary to follow rapid variations. The triangular wave is smoother (less jerky) than is the square wave thus it has less high frequency (fast-wiggle) behavior than does the square wave.

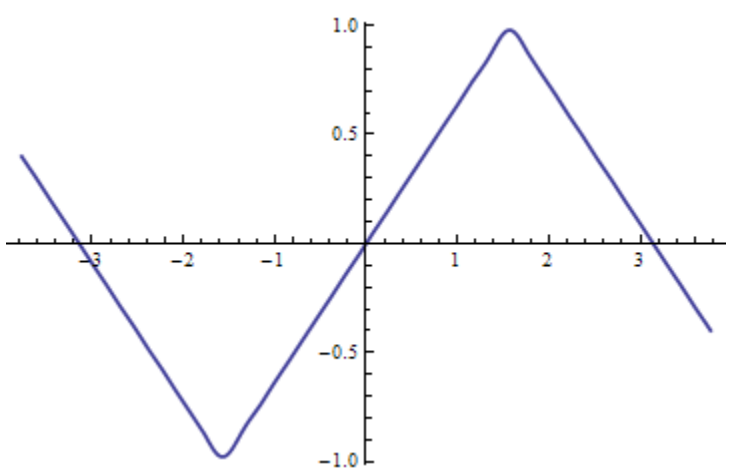
Conjecture: If a function *f(t)* is continuous and has continuous derivatives through p^{th} order, then its Fourier expansion coefficients will vanish **<u>no faster than</u>** $1/m^{(p+1)}$ for large *m*. The smoother the function is, the less prominent the very high frequency contributions are. See problem 41.

```
Plot[(8/Pi^2) * ( Sin[x] - Sin[3x]/3^2 + Sin[5 x]/5^2 ),{x,-1.4 Pi,1.4 Pi}]
```



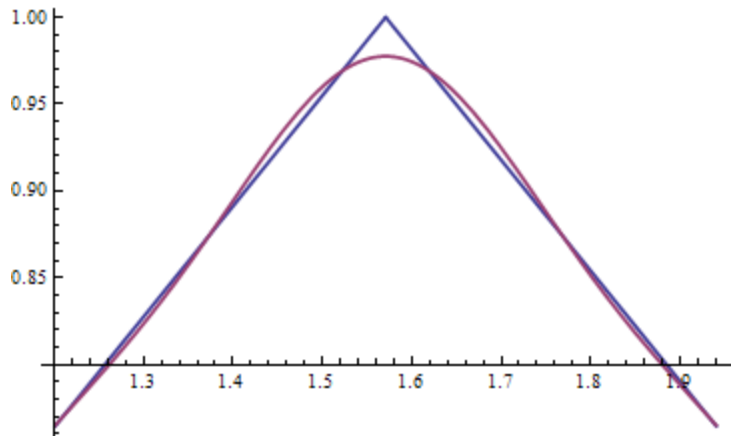
The first three expansion terms provide a fairly faithful rendition of the triangular wave. It wobbles a little, but it is not too shabby. There are no discontinuities so the series appears to smoothly approach the desired function. In particular, no overshoot phenomenon is evident. As a more complete comparison, the nine-term representation is displayed below. It is just plain great. One can point at the corners being rounded, but that is to be expected. A 17 ω_0 wave corners 34 times in the period of f(t). A corner is therefore expected to make its bend in a fraction $^1/_{34}$ of the period. A plot with expanded detail and dimensionless period 2π verifies this expectation qualitatively. The bend is expected to require about $^{2\pi}/_{34}$ or one-fourth of the $^{4\pi}/_{17}$ range plotted. The important observation is that, as the number of terms increases, the regions in which the sum is a poor representation of the function shrink <u>and</u> the magnitude of the error in these regions also shrinks. Contrast this with the Gibbs phenomenon at discontinuities.

```
Plot[(8/Pi^2) * ( Sin[x] - Sin[3x]/3^2 + Sin[5 x]/5^2 -Sin[7 x]/7^2 + Sin[9
x]/9^2 - Sin[11 x]/11^2 +Sin[13 x]/13^2-Sin[15 x]/15^2+Sin[17
x]/17^2),{x,-1.2  Pi,1.2 Pi}]
```

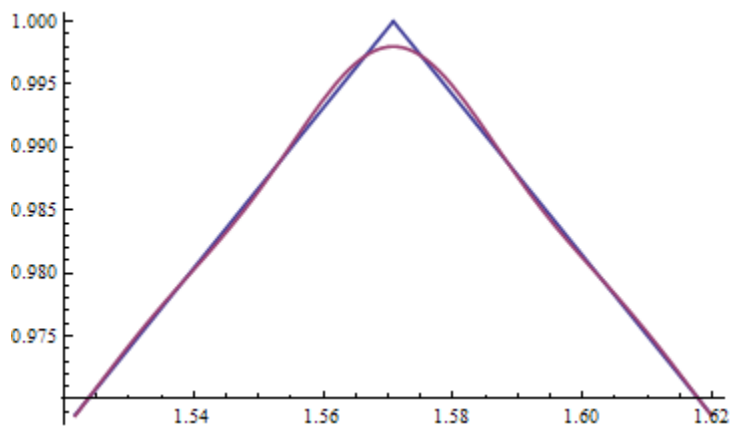


Plot through m = 17.

```
Plot[{ 2 x/Pi + UnitStep[ x -Pi/2 ]  (2 - 4 x /Pi),
(8/Pi^2)*(Sin[x]-Sin[3x]/3^2+Sin[5 x]/5^2-Sin[7
x]/7^2+Sin[9 x]/9^2-Sin[11 x]/11^2+Sin[13 x]/13^2-
Sin[15 x]/15^2+Sin[17 x]/17^2)},{x,Pi/2-
2*Pi/17,Pi/2+2*Pi/17}, PlotStyle -> Thickness[0.005]]
```



Plot through m = 17, Corner Detail



Plot through m = 201, Corner Detail

Exercise: Give the Fourier coefficients for a triangular wave similar to the one above, but with amplitude A rather than one.

Exercise: Give the Fourier coefficients for a function that is the sum of the unit amplitude square wave plus the unit amplitude triangular wave discussed above. How fast do the Fourier coefficients vanish for large m? Why is this dependence

expected?

Dirichlet Conditions: Under what conditions can a function be represented well by its Fourier series? The following is a statement of sufficient conditions.

If $f(t)$ is a periodic function of period T, and if between $-T/2$ and $T/2$, the function is single-valued, has a finite number of finite discontinuities, has a finite number of maxima and minima, and if $\int_{-T/2}^{T/2} |f(t)|^2 \, dt$ is finite then the Fourier series as defined above converges to $f(t)$ at all points where $f(t)$ is continuous, and, at discontinuities, the series converges to the average of the left and right limiting values of the function as it approaches the discontinuity; *it converges to the mean value at a discontinuity*.

Convergence and Parseval Relations

Several convergence properties of Fourier series have already been noted. If the function has a discontinuity at $t = t_{\text{jump}}$, the series converges to the average of the limits of the function as t approaches t_{jump} from smaller and from larger values.

To discuss the overall convergence of the series for $f(t)$, the sequence of partial sums is considered where $s_n(t)$ is the sum of the $2n + 1$ lowest frequency terms of the series. It includes the constant term and all the sine and cosine term with frequencies up to $n\omega_o$. A ***proper partial sum*** includes all the terms in the Fourier series with frequencies up to a maximum $n\omega_o$ and no terms with frequency higher than that value. Never use an improper partial sum. (A later section on the Gibbs phenomenon demonstrates the importance of using partial sums that consistently include all terms up to a set maximum frequency.)

$$s_n(t) \;=\; c_{f0}\left(1\right) \;+\; \sum_{m=1}^{n} a_{fm}\cos\left[\omega_m t\right] \;+\; \sum_{m=1}^{n} b_{fm}\sin\left[\omega_m t\right] \qquad \text{[FS.15]}$$

If the series converges uniformly to the function $f(t)$, then for any given ε there exists an N such that $|f(t) - s_N(t)| < \varepsilon$ for all $t \in [-\frac{1}{2}T, \frac{1}{2}T]$. That is, if and only if life is great, the sequence of partial sums converges uniformly to $f(t)$ in the interval $[-\frac{1}{2}T, \frac{1}{2}T]$. If a series converges uniformly, then there is always an N term (finite sum) that represents the infinite series to any accuracy desired. Working in terms of the finite sum, interchanging the order of operations of summing series and integrating (or differentiating) is clearly allowed. That would be great, but uniform convergence just does not happen in all cases. At a discontinuity, the series converges to the average of the left and right limits of the function, and, in the Gibbs overshoot region, the series diverges from the value of the function by 8.95% of the magnitude of the discontinuity for any large, but finite sum of terms.

Fourier partial sum sequences do **converge in the mean**. For every ε there exists an N such that $\int_{-T/2}^{T/2}\left|f(t)-s_N(t)\right|^2 dt < \varepsilon$ where the limits are just those for any complete period of $f(t)$. This can be rewritten as:

$$\lim_{N\to\infty}\left[\int_{-T/2}^{T/2}\left|f(t)-s_N(t)\right|^2 dt\right] = 0 \qquad \text{[FS.16]}$$

For periodic functions $f(t)$ with 'not too many' discontinuities and that are square integrable [the integral $\int_{-T/2}^{T/2}\left|f(t)\right|^2 dt$ is defined and finite], the corresponding Fourier series converges in the mean to $f(t)$.

The convergence of the sequence of partial sums to a discontinuous function $f(t)$ is **weak**. Consider the Gibbs 9% overshoot at discontinuities (≈ 8.95% of the discontinuity). For the Nth partial sum, the region in which the sum differs from the function by an amount of order 9% of the discontinuity has the approximate extent of $T/2N$. Even for large N, there are tiny regions in which the partial sum

misrepresents the function by a finite amount.

A Fourier series expansion **converges weakly** to a discontinuous function.

Replace the function with its Fourier series: Parseval and beyond

Once you have the Fourier series for a function, that series can be used in the place of the function. That is: the Fourier series is a robust representation of the function. Use the orthogonality relations to consider $\int_{-T/2}^{T/2} |f(t)|^2\, dt$.

The summation indices are dummy labels. They may be relabeled at will. At minimum, rename the labels in a product to ensure that the labels are distinct.

Represent $f(t)^*$ and $f(t)$ using distinct summation index labels throughout.

$$\int_{-T/2}^{T/2} |f(t)|^2\, dt = \int_{-T/2}^{T/2} \left(c_{f0} + \sum_{m=1}^{\infty} a_{fm} \cos[\omega_m t] + \sum_{q=1}^{\infty} b_{fq} \sin[\omega_q t] \right)^*$$

$$* \left(c_{f0} + \sum_{p=1}^{\infty} a_{fp} \cos[\omega_p t] + \sum_{r=1}^{\infty} b_{fr} \sin[\omega_r t] \right) dt =$$

$$= \int_{-T/2}^{T/2} \left(c^*_{f0}\, c_{f0} + c^*_{f0} \sum_{p=1}^{\infty} a_{fp} \cos[\omega_p t] + \sum_{m=1}^{\infty} \sum_{q=1}^{\infty} b_{fq} \sin[\omega_q t]\, a^*_{fm} \cos[\omega_m t] + \ldots \right) dt$$

nine terms in the complete integrand! Use orthogonality.

Finally $$\int_{-T/2}^{T/2} |f(t)|^2\, dt = T \left[|c_{f0}|^2 + \tfrac{1}{2} \sum_{m=1}^{\infty} |a_{fm}|^2 + \tfrac{1}{2} \sum_{m=1}^{\infty} |b_{fm}|^2 \right]$$

where the factors of one half are an annoying consequence of the choice of expansion set functions (or **basis** functions): $\{1; \{\cos[\omega_m t]\}; \{\sin[\omega_p t]\}\}$ which have uneven square integrals. This relation is **Parseval's Equality**:

$$\int_{-T/2}^{T/2} |f(t)|^2\, dt = T \left[|c_{f0}|^2 + \tfrac{1}{2} \sum_{m=1}^{\infty} |a_{fm}|^2 + \tfrac{1}{2} \sum_{m=1}^{\infty} |b_{fm}|^2 \right]$$

Equally as cool and proven using the same techniques, the inner product relation can be re-expressed:

$$\left(\tfrac{1}{T}\right)\int_{-T/2}^{T/2}\left[g(t)\right]^{*} f(t)dt = c_{g0}^{*}c_{f0} + \tfrac{1}{2}\sum_{m} a_{gm}^{*}a_{fm} + \tfrac{1}{2}\sum_{p} b_{gp}^{*}b_{fp} \qquad \text{[FS.17]}$$

The coordinate and the Fourier representations for the inner product agree at least as long as our functions are well-behaved. Compare these relations with the expressions $\left|\vec{A}\right|^2 = \vec{A}\cdot\vec{A} = A_x A_x + A_y A_y + A_z A_z$ and $\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$.

Notation: in other references expect:

$$\langle g | f \rangle = \tfrac{1}{4} a_{g0}^{*} a_{f0} + \tfrac{1}{2}\sum_{m} a_{gm}^{*} a_{fm} + \tfrac{1}{2}\sum_{p} b_{gp}^{*} b_{fp}$$

Recall the average value of the function c_0 is to be replaced by $\tfrac{1}{2}\ a_0$ to agree with notations commonly used.

Fourier series provide a robust representation of the function:

Operations can be executed on the Fourier series for a function as well as on the function itself. Examples: differentiation and integration. Properly, the techniques are guaranteed valid only if the series converge uniformly.

Start with the anti-symmetric triangular wave:

$$f(t) = \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\ \sin\left[(2m+1)\frac{2\pi\, t}{T}\right]$$

Compute the series for the derivative of *f(t)* by differentiating the series term by term. Sketch the derivative. The series represents a symmetric square wave. Compare the frequency content with that of the anti-symmetric square wave.

$$\frac{df}{dt} = \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)}\ \left(\frac{2\pi}{T}\right)\left(\cos\left[(2m+1)\frac{2\pi\, t}{T}\right]\right)$$

Exercise: Compare the frequency content of the symmetric square wave that results from the term-by-term differentiation of the anti-symmetric triangular wave

with the frequency content of the anti-symmetric square wave.

Next, $f(t)$ is to be integrated. Sketch the integral of triangular wave $f(t)$.

$$g(t) = \int_0^t f(t')\,dt' = \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\int_0^t \sin\left[(2m+1)\frac{2\pi\, t'}{T}\right]dt'$$

$$= \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3}\left(T/2\pi\right)\left[1-\cos\left[(2m+1)\frac{2\pi\, t}{T}\right]\right]$$

Exercise: Sketch the integral of the triangular wave. Note that the coefficients of its Fourier series vanish as m^{-3} for large m. This dependence suggests that the function has a continuous first derivative. Does your sketch support this conclusion?

Relating Fourier series derived for different periods

Given a Fourier series for a periodic function $f(t)$ with period T_1, the Fourier series for $g(t) = f(\frac{T_1}{T_2}t)$, a function with the same '*shape*' that has been argument stretched to have a period of T_2 can been generated by scaling the independent variable in the original series. The independent variable in the original series can be scaled in the same manner to yield the Fourier series for $g(t)$. [$\omega_1 = 2\pi/T_1$]

$$f(t) = c_0 + \sum_{m=1}^{\infty} a_m \cos\left[m\omega_1 t\right] + \sum_{m=1}^{\infty} b_m \sin\left[m\omega_1 t\right]$$

$$g(t) = c_0 + \sum_{m=1}^{\infty} a_m \cos\left[m\omega_1\left(\tfrac{T_1}{T_2}\right)t\right] + \sum_{m=1}^{\infty} b_m \sin\left[m\omega_1\left(\tfrac{T_1}{T_2}\right)t\right]$$

The function $g(t)$ has the a period of T_2 , and the same *amplitude* as $f(t)$.

Many references assume a period of 2π. The result is a series of the form:

$$f(x) = c_0 + \sum_{m=1}^{\infty} a_m \cos\left[m\,x\right] + \sum_{m=1}^{\infty} b_m \sin\left[m\,x\right] \qquad \text{period } 2\pi$$

The series for the *same shape* and a period of T is generated by replacing x by $\frac{T_1}{T_2}t$ or $\frac{2\pi}{T}t$. Mathematica assumes a range -½ to ½ or an initial period of $T_1 = 1$. In a Mathematica generated series, replace the independent variable by $\frac{T_1}{T_2}t \equiv \frac{t}{T}$.

Exercise: Given a periodic function $f(t)$ with period T_1, show that $g(t) = f(\frac{T_1}{T_2}t)$ is a periodic function with a period of T_2.

Physical Application: The Damped Driven Oscillator

Mechanical systems usually have an inertial element that '*resists acceleration*' corresponding to second time derivative in the displacement equation ensuring a $1/\omega^2$ dependence of the amplitude response to an excitation in the high frequency limit. This dependence means that the series for the response converges more rapidly than that for the excitation and that high frequency weirdness is usually not a practical issue.

The steady-state response of the oscillator to a sinusoidal driving force is assumed to be known. See, for example, the differential equations portion of this handout set. A condensed derivation follows. The damped driven mechanical oscillator obeys the differential equation $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + k\,x = F(t)$ which is rewritten as

$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = f(t)$ where $\gamma = b/2m$, $\omega_0^2 = k/m$ and $f(t) = F(t)/m$.

MEGACEPT: A linear system responds at the frequency at which it is driven. If the system is driven at a single (pure) frequency, its <u>steady-state</u> response is at that same pure frequency with at most a phase shift. The relative amplitude

of the response can depend on frequency. The single pure frequency functions are $\sin(\omega t), \cos((\omega t), e^{\pm i\omega t}$. Linear systems obey the superposition principle so the response to any periodic function can be found by decomposing the driving function into its Fourier (pure frequency) components, computing the response to each component and then summing the responses to find the response to the full periodic driving function. NOTE: The transient behavior of a linear system has a frequency character determined by the parameters of the system, not those of the driving force. The steady-state solution is the long-time behavior after the transients have decayed. The transient behavior satisfies the homogeneous equation.

Consider a single frequency driving function: $f(t) = f_0 \cos(\omega t)$. The response is at the same frequency, but with a phase shift: $x_\omega(t) = R(\omega)\, f_0 \cos(\omega t - \theta)$. The sign in expression: ($\omega t - \theta$) is chosen to be negative as the response lags behind the drive or cause. The manipulations are more compact if the clock for t is reset by $+\theta/\omega$. This yields $f(t) = f_0 \cos(\omega t + \theta)$ and $x_\omega(t) = R(\omega)\, f_0 \cos(\omega t)$ which states that the drive leads the response by θ which is the same as saying that the response lags the drive by θ. Substituting these expressions into the differential equation:

USE: $\cos(\omega t + \theta) = \cos(\theta)\cos(\omega t) - \sin(\theta)\sin(\omega t)$

$$R(\omega) f_0 \left[-\omega^2 \cos(\omega t) - 2\gamma\omega \sin(\omega t) + \omega_0^2 \cos(\omega t) \right] = f_0 \left[\cos(\theta)\cos(\omega t) - \sin(\theta)\sin(\omega t) \right]$$

The functions $\sin(\omega t)$ and $\cos(\omega t)$ are linearly independent so the coefficients of each must match on the two sides of the equation.

$\cos(\omega t)$: $\quad R(\omega)\left[\omega^2 - \omega_0^2\right] = \cos(\theta)$ $\quad$ and $\quad$ $\sin(\omega t)$: $\quad R(\omega)\left[-2\omega\gamma\right] = \sin(\theta)$

After applying the two most powerful trigonometric identities [$\sin^2(\theta) + \cos^2(\theta) = 1$ and $\tan(\theta) = \sin(\theta)/\cos(\theta)$], it follows that:

$$\left[R(\omega)\right]^{-1} = D(\omega) = \sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(2\gamma\omega\right)^2} \qquad \text{and} \qquad \theta = \tan^{-1}\left[\frac{2\omega\gamma}{\omega_0^2 - \omega^2}\right]$$

The steady-state response to a driving function $f(t) = f_0 \cos(\omega t)$ is:

$$x_\omega(t) = R(\omega)\, f_0 \cos(\omega t - \theta) = \frac{f_0}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(2\gamma\omega\right)^2}} \cos\left(\omega t - \tan^{-1}\left[\frac{2\omega\gamma}{\omega_0^2 - \omega^2}\right]\right)$$

Note that the result above supports the conjecture that a linear system driven by a pure frequency responds at that same frequency. The amplitude of the response does depend on that frequency. Transients associated with 'turn-on' or start-up' require that additional frequencies be added to the representation of $f(t)$. Invoking superposition for linear systems, the response for an arbitrary periodic driving force, $f(t) = c_0\,(1) + \sum_{m=1}^{\infty} a_m \cos\left[\omega_m t\right] + \sum_{m=1}^{\infty} b_m \sin\left[\omega_m t\right]$,

is: **[Recall that: $f(t) = m^{-1} F(t)$.]**

$$x(t) = \frac{c_0}{\omega_0^2} + \sum_{m=1}^{\infty} \frac{a_m \cos\left[\omega_m t - \theta_m\right]}{\sqrt{\left(\omega_m{}^2 - \omega_0^2\right)^2 + \left(2\gamma\omega_m\right)^2}} + \sum_{m=1}^{\infty} \frac{a_m \sin\left[\omega_m t - \theta_m\right]}{\sqrt{\left(\omega_m{}^2 - \omega_0^2\right)^2 + \left(2\gamma\omega_m\right)^2}} \qquad \text{[FS.18]}$$

$$\theta_m = \tan^{-1}\left[\frac{2\omega_m\gamma}{\omega_0^2 - \omega_m{}^2}\right]$$

It looks hopeless. The result is nonetheless very useful, particularly when the system has a sharp resonance that selects from all the possible ω_m only a few near the natural frequency ω_o with the response to frequencies further from resonance being negligible by comparison.

Why expand in a Fourier series rather than in (say) a Taylor's series? Series expansions come in many varieties, and one must choose the type that best suits the problem. The method of Frobenius revealed the awesome power of Taylor's series expansions. The individual expansion base functions (.. x^n ..) are simpler

than sines and cosines. Why would you ever bother? For many important problems that you study, there exists a set of characteristic solutions, solutions that just fit the problem. You can recognize their appearance once you know that the German for characteristic is *eigenschaft* and that the functions are eigen-functions. For linear response problems such as the oscillator studied above, the system responds at the frequency at which it is driven. A pure frequency drive leads to a response at that same pure frequency. Obviously one wants a series that represents the driving input in terms of its pure frequency components. Each pure frequency component can be analyzed independently with no mixing with components at other frequencies. The pure single frequency functions are sin(ωt), cos(ωt), $e^{i\omega t}$ and $e^{-i\omega t}$. These functions are the bases for Fourier expansions. Fourier expansions are a leading choice for most linear problems for this reason. A second strong indicator is that the function is periodic (or can be extended as a periodic) function. The functions sin(ωt), cos(ωt), $e^{i\omega t}$ and $e^{-i\omega t}$ are periodic and hence share a strong characteristic behavior with the function of interest. Expansions in terms of basis functions that share behaviors with the function to be expanded are more natural.

German: Eigenshaft ⇒ nature, virtue, feature, quality, property, attribute, character, characteristic, characteristic quality

Other Fourier Series: The Sine, Cosine and Complex Exponential series

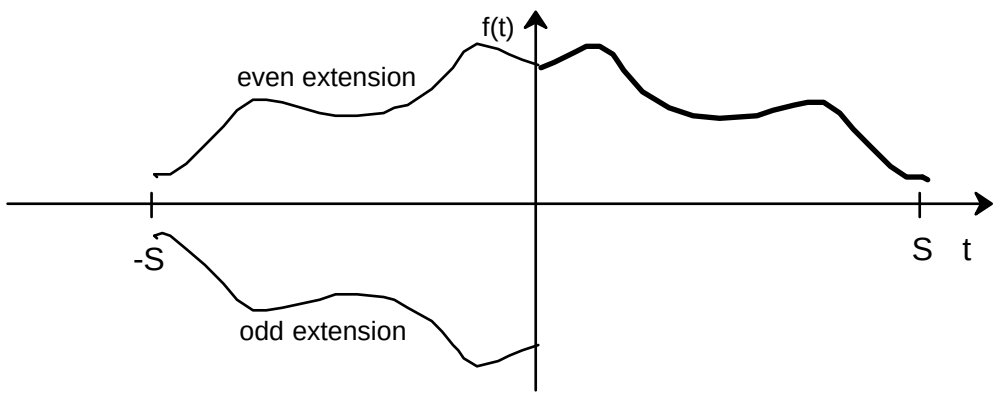
Consider any function defined from 0 to S. That function can be extended as an even or odd function to the interval -S to S and beyond by assuming that the function is periodic with period 2 S. If the anti-symmetric extension is adopted, the Fourier series will only have sine terms.

$$f(t) \;=\; \sum_{m=1}^{\infty} b_m \sin\left[\omega_m t\right] \;\text{ where } b_m \;=\; \frac{4}{2S}\int_0^S \sin\left[p\left(2\pi/2S\right)t\right] f(t)\,dt$$

If the even extension is adopted, the function may be represented as

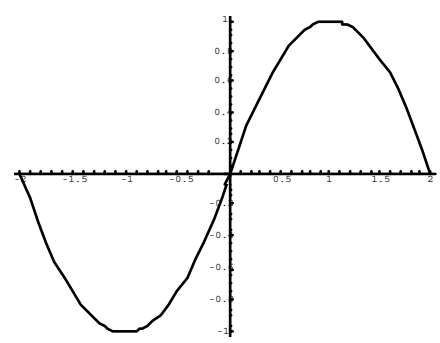
$$f(t) = c_0 + \sum_{m=1}^{\infty} a_m \cos[\omega_m t] \quad \text{where } c_0 = \frac{1}{S}\int_0^S f(t)\,dt \text{ and } a_m = \frac{4}{2S}\int_0^S \cos[m(2\pi/2S)t]\, f(t)\,dt$$

The penalty for using the odd extension is that the extended function will have additional discontinuities at $t = 0$ and at t = -S and + S unless $f(t)$ vanishes at these points. The even extension does not introduce additional discontinuities. Recall that discontinuities ensure that the coefficients vanish no faster than $1/m$ for large m.



Exercise: Consider the function $f(t)$ that consists of two slope-matched parabolic sections. Consider expanding the function as defined over the interval $0 < t < 2$ as an even or odd function to complete the definition over the period $-2 < t < 2$. Would a sine of cosine series converge more rapidly? Assuming that the coefficients would decrease as m^{-n} for large m predict the value of n for a sine series expansion and for a cosine series expansion.

$$f(t) = \begin{cases} 1-(t-1)^2 = 2t - t^2 & for \quad 0 < t < 2 \\ -1+(t+1)^2 = 2t + t^2 & for \quad -2 < t < 0 \end{cases}$$



[FS.19]

Euler's identity and tons of perspiration can be used to develop the complex Fourier series with the result that one just uses the same set of frequencies for the $\left[-T/2, T/2\right]$ interval. $f(t) = \sum_{m=-\infty}^{\infty} d_m \exp\left[i\, m \left(2\pi/T\right) t\right]$ A typical coefficient c_p is projected out of the sum by first multiplying both sides by $\exp\left[-i\, p \left(2\pi/T\right) t\right]$

$$\int_{-T/2}^{T/2} \exp\left[-i\, p \left(2\pi/T\right) t\right] f(t)\, dt \quad = \quad \sum_{m=-\infty}^{\infty} d_m \int_{-T/2}^{T/2} \exp\left[-i\, p \left(2\pi/T\right) t\right] \exp\left[i\, m \left(2\pi/T\right) t\right] dt = T\, d_p$$

$$\text{or} \quad d_p = \left(1/T\right) \int_{-T/2}^{T/2} \exp\left[-i\, p \left(2\pi/T\right) t\right] f(t)\, dt\,.$$

NOTE: To project out the coefficient of a complex-valued function $e^{i\omega t}$*, the series is pre-multiplied by the complex conjugate of that function,* $\left(e^{i\omega t}\right)^* = e^{-i\omega t}$*. The same procedure was followed for the real functions. In the real case,* $*$ *just did not matter. The procedure follows from the form of the inner product. The complex conjugate of g(t) is required.*

$$\langle g(t) | f(t) \rangle = \left(1/T\right) \int_{-T/2}^{T/2} \left(g(t)\right)^* f(t) dt$$

Exercise: Show that:

$$\left(1/T\right) \int_{-T/2}^{T/2} \exp\left[-i\, p \left(2\pi/T\right) t\right] \exp\left[i\, m \left(2\pi/T\right) t\right] dt \; = \delta_{pm} \qquad \text{[FS.20]}$$

Complex representations are not necessary to describe classical physics. Certain computational economies result from adopting complex notation, but care must be taken to avoid interpreting or including the imaginary parts of the results. Few problems arise as long as only linear operations are involved. Non-linear operations however mix the real and imaginary (the physical and not-physical)

parts. Evil lurks in the shadows with the imaginary part. [Quantum Mechanics requires complex values to represent the physics.]

Offbeat Applications: Summing Series

Sum 1: The square wave example yielded the representation:

$$f(t) \quad = \frac{4}{\pi} \quad \sum_{m=1}^{\infty}\left(\frac{1}{2m+1}\right)\sin\left[(2m+1)\frac{2\pi\, t}{T}\right] = \begin{cases} +1 & for \quad 0 < t < T/2 \\ -1 & for \quad -T/2 < t < 0 \end{cases}$$

Examining $f(T/4) = 1 = \frac{4}{\pi} \quad \sum_{m=1}^{\infty}\left(\frac{1}{2m+1}\right)\sin\left[(2m+1)\frac{\pi}{2}\right] = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right]$ provides the sum of the inverses of the odd integers with alternating signs.

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \quad = \quad \frac{\pi}{4} \qquad \text{[FS.21]}$$

Sum 2: The triangular wave example

$$f(t) = \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\left[(2m+1)\frac{2\pi\, t}{T}\right] = \begin{cases} 2 - 4x/T & for \quad T/4 < t < T/2 \\ 4x/T & for \quad -T/4 < t < T/4 \\ -2 - 4x/T & for \quad -T/2 < t < -T/4 \end{cases}$$

The trick is to find a value of the argument that allows the sines to be easily evaluated. Choose $t = {}^{T}/_{4}$ and use $f({}^{T}/_{4}) = 1$.

$$f(T/4) = 1 = \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\left[(2m+1)\frac{\pi}{2}\right] = \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(+1)}{(2m+1)^2}$$

$$\sum_{m=0}^{\infty}\left(\frac{(+1)}{2m+1}\right)^2 = \frac{\pi^2}{8} \qquad \text{[FS.22]}$$

Additional sums can be extracted from completed Fourier series by invoking Parseval's equalities and the identity for inner products of functions.

$$\int_{-T/2}^{T/2} |f(t)|^2\, dt \;=\; T \left[|c_{f0}|^2 + \tfrac{1}{2}\sum_{m=1}^{\infty} |a_{fm}|^2 + \tfrac{1}{2}\sum_{m=1}^{\infty} |b_{fm}|^2 \right]$$

$$\left(\tfrac{1}{T}\right)\int_{-T/2}^{T/2} \left[g(t)\right]^* f(t)dt = c_{g0}^* c_{f0} + \tfrac{1}{2}\sum_{m} a_{gm}^* a_{fm} + \tfrac{1}{2}\sum_{p} a_{gp}^* a_{fp}$$

Sum 3: The square wave and Parseval's equality.

$$f(t) \;=\frac{4}{\pi}\;\sum_{m=1}^{\infty}\left(\frac{1}{2m+1}\right)\sin\left[(2m+1)\frac{2\pi\, t}{T}\right] = \begin{cases} +1 & for \quad 0 < t < T/2 \\ -1 & for \quad -T/2 < t < 0 \end{cases}$$

$$\int_{-T/2}^{T/2} |f(t)|^2\, dt = T \;=\; T \left[|c_{f0}|^2 + \tfrac{1}{2}\sum_{m=1}^{\infty} |a_{fm}|^2 + \tfrac{1}{2}\sum_{m=1}^{\infty} |b_{fm}|^2 \right] = T\left(\tfrac{1}{2}\right)\left(\frac{16}{\pi^2}\right)\sum_{m=1}^{\infty}\left(\frac{1}{2m+1}\right)^2$$

It follows that: $$\sum_{m=0}^{\infty}\left(\frac{(+1)}{2m+1}\right)^2 = \frac{\pi^2}{8}$$

Before you cry foul, think about it. It would have been much worse if the two approaches were in conflict!

Exercise: Relating sums of odd and even index terms for inverse powers.

$$\sum_{m=0}^{\infty}\left(\frac{(+1)}{2m+1}\right)^2 = \frac{\pi^2}{8} = \sum_{n\ odd}\frac{1}{n^2} \quad \text{and ....} \quad \sum_{n\ even}\frac{1}{n^2} = \sum_{all\ \ m}^{\infty}\frac{1}{(2m)^2} = \frac{1}{2^2}\sum_{all\ \ m}^{\infty}\frac{1}{m^2}$$

$$\sum_{all\ \ n}\frac{1}{n^2} = \sum_{n\ even}\frac{1}{n^2} + \sum_{n\ even}\frac{1}{n^2} = \frac{1}{2^2}\sum_{n\ odd}\frac{1}{n^2} + \sum_{n\ odn}\frac{1}{n^2} = \frac{5}{4}\sum_{n\ odd}\frac{1}{n^2}$$

Give the values of $\sum_{all\ \ n}\frac{1}{n^2}$ and $\sum_{n\ even}\frac{1}{n^2}$ where all n means all positive integers.

Exercise: Use the Fourier series for the unit amplitude triangular wave and Parseval's equality to develop the sum of an infinite series.

Recall the term-by-term integration of series. The example was the triangular

wave $f(t)$. Sketch $f(t)$ and its integral.

$$g(t)=\int_0^t f(t')\,dt' = \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\int_0^t \sin\left[(2m+1)\frac{2\pi\, t'}{T}\right]dt'$$

$$= \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3}\left(T/2\pi\right)\left[1-\cos\left[(2m+1)\frac{2\pi\, t}{T}\right]\right]$$

Note that the average value of $g(t)$ is:

$$\langle g(t)\rangle = \frac{4}{\pi^3}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3}$$

Spatial Fourier series

Most of this handout has been directed at temporal Fourier series. The relations are to be restated for spatial series. In the space-spatial frequency realm, Fourier's conjecture takes the form: Any function $f(x)$ which is periodic with period L can be represented as a sum of a constant plus sinusoidal terms with spatial frequencies $\omega_m = m(2\pi/L)$. The sinusoids are the sine and cosine. The set of frequencies ω_m ensure that each sinusoid completes an integer m cycles in the distance L, the period of $f(x)$ The base frequency for the expansion is $\omega_0 = \omega_1 = (2\pi/L)$ which is called the **fundamental spatial (angular) frequency**. This frequency is determined by the **period of the function to be expanded alone**. The frequency of every term in the expansion is an integer multiple (or harmonic) of this fundamental frequency.

$$f(x) \;=\; c_0\,(1) \;+\; \sum_{m=1}^{\infty} a_m \cos[\omega_m x] \;+\; \sum_{m=1}^{\infty} b_m \sin[\omega_m x] \quad \text{where } \omega_m = m(2\pi/L).$$

Each term in the expansion represents a distinct behavior of a function with period L. The periodicity restricts the function to this set of behaviors. The first term c_0 (1), a fancy way to write c_0, is the average value of the function with the (1)

indicating that it represents the constant behavior of the function. The terms $a_m \cos\left[\omega_m x\right]$ represent the independent or distinct ways in which an even function of period L can wiggle, and the terms $b_m \sin\left[\omega_m x\right]$ represent the independent or distinct ways in which an odd function of period L can wiggle. It is crucial to recognize than each term represents an **independent behavior**.

Begin with 1. Find out how much constant behavior is in the function.

$$\frac{1}{L}\int_{-L/2}^{L/2} 1\, f(x)\, dx \quad = \quad \frac{1}{L}\int_{-L/2}^{L/2} 1\, c_0\, dx \quad + \quad \sum_{m=1}^{\infty} a_m \frac{1}{L}\int_{-L/2}^{L/2} 1 \cos\left[\omega_m x\right] dx \quad + \quad \sum_{m=1}^{\infty} b_m \frac{1}{L}\int_{-L/2}^{L/2} 1 \sin\left[\omega_m x\right] dx$$

The integrals of the sine and cosines are over an integral number of cycles - hence they vanish. Wiggling like a sine or cosine for an integral number of cycles is nothing like being constant. So:

$$c_0 = \frac{1}{L}\int_{-L/2}^{L/2} 1\, f(x)\, dx \quad = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, dx = \left\langle f(x) \right\rangle \qquad \text{[FS.23]}$$

The constant part of the function is just its average value over the interval $\left[-L/2, L/2\right]$. Periodic implies all single intervals of length L give the same result.

The coefficients for the independent wiggle modes become:

$$a_p = \frac{2}{L}\int_{-L/2}^{L/2} \cos\left[\omega_p x\right] f(x)\, dx \quad \text{and} \quad b_p = \frac{2}{L}\int_{-L/2}^{L/2} \sin\left[\omega_p x\right] f(x)\, dx \quad \text{for } p \geq 1 \qquad \text{[FS.24]}$$

The factor of two arises because the average value of $\cos^2\left[\omega_p x\right]$ and of $\sin^2\left[\omega_p x\right]$ is just one half (while the average value of 1 is 1 !).

Spatial Fourier series have the same Parseval, inner product and convergence properties as do the temporal series.

An ALTERNATIVE NOTATION replaces ω_m by k_m when the Fourier series is for

function of a spatial variable. The terms *spatial frequency* and *wave number* are used for k_m. It is common to expand functions dependent on several spatial dimensions in which cases the collection of spatial frequencies forms a vector. Hence the final term for $\vec{k}_{\ell mn}$ is *wave vector* where the number of subscripts matches the dimensions of the function's argument.

Beyond periodic functions: THE FOURIER TRANSFORM

The generalization of Fourier series to forms appropriate for more general functions defined from -∞ to +∞ is not as painful as it first appears. The process is presented here even though it lies beyond the goals for this section. The process illustrates the transition from a sum to an integral, an important thing to understand. The functions to be expanded are restricted to piecewise continuous square integrable functions. (Less restrictive developments may be found elsewhere.)

The complex form of the Fourier series is the starting point.

$$f(x) \;=\; \sum_{m=-\infty}^{\infty} d_m \exp\left[i\, m \left(2\pi/L\right) x\right] \qquad \text{[FS.25]}$$

A typical coefficient d_p is projected out of the sum by first multiplying both sides by $\exp\left[-i\, p \left(2\pi/L\right) x\right]$, the complex conjugate of $\exp\left[i\, p \left(2\pi/L\right) x\right]$, the function d_p multiplies.

$$\int_{-L/2}^{L/2} \exp\left[-i\, p \left(2\pi/L\right) x\right] f(x)\, dx \;=\; \sum_{m=-\infty}^{\infty} d_m \int_{-L/2}^{L/2} \exp\left[-i\, p \left(2\pi/L\right) x\right] \exp\left[i\, m \left(2\pi/L\right) x\right] dx = L\, d_p$$

$$\text{or} \quad d_p = \left(1/L\right) \int_{-L/2}^{L/2} \exp\left[-i\, p \left(2\pi/L\right) x\right] f(x)\, dx\,.$$

The trick is to define $k = 2\pi\, m / L$ in $f(x) \;=\; \sum_{m=-\infty}^{\infty} d_m \exp\left[i\, m \left(2\pi/L\right) x\right]$ to yield

$$f(x) \;=\; \sum_{m=-\infty}^{\infty} d_m \left(L/2\pi\right)\exp\left[i\,m\left(2\pi/L\right)x\right]\left(2\pi/L\right) = \sum_{m=-\infty}^{\infty} d_m \left(L/2\pi\right)\exp\left[i\,kx\right]\Delta k$$

where $\Delta k = 2\pi/L$, the change in k as the index m increments by one from term to term in the sum. In the limit $L \to \infty$, Δk becomes an infinitesimal, k becomes a 'continuous' rather than a discrete variable [$d_m \to d(k)$], and the sum of a great many small contributions becomes an integral.

$$f(x) \;=\; \left(1/2\pi\right)\int_{-\infty}^{\infty}\left[d(k)\,L\right]e^{ikx}dk = \left(1/2\pi\right)\int_{-\infty}^{\infty}\left[g(k)\right]e^{ikx}dk$$

where from the equation for the d_{p}

$$d(k)\,L = \int_{-L/2}^{L/2} e^{-ikx}\,f(x)\,dx = g(k)$$

The function g(k) is the Fourier transform of $f(x)$ which is the amplitude to find wiggling at the spatial frequency k in the function $f(x)$. The Fourier transform of $f(x)$ is to be represented as $\tilde{f}(k)$. Sadly, there is no universal treaty covering, the Fourier transform, and factors of 2π are shuttled from place to place in different treatments. Some hold that the balanced definition is the only true definition:

$$f(x) \;=\; \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\tilde{f}(k)\right]e^{ikx}dk \qquad \tilde{f}(k) \;=\; \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[f(x)\right]e^{-ikx}dx \qquad \text{[FS.26]}$$

The twiddle applied to the function symbol denotes the Fourier transform of that function, and the argument becomes the Fourier conjugate variable. $t \leftrightarrow \omega; x \leftrightarrow k$

In truth the inverse of 2π must appear, but it can be split up in any fashion.

$$f(x) \;=\; \left(1/2\pi\right)^{1-S}\int_{-\infty}^{\infty}\left[\tilde{f}(k)\right]e^{ikx}dk \qquad \tilde{f}(k) \;=\; \left(1/2\pi\right)^{S}\int_{-\infty}^{\infty}\left[f(x)\right]e^{-ikx}dx$$

The common choices are $S = 0$, $S = 1$ and $S = ½$.

The Fourier transform has a rich list of properties. Suffice it to say that analogs of the convergence properties, inner products and Parseval relations found for the

Fourier series exist and much more. The current goal is only to show that the restriction to periodic functions can be relaxed.

Fourier methods are extremely mysterious on first encounter. Pay the price. The rewards for mastering Fourier methods are enormous and cool. In the time domain, the Fourier transform identifies the frequency content of a function of time. Modern SONAR and passive acoustic monitoring systems depend on examining the received signal transformed into frequency space. Many systems are identified by their tonals, distinct frequency combinations in their acoustic emissions. In quantum mechanics, the spatial Fourier transform of the wave function reveals its plane wave or momentum content. In optics, the spatial Fourier transform of EM field amplitude in the plane of an aperture predicts the Fraunhofer diffraction pattern of that aperture. Scaling up to radar wavelengths, the spatial Fourier transform of an antenna predicts the far-field radiation pattern of that antenna. A result that also applies to hydrophone arrays in acoustics. There are problems that defy solution in the time domain that yield results freely when transformed to the (Fourier) frequency domain.

Fourier Series and the Vector Space of functions with period *T*

REVIEW: Displacement Model for a Vector Space

As an introduction, we will review the properties of the collection of all possible displacements $\Delta\vec{r}_i$ of a particle in our familiar model, a flat (Euclidean) infinite three-dimensional universe.

Rules for the Addition of Displacements

A1. **Closure**: the sum of two displacements is another displacement.

$$\Delta\vec{r}_1 + \Delta\vec{r}_2 = \Delta\vec{r}_3$$

A2. Additions can be regrouped without changing the resultant displacement.

$$\Delta\vec{r}_1 + \left(\Delta\vec{r}_2 + \Delta\vec{r}_3\right) = \left(\Delta\vec{r}_1 + \Delta\vec{r}_2\right) + \Delta\vec{r}_3$$

A3. The order of additions can be changed without changing the sum.

$$\Delta\vec{r}_1 + \Delta\vec{r}_2 = \Delta\vec{r}_2 + \Delta\vec{r}_1$$

A4. There is a displacement that does not change the position of the particle.

$$\Delta\vec{r}_i + \vec{0} = \Delta\vec{r}_i \quad \text{for all } \Delta\vec{r}_i .$$

A5. For any displacement $\Delta\vec{r}_i$, there is another $-\Delta\vec{r}_i$ that has the opposite action on a particle's position.

$$\text{for all } \Delta\vec{r}_i \ , \quad \Delta\vec{r}_i + \left(-\Delta\vec{r}_i\right) = \vec{0} .$$

Rules for the Multiplication of Displacements by Scalars

M1. If $\Delta\vec{r}_i$ is an allowed displacement, then $c\,\Delta\vec{r}_i$ is also an allowed displacement.

M2. The scalar multiple of a sum of displacements is the same as the sum of the same multiple of the individual displacements.

$$c\left(\Delta\vec{r}_1 + \Delta\vec{r}_2\right) = c\Delta\vec{r}_1 + c\,\Delta\vec{r}_2 \text{, an allowed displacement}$$

M3. A sum of scalars times a displacement is the sum of the individual scalars times that displacement.

$$\left(c + d\right)\Delta\vec{r}_i = c\,\Delta\vec{r}_i + d\,\Delta\vec{r}_i \text{, an allowed displacement}$$

M4. The scalar multiplication of displacements is associative.

$$c\left(d\,\Delta\vec{r}_i\right)\Delta\vec{r}_i = \left(c\,d\right)\Delta\vec{r}_i .$$

M5. One times any displacement is that same displacement.

$$1\,\Delta\vec{r}_i = \Delta\vec{r}_i \ \forall\ \Delta\vec{r}_i .$$

Displacements have other features such as a scalar inner (dot) product that yields a measure of the component of one displacement along the direction of another. The inner product of a displacement with itself is always positive. Thus the inner product can be used to assign lengths to all the displacements in the space (to set a metric for the space). [$\left|\Delta\vec{r}_i\right| = \sqrt{\Delta\vec{r}_i \cdot \Delta\vec{r}_i}$ |] Further, we can find a set of three displacements $\left\{\vec{b}_1, \vec{b}_2, \vec{b}_3\right\}$ such that all the allowed displacements can be represented as a sum of multiples of the members of this set, a linear combination. That is: all $\Delta\vec{r}_i = c_1\ \vec{b}_1 + c_2\ \vec{b}_2 + c_3\ \vec{b}_3$ for some scalars $\left\{c_1, c_2, c_3\right\}$. This set of displacements $\left\{\vec{b}_1, \vec{b}_2, \vec{b}_3\right\}$ is a basis for our three dimensional universe. An enormous simplification follows if the basis set $\left\{\vec{b}_1, \vec{b}_2, \vec{b}_3\right\}$ is chosen to be orthogonal and unity normalized. For example: the familiar set $\left\{\hat{i}, \hat{j}, \hat{k}\right\}$ is orthogonal $\left\{\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0\right\}$ in the sense that each represents a unit displacement that is completely independent of the other two. The set is also

unity normalized as: $\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$. The components of a vector can be found by projection using the inner product.

$$c_x=\left(\Delta\vec{r}\right)_x=\hat{i}\cdot\Delta\vec{r};\quad c_y=\left(\Delta\vec{r}\right)_y=\hat{j}\cdot\Delta\vec{r};\quad c_z=\left(\Delta\vec{r}\right)_z=\hat{k}\cdot\Delta\vec{r}$$

The inner product of two displacements can be expressed in terms of their components.

$$\Delta\vec{r}_1\cdot\Delta\vec{r}_2=c_{1x}c_{2x}+c_{1y}c_{2y}+c_{1z}c_{2z}$$

The Fourier Series basis set is orthogonal, but it is not unity normalized.

The Fourier vector space of periodic functions

Every *well-behaved* function of period T can be expanded in a series of the form

$$f(t)\;=\;c_0(1)\;+\;\sum_{m=1}^{\infty}a_m\cos\left[\omega_m t\right]\;+\;\sum_{m=1}^{\infty}b_m\sin\left[\omega_m t\right]\quad\text{where }\omega_m=m\left(2\pi/T\right).\qquad\text{[FS.27]}$$

The sum of two functions of period *T* is another function of period *T* is also a function of period *T* is also a function of period *T*. Any scalar multiple of a function of period *T* is also a function of period *T*. That is: the addition closure condition A1 and the multiplication closure condition M1 for vector spaces are met. Subject to the Dirichlet conditions, the Fourier expansion set

$$\left\{1;\left\{\cos\left[\omega_m t\right]\right\};\left\{\sin\left[\omega_p t\right]\right\}\right\}\text{ where }\omega_m=m\left(2\pi/T\right)\qquad\text{[FS.28]}$$

provides a complete basis for the expansion. Complete means that basis includes all behaviors necessary to accurately represent (converge in the mean to) all reasonable functions with period T. Orthogonal means that each member of the basis set has a vanishing inner product with any other member of the set. Each member of the set represents an independent behavior that is not shared with any other member of the set. The expansion coefficients (components with respect to the chosen basis) can then be found by using the inner product to **project out a behavior**. Choose the $\cos\left[\omega_p t\right]$ behavior. Multiply the series by $\left(\cos\left[\omega_p t\right]\right)^*$ and use the orthogonality relations. (! Omit the complex conjugation for real functions.)

$$\int_{-T/2}^{T/2} \cos\left[\omega_p t\right] f(t)\,dt \;=\; \int_{-T/2}^{T/2} \cos\left[\omega_p t\right] c_0\,dt$$
$$+\sum_{m=1}^{\infty} a_m \int_{-T/2}^{T/2} \cos\left[\omega_p t\right]\cos\left[\omega_m t\right]dt + \sum_{m=1}^{\infty} b_m \int_{-T/2}^{T/2} \cos\left[\omega_p t\right]\sin\left[\omega_m t\right]dt$$

Each function in the basis behaves like itself and not at all like any other so only the cos times cos integral with $m = p$ survives.

$$\int_{-T/2}^{T/2} \cos\left[\omega_p t\right] f(t)\,dt \;=\; \sum_{m=1}^{\infty} a_m \int_{-T/2}^{T/2} \cos\left[\omega_p t\right]\cos\left[\omega_m t\right]dt = \sum_{m=1}^{\infty} a_m \frac{T}{2}\delta_{\ell m} = a_p \frac{T}{2}$$

The familiar result follows:

$$a_p = \frac{2}{T}\int_{-T/2}^{T/2} \cos\left[\omega_p t\right] f(t)\,dt$$

All operations on the function can be defined in terms of actions on the Fourier components (coefficients) of the functions. The series for the sum of two functions (with period T) is the series with components that are the sum of the corresponding components in the series for the individual functions. The scalar multiple of a function has Fourier components that are that same scalar multiple of the Fourier components of the original function. The inner product of two functions can be expressed in terms of their Fourier components using Parseval's equality. Given:

$$f(t) \;=\; c_{f0}(1) \;+\; \sum_{m=1}^{\infty} a_{fm}\cos\left[\omega_m t\right] \;+\; \sum_{m=1}^{\infty} b_{fm}\sin\left[\omega_m t\right]$$

$$g(t) \;=\; c_{g0}(1) \;+\; \sum_{m=1}^{\infty} a_{gm}\cos\left[\omega_m t\right] \;+\; \sum_{m=1}^{\infty} b_{gm}\sin\left[\omega_m t\right] \qquad \text{[FS.29]}$$

$$\langle g|f\rangle = \left(1/T\right)\int_{-T/2}^{T/2} \left[g(t)\right]^* f(t)dt = c_{g0}^* c_{f0} + 1/2\sum_m a_{gm}^* a_{fm} + 1/2\sum_p b_{gp}^* b_{fp}$$

The annoying factors of $^1/_2$ appear because the Fourier basis set is not unity normalized. Recall that the average value of $\cos^2\left[\omega_p t\right]$ or of $\sin^2\left[\omega_p t\right]$ is one half (while the average value of 1 is 1 !). Forgive Fourier; he did not realize that he was defining a basis for a vector space.

Dimension is another issue. The dimension of a vector space is the smallest number of vectors that span the space - can be formed in a sums representing every vector in the space. For the Fourier series space that number is infinite. It is not very infinite however; it is a mere countable infinity. An infinite set is countable, if every element can be placed into a one-to-one correspondence with the counting numbers, the positive integers. One such correspondence is:

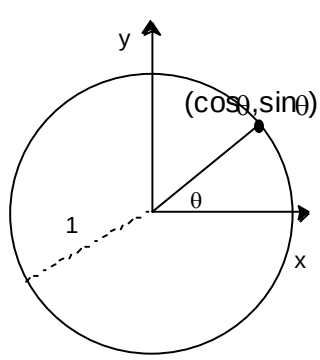
$$c_{f0}(1) \to 1; \qquad a_{fm}\cos\left[\omega_m t\right] \to 2m; \qquad b_{fm}\sin\left[\omega_m t\right] \to 2m+1$$

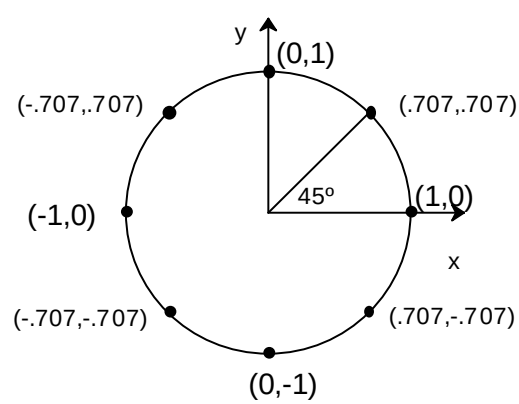
In a similar fashion, the number of bound states for a hydrogen atom is countable. Don't sweat it! Several mathematicians assure me that a countable infinity is hardly more than a finite number in the sense that it causes no problems beyond the headache currently throbbing between your temples.

Tools of the Trade:

Evaluating Trig Functions using the Unit Circle:

The evaluation of Fourier coefficients often involves specifying values for expressions of the form: $\sin[m\theta_o]$ where $\theta_o = \pi/p$. The answers can literally be read off the unit circle. The unit circle is a circle of radius one centered on the origin of the *x*-*y* plane. The coordinates of each point on the circle are therefore (cosθ, sinθ) where θ is the angle measured CCW from the x-axis.





Unit Circle for SIN and COS; .707 is a shorthand for $1/\sqrt{2}$

$$\sin[m\pi] = \begin{cases} 0 \ for\ m\ even;\ m = 2p \\ 0 \ for\ m\ odd;\ m = 2p+1 \end{cases} \qquad \cos[m\pi] = (-1)^m = \begin{cases} 1 \ for\ m\ even;\ m = 2p \\ -1 \ for\ m\ odd;\ m = 2p+1 \end{cases}$$

$$\sin[m\left(\frac{\pi}{2}\right)] = \begin{cases} 0 \quad for \quad m = 4p = 0,4,8,12,... \\ 1 \ for\ m = 4p+1 = 1,5,9,... \\ 0 \ for\ m = 4p+2 = 2,6,10,... \\ -1 \ for\ m = 4p+3 = 3,7,11,... \end{cases} \quad \cos[m\left(\frac{\pi}{2}\right)] = \begin{cases} 1 \quad for \quad m = 4p = 0,4,8,12,... \\ 0 \ for\ m = 4p+1 = 1,5,9,... \\ -1 \ for\ m = 4p+2 = 2,6,10,... \\ 0 \ for\ m = 4p+3 = 3,7,11,... \end{cases}$$

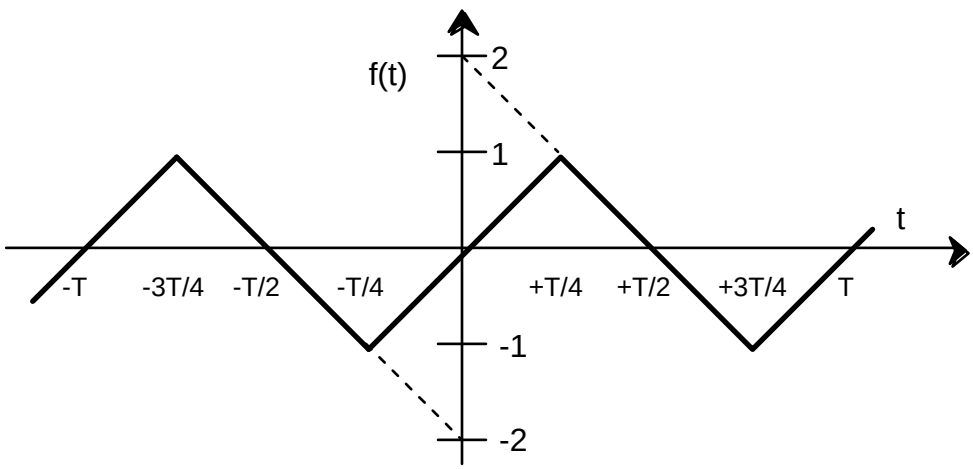
$$\sin[m\left(\frac{\pi}{4}\right)] = \begin{cases} 0 \quad for \quad m = 8p \\ 1/\sqrt{2} \ for\ m = 8p+1 \\ 1 \ for\ m = 8p+2 \\ 1/\sqrt{2} \ for\ m = 8p+3 \\ 0 \ for\ m = 8p+4 \\ -1/\sqrt{2} \ for\ m = 8p+5 \\ -1 \ for\ m = 8p+6 \\ -1/\sqrt{2} \ for\ m = 8p+7 \end{cases} \qquad \cos[m\left(\frac{\pi}{4}\right)] = \begin{cases} 1 \ for \ m = 8p \\ 1/\sqrt{2} \ for\ m = 8p+1 \\ 0 \ for\ m = 8p+2 \\ -1/\sqrt{2} \ for\ m = 8p+3 \\ -1 \ for\ m = 8p+4 \\ -1/\sqrt{2} \ for\ m = 8p+5 \\ 0 \ for\ m = 8p+6 \\ 1/\sqrt{2} \ for\ m = 8p+7 \end{cases}$$

Exercise: Use the methods above to evaluate $\left(1/m\right)\sin(m\pi/2)$ and $\left(1/m\right)\left(1-\cos(m\pi/2)\right)$ where m is an integer. See problem 16.

Dimensions in integrals and of functions: Integrals represent dimensioned values in science and engineering while functions and integral are considered dimensionless (*pure numbers*) in mathematics. Consider the function $b_m \sin\left[\omega_m t\right]$. The sine function returns a pure number so the coefficient b_m must have the dimensions of the result. Similarly, the arguments of mathematical functions must be dimensionless. For example $\omega_m t$ is the product of ω that has the units $^{rad}/_{s}$ and t which has the units s. The overall units of *radians* are not a problem as a radian is

the ratio of an arc length measured along a circle to its radius. That is a radian is itself *dimensionless* ($= {}^{m}/_{m}$).

Recall the triangular wave example:



The function $f(t)$ has a maximum value of 1. That is $f(t)$ is dimensionless. A coefficient b_p has the representation.

$$\frac{2}{T}\int_{-T/4}^{T/4}\sin\left[p\left(2\pi/T\right)t\right]\left({}^{4t}\!/_{T}\right)dt$$

Pull all the dimensioned quantities out of the integral. Choose $u = p\left(2\pi/T\right)t$. Here T is the period and t is the time so u is dimensionless.

The recommended choice for the change of variable is the argument of the most complicated function in the integrand. Above, the argument of the sine function was chosen. There are some variants of this rule as a form $e^{-\left[{}^{(x-x_0)^2}\!/_{a^2}\right]}$ requires $u^2 = {}^{(x-x_0)^2}\!/_{a^2}$.

$$\frac{2}{T}\int_{-T/4}^{T/4}\sin\left[p\left(2\pi/T\right)t\right]\left({}^{4t}\!/_{T}\right)dt = \frac{2}{T}\int_{-T/4}^{T/4}\sin\left[p\left(2\pi/T\right)t\right]\left(4\left\{\frac{2\pi\,p}{2\pi\,p}\right\}{}^{t}\!/_{T}\right)\left\{p\left(2\pi/T\right)\ \left(T/2\pi p\right)\right\}dt$$

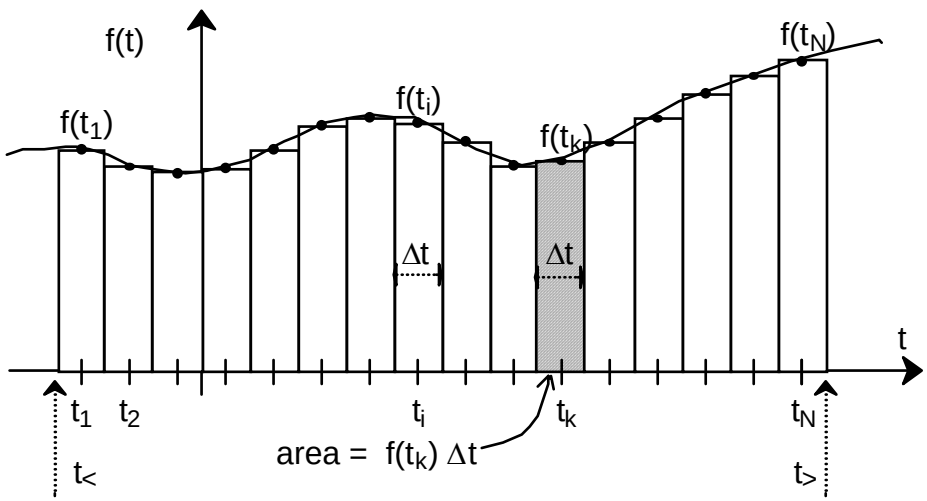
$$\frac{2}{T}\int_{-T/4}^{T/4}\sin\left[p\left(2\pi/T\right)t\right]\left({}^{4t}\!/_{T}\right)dt = \frac{2}{T}\int_{-p\pi/2}^{p\pi/2}\sin\left[u\right]\left(\left\{\frac{4}{2\pi\,p}\right\}u\right)\left\{\left(T/2\pi p\right)\right\}du$$

$$\frac{2}{T}\int_{-T/4}^{T/4}\sin\left[p\left(2\pi/T\right)t\right]\left(4t/T\right)dt=\left\{\frac{2}{\pi^2p^2}\right\}\left\{\int_{-p\pi/2}^{p\pi/2}\sin\left[u\right]u\,du\right\}$$

In the final form the quantity inside the first set of braces is to include all the dimensioned quantities and the integral in the second set of braces is to be dimensionless, a pure number. In this example $\left\{\frac{2}{\pi^2p^2}\right\}$ happens to be dimensionless; the first brace in other problems will carry the dimensions appropriate for the problem.

Converting Sums to Integrals

It is said that an integral is a sum of little pieces, but some precision is required before the statement becomes useful. Beginning with a function *f(t)* and a sequence of values for $t = \{t_1, t_2, t_3, \ldots, t_N\}$, the sum $\sum_{i=1}^{i=N} f(t_i)$ does not represent the integral $\int_{t_<}^{t_>} f(t)\,dt$ even if a great many closely spaced values of t are used. Nothing has been included in the sum to represent *dt*. One requires $\sum_{i=1}^{i=N} f(t_i)\,\Delta t_i$ where $\Delta t_i = \left(1/2\right)\left[t_{i+1} - t_{i-1}\right]$ is the interval between sequential values of *t* values at t_i. For well-behaved cases, the expression $\sum_{i=1}^{i=N} f(t_i)\,\Delta t_i$ approaches the Riemann sum definition of an integral as the *t*-axis is chopped up more and more finely. As illustrated below, in the limit Δt goes to zero, the sum $\sum_{i=1}^{i=N} f(t_i)\,\Delta t_i$ approaches the area under the curve between $t_<$ and $t_>$. That is: it represents $\int_{t_<}^{t_>} f(t)\,dt$ provided the sequence of sums converges, and life is good. The theory of integration is not the topic of this passage. The goal is simply to remind you that the Δt must be factored out of each term that is being summed in order to identify the integrand.



For the discussion of the inner product, the function $h(t) = f(t)\, g(t)$ was considered at N equally spaced points between $^{-T}/_{2}$ and $^{+T}/_{2}$. This leads to the sum $\sum_{m=1}^{m=N} h(t_m)$ where the points $t_m = -\left(T/2\right) - \left(T/2N\right) + m\left(T/N\right)$ have equal spacing $\Delta t = T/N$. As the number of terms gets large, the sum must be divided by N to keep the result finite leading to $\sum_{m=1}^{m=N} h(t_m)\left(1/N\right)$. The rule for converting sums to integral requires that $\Delta t = T/N$ be explicitly factored from each term in the sum. Thus

$\sum_{m=1}^{m=N} h(t_m)\left(1/N\right) \rightarrow \sum_{m=1}^{m=N} \left[\left(1/T\right) h(t_m)\right] \left(T/N\right) = \sum_{m=1}^{m=N} \left[\left(1/T\right) h(t_m)\right] \Delta t$ which becomes

$\int_{-T/2}^{+T/2} \left[\left(1/T\right) h(t)\right] dt = \left(1/T\right) \int_{-T/2}^{+T/2} \left[g(t)\, f(t)\right] dt$ as N gets large and Δt small.

Converting alternating term sums: $\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4} = 0.7853982$

Summing through m = 7: $\sum_{m=0}^{7} \frac{(-1)^m}{2m+1} = 0.7542680$

The goal is to estimate $\sum_{m=8}^{\infty} \frac{(-1)^m}{2m+1}$ by converting the sum to an integral.

Converting a sum to an integral is sure fail unless the change in the value of terms is small compared to the absolute value of the terms from one term to the next. The conversion may fail anyway, but unless this condition is met, it's hopeless. The series above is an alternating (sign) term series. The magnitude of the change in value of the sum from term to term is twice the magnitude of the terms. A trick is needed. Combine positive and negative terms in pairs. It is a trick; think about the sums below to verify that they are equal.

$$\sum_{m=8}^{\infty} \frac{(-1)^m}{2m+1} = \sum_{\ell=4}^{\infty} \left[\frac{1}{4\ell+1} - \frac{1}{4\ell+3}\right] = \sum_{\ell=4}^{\infty} \left[\frac{2}{16\ell^2+16\ell+3}\right]$$

The last sum has the advantage that it is a sum of positive terms that vanish more rapidly than do the alternating terms in the original series. It has better convergence properties.

To convert to an integral, $\Delta\ell$ must be factored from each term, but ℓ takes on ***every*** integer value greater than 4, the starting point. We conclude that $\Delta\ell = 1$.

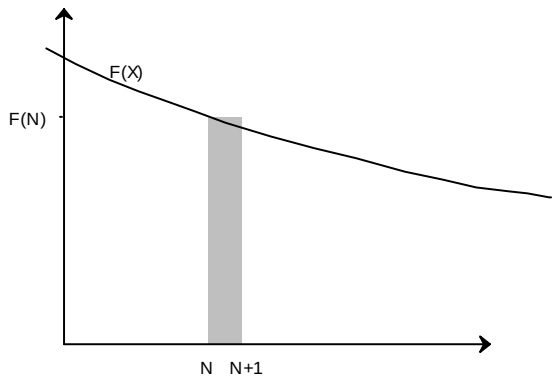
$$\sum_{\ell=4}^{\infty} \left[\frac{2}{16\ell^2+16\ell+3}\right] \equiv \sum_{\ell=4}^{\infty} \left[\frac{2}{16\ell^2+16\ell+3}\right]\Delta\ell \approx \int_4^{\infty} \frac{2}{16\ell^2+16\ell+3}\,d\ell = \frac{1}{4}\ln\left[\frac{19}{17}\right] \approx .0278064$$

$$\sum_{m=0}^{7} \frac{(-1)^m}{2m+1} + \int_4^{\infty} \frac{2}{16\ell^2+16\ell+3}\,d\ell \approx 0.7821$$ With patience one can do better.

Exercise: Show that: $\int_N^{\infty} \frac{2}{16\ell^2+16\ell+3}\,d\ell = \int_N^{\infty}\left[\frac{1}{4\ell+1} - \frac{1}{4\ell+3}\right]d\ell = \int_{4N+1}^{4N+3} u^{-1}\left(du/4\right)$. Use the result to evaluate the integral above.

Bounding the sum: Suitable choices of the integration limits can provide both

upper and lower bounds for the sum. A careful sketch shows that for a monotone decreasing function *F(n)*, $\int_N^\infty F(x)\,dx < \sum_{m=N}^{\infty} F(n) < \int_{N-1}^\infty F(x)\,dx$.



The area of the shaded region is *F(N)*, and *F(x)* as drawn is a monotone decreasing function. It follows that:

$$\int_N^{N+1} F(x)\,dx < F(N) < \int_{N-1}^{N} F(x)\,dx$$

Exercise: Argue that for a monotone decreasing function,

$$\int_N^\infty F(x)\,dx < \sum_{m=N}^{\infty} F(n) < \int_{N-1}^\infty F(x)\,dx$$

$$\sum_{m=0}^{7} \frac{(-1)^m}{2m+1} + \int_4^\infty \frac{2}{16\ell^2+16\ell+3}\,d\ell < \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} < \sum_{m=0}^{7} \frac{(-1)^m}{2m+1} + \int_3^\infty \frac{2}{16\ell^2+16\ell+3}\,d\ell$$

or equivalently $0.7821 < \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} < 0.7900$. These results are consistent with a sum of $\pi/4 \approx 0.7854$. A current estimate might be chosen to be:

$$\sum_{m=0}^{7} \frac{(-1)^m}{2m+1} + \int_{3.5}^\infty \frac{2}{16\ell^2+16\ell+3}\,d\ell \approx 0.7542680 + \left(\frac{1}{4}\right)\ln\left(\frac{17}{15}\right) \approx 0.7855587$$

It seems that splitting the difference is reasonable so long as one remembers the bounds on the proven range.

Summing Series and Special Tricks:

Sine and cosine series can be rich sources of Fourier coefficients to use in evaluating sums either by evaluating the series for a particular argument or by employing a Parseval relation for the square of a function of the product of a pair of functions. Use the convergence properties as a guide. Consider a function like:

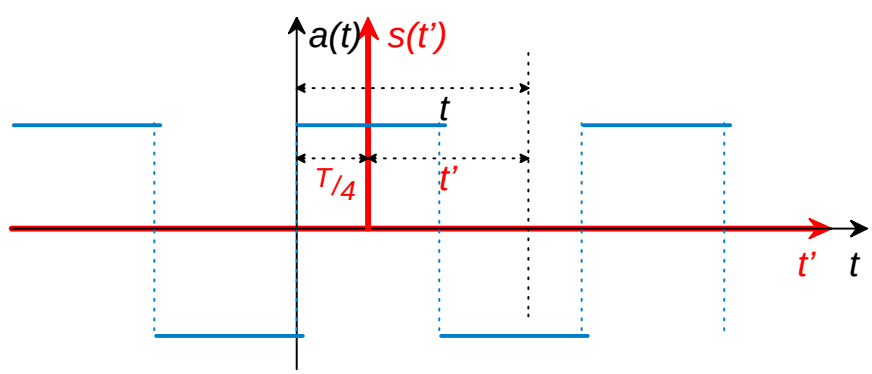
$f(x) = x(\pi - x)$ for the interval $[0, \pi]$. Evaluated as a cosine series, it has a a discontinuity in the first derivative when extended so the coefficients vanish no faster than m^{-2} indicating the powers in the sums that can be directly evaluated and that m^{-4} sums can be found using a Parseval relation. If the function is evaluated as a sine series, its anti-symmetric extension has a continuous derivative so the coefficients might vanish like m^{-3} so that m^{-3} and m^{-6} sums can be evaluated.

The Riemann zeta is particularly important $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$. As many Fourier series consist of only even or odd index coefficients, it is helpful to note that:

$$\sum_{n=even>0}^{\infty} n^{-p} = 2^{-p}\sum_{n=1}^{\infty} n^{-p} \qquad \sum_{n=odd>0}^{\infty} n^{-p} = (1-2^{-p})\sum_{n=1}^{\infty} n^{-p}$$

Translating a Function: Fourier series for a function with an offset translation

How can a representation for a symmetric square $s(t)$ wave be built using a representation of an anti-symmetric square wave $a(t)$? A reliable procedure for translating the argument is required. To begin, prepare a figure.



$$t = t' + {}^{T}/_{4} \quad \text{and} \quad a(t) \equiv s(t') \Rightarrow s(t') = a(t' + {}^{T}/_{4})$$

$$a(t) = \frac{4}{\pi}\sum_{m=0}^{\infty}\left(\frac{1}{2m+1}\right)\sin\left[(2m+1)\frac{2\pi t}{T}\right] \qquad \text{anti-symmetric square wave}$$

$$s(t') = a(t' + {}^{T}\!/_{4}) = \frac{4}{\pi}\sum_{m=0}^{\infty}\left(\frac{1}{2m+1}\right)\sin\left[(2m+1)\frac{2\pi\left(t'+{}^{T}\!/_{4}\right)}{T}\right] \quad \text{symmetric square wave}$$

$$s(t') = \frac{4}{\pi}\sum_{m=0}^{\infty}\left(\frac{1}{2m+1}\right)\sin\left[(2m+1)\frac{2\pi\, t'}{T} + m\pi + \pi/2\right]$$

The sine changes sign for each shift of π, and sin[θ + $^{\pi}/_{2}$] = cos[θ].

$$\Rightarrow \sin\left[(2m+1)\frac{2\pi\, t'}{T} + m\pi + \pi/2\right] = (-1)^m \cos\left[(2m+1)\frac{2\pi\, t'}{T}\right]$$

$$s(t') = \frac{4}{\pi}\sum_{m=0}^{\infty}\left(\frac{(-1)^m}{2m+1}\right)\cos\left[(2m+1)\frac{2\pi\, t'}{T}\right] \text{or simply}$$

$$s(t) = \frac{4}{\pi}\sum_{m=0}^{\infty}\left(\frac{(-1)^m}{2m+1}\right)\cos\left[(2m+1)\frac{2\pi\, t}{T}\right].$$

The Fourier series for the symmetric square wave has been deduced from the series for the anti-symmetric square wave by carefully applying an argument translation.

Vector Analog of Inner Product Concepts:

The inner product of two 3D vectors is to be defined as: $\vec{A}\cdot\vec{B} = A\,B\cos\theta_{AB}$.

It is assumed that the following properties are known:

The inner product of two vectors returns a scalar value.

i.) $\vec{A}\cdot\left(c\,\vec{B}\right) = c\left(\vec{A}\cdot\vec{B}\right)$ where *c* is a scalar multiplier.

ii.) $\vec{A}\cdot\left(\vec{B}+\vec{C}\right) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$

iii.) $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ for all $\vec{A}$ and $\vec{B}$.

iv.) $\vec{A}\cdot\vec{A} \geq 0;\ \vec{A}\cdot\vec{A} = 0$ *iffi* $\vec{A} = 0$

FS1. Establish the orthogonality of the basis set $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \left\{\hat{i}, \hat{j}, \hat{k}\right\}$.

$$\hat{i}\cdot\hat{i} = (1)(1)\cos(0) = 1 \qquad \hat{i}\cdot\hat{j} = (1)(1)\cos(\pi/2) = 0 \qquad \hat{i}\cdot\hat{k} = (1)(1)\cos(\pi/2) = 0$$

and so on until we have $\hat{e}_i \cdot \hat{e}_j = (1)(1)\cos(\theta_{ij}) = \delta_{ij}$.

FS2. The claim is that all of our vectors can be *adequately* represented as linear combinations of the basis *expansion* set $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{\hat{i}, \hat{j}, \hat{k}\}$.

$$\vec{A} = \sum_{i=1}^{3} A_i \hat{e}_i \quad \text{and} \quad \vec{B} = \sum_{j=1}^{3} B_j \hat{e}_j$$

If the expansions are robust, they can be used in place of the vectors.
Note the use of distinct dummy summation indices.

$$\vec{A} \cdot \vec{B} \rightarrow \left(\sum_{i=1}^{3} A_i \hat{e}_i \right) \cdot \left(\sum_{j=1}^{3} B_j \hat{e}_j \right) = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \left(\hat{e}_i \cdot \hat{e}_j \right)$$

$$\vec{A} \cdot \vec{B} \rightarrow \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \left(\delta_{ij} \right) = \sum_{i=1}^{3} A_i B_i$$

Summing over j, the Kronecker delta picks out the one term with $j = i$.
Hence we have developed a second representation of the inner product that is expressed in terms of the expansion coefficients.

$$\vec{A} \cdot \vec{B} \equiv A B \cos\theta_{AB} = \sum_{i=1}^{3} A_i B_i = A_x B_x + A_y B_y + A_z B_z$$

FS3. Suppose that a vector has two expansions in terms of the set.

$$\vec{A} = \sum_{i=1}^{3} A_i \hat{e}_i = \sum_{i=1}^{3} A_i^{/} \hat{e}_i$$

Show that the coefficients must be equal term by term. Take the inner product on both sides with an *independent* expansion set member.

$$\hat{e}_j \cdot \sum_{i=1}^{3} A_i \hat{e}_i = \sum_{i=1}^{3} A_i \hat{e}_j \cdot \hat{e}_i = \hat{e}_j \cdot \sum_{m=1}^{3} A_m^{/} \hat{e}_m = \sum_{m=1}^{3} A_m^{/} \hat{e}_j \cdot \hat{e}_m$$

$$\sum_{i=1}^{3} A_i \delta_{ji} = \sum_{m=1}^{3} A_m^{/} \delta_{jm} \Rightarrow A_j = A_j^{/}$$

In the sum over i, the Kronecker delta picks out the one term with $i = j$. The

analogous collapse occurs for the sum over *m*. The conclusion is that the expansion coefficients must be equal term by term so that the expansion in terms of an orthogonal expansion set is unique.

FS4. For a vector $\vec{H} = c_f \vec{F} + c_g \vec{G}$, one finds the expansion coefficients using the canonical prescription.

$$H_j = \frac{\hat{e}_j \cdot \vec{H}}{\hat{e}_j \cdot \hat{e}_j} = \frac{\langle \hat{e}_j | \vec{H} \rangle}{\langle \hat{e}_j | \hat{e}_j \rangle} = \frac{\hat{e}_j \cdot \left(c_f \vec{F} + c_g \vec{G} \right)}{\hat{e}_j \cdot \hat{e}_j} = \frac{c_f \left(\hat{e}_j \cdot \vec{F} \right)}{\hat{e}_j \cdot \hat{e}_j} + \frac{c_g \left(\hat{e}_j \cdot \vec{G} \right)}{\hat{e}_j \cdot \hat{e}_j} = c_f F_j + c_g G_j$$

The expansion coefficients are the same linear combination as are the vectors coefficient by coefficient. $H_j = c_f F_j + c_g G_j$ for j = 1, 2, 3.

Compare $H_j = \frac{\hat{e}_j \cdot \vec{H}}{\hat{e}_j \cdot \hat{e}_j} = \frac{\langle \hat{e}_j | \vec{H} \rangle}{\langle \hat{e}_j | \hat{e}_j \rangle}$ with:

$$a_{f\,p} = \frac{\langle \cos[\omega_p t] | f(t) \rangle}{\langle \cos[\omega_p t] | \cos[\omega_p t] \rangle} = \frac{\frac{1}{T} \int_{-T/2}^{T/2} \cos[\omega_p t]\, f(t)\, dt}{\frac{1}{T} \int_{-T/2}^{T/2} \cos[\omega_p t] \cos[\omega_p t]\, dt} = \frac{2}{T} \int_{-T/2}^{T/2} \cos[\omega_p t]\, f(t)\, dt$$

Discovery Exercises:

ALERT! Mathematica and Maple templates are included just after the discovery exercises.

D1.) The goal is to represent 4 $sin^3(x)$ as a sum of sine functions.

$$f(x) = 4\, sin^3(x) = a_1\, sinx + a_2\, sin(2x) + a_3\, sin(3x) + a_4\, sin(4x)$$

where the coefficients a_i are integers in the range –5 to 5.

Method 1: Plot 4 $sin^3(x)$ for the range -π to π. Set the a_i = {1,0,0,0} and plot

$g(x) = 4\, sin^3(x) - [a_1\, sinx + a_2\, sin(2x) + a_3\, sin(3x) + a_4\, sin(4x)]$

Adjust the a_i until $g(x) = 0$.

Method 2: Plot $|g(x)|^2$ for the range -π to π. Set the $a_i = \{1,0,0,0\}$ and then adjust the a_1 to minimize the area under the curve for $|g(x)|^2$. After finding the best value for a_1, move on to a_2. Is method 2 more or less efficient than method 1?

Method 3: The functions $\sin(n\,x)$ represent the independent ways that an anti-symmetric function of period 2π can wiggle. Plot the functions pairs $f(x)$ and $\sin(n\,x)$ over the interval -π to π. Estimate the degree to which the functions in each pair vary together over the interval. Compute the integrals $(1/\pi)\int_{-\pi}^{\pi}\sin(n\,x)\ f(x)\,dx$ for n = 1, 2, 3 and 4. These integrals are proportional the inner product of $\sin(n\,x)$ and $f(x)$ and gauge the amount of $\sin(n\,x)$ behavior in $f(x)$.

D2.) The goal is to represent $8\cos^4(x) - 8\cos^3(x) - 3$ as a sum of sine functions.

$$f(x) = 8\cos^4(x) - 8\cos^3(x) - 3 = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + a_4\cos(4x)$$

where the coefficients a_i are integers in the range –9 to 9.

Method 1: Plot $f(x)$ for the range -π to π. Set the $a_i = \{1,0,0,0,0\}$ and plot

$$g(x) = 8\cos^4(x) - 8\cos^3(x) - 3 - [a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + a_4\cos(4x)]$$

Adjust the a_i until $g(x) = 0$.

Method 2: Plot $|g(x)|^2$ for the range -π to π. Set the $a_i = \{1,0,0,0,0\}$ and then adjust the a_1 to minimize the area under the curve for $|g(x)|^2$. After finding the best value for a_0, move on to a_1. Is method 2 more or less efficient than method 1?

Method 3: The functions cos*(n x)* represent the independent ways that a symmetric function of period 2π can wiggle. Plot the functions pairs *f(x)* and cos(*n x*) over the interval -π to π. Estimate the degree to which the functions in each pair vary together over the interval. Compute the integrals $(1/\pi)\int_{-\pi}^{\pi}\cos(n\,x)\ f(x)\,dx$ for n = 1, 2, 3 and 4. These integrals are proportional the inner product of cos(*n x*) and *f(x)* and gauge the amount of cos(*n x*) behavior in *f(x)*. The *correct* method to compute a_n is: $a_n = \dfrac{\int_{-\pi}^{\pi}\cos(n\,x)\ f(x)\,dx}{\int_{-\pi}^{\pi}\cos(n\,x)\ \cos(n\,x)\,dx}$. For which n might this equation yield a different result than $(1/\pi)\int_{-\pi}^{\pi}\cos(n\,x)\ f(x)\,dx$?

Mathematica Templates for D1 and D2: *Mathematica is case sensitive.*

Mathematica 5.2 Syntax ` is to the left of the 1 key

<<Calculus`FourierTransform` loads the Fourier package

Mathematica 6 Syntax

Fourier transform library is preloaded

`<<FourierSeries`` New load command needed to load the Fourier series library

`FourierTrigSeries[f(x),x,terms]`

generates <terms> terms of the Fourier series for f(x) with period 2.

UnitStep[x] = 0 for x < 0; = 1 for x > 1

```
Plot[4 (Sin[x])^3, {x,-Pi,Pi}, PlotRange -> {-4,4}, PlotStyle ->RGBColor[1,0,0]]
```

a0=1; a1=1; a2=0; a3 = 0; a4 = 0 **execute this line whenever the a's are changed*

ENTER of SHIFT-RETURN executes a line

```
Plot[4 (Sin[x])^3 - (a1 * Sin[x] + a2 * Sin[2 x] + a3 Sin[3 x] + a4 Sin[4 x]),
  {x,-Pi,Pi}, PlotRange -> {-4,4}, PlotStyle ->RGBColor[1,0,0]]
```

```
Plot[(4 (Sin[x])^3 -
      (a1 * Sin[x] + a2 * Sin[2 x] + a3 Sin[3 x] + a4 Sin[4 x]))^2,
  {x,-Pi,Pi}, PlotRange -> {-4,4}, PlotStyle ->RGBColor[1,0,0]]
```

```
Plot[{4 (Sin[x])^3 , Sin[3 x] }, {x,-Pi,Pi}, PlotRange -> {-4,4},
  PlotStyle ->{RGBColor[1,0,0],RGBColor[0,0,1]}]
```

Note that the function to be plotted has been replaced by a list of two functions

```
(1/Pi) * Integrate[ Sin[ 3 x] *(4 (Sin[x])^3  ),{x,-Pi,Pi}]
```

Cos[x] is the cosine function

A space or an * represent multiplication

Plot[4 (Sin[x])^3, {x,-Pi,Pi}, PlotRange -> {-4,4}, PlotStyle ->RGBColor[1,0,0]]

[…] use square brackets to enclose function arguments

(…) use parentheses to group terms

{ … } use braces to enclose a list Pi = π

Plot[function[x], { abscissaVariable, lower, upper}, *optional properties*]

PlotRange -> {y_{min},y_{max}} set the vertical or ordinate range

PlotStyle -> RGBColor[red, green, blue] *sets the plot line color*

Integrate[Sin[3 x] *(4 (Sin[x])^3),{x,-Pi,Pi}]

Integrate [function[x], { integrationVariable, lower, upper}]

Maple Sample Code:

```
> a0:=1; a1:=0; a2:=0; a3:=0; a4:=0;

> f:=(x)-> 4*(sin(x))^3- (a1*sin(x)+a2*sin(2*x)+a3*sin(3*x)+a4*sin(4*x));

> with(plots):plot(f(x),x=-Pi..Pi);

> with(plots):plot({4*(sin(x))^3, sin(1 * x)},x=-Pi..Pi);

>  (1/Pi)* int(sin(x)*f(x),x=-Pi..Pi);
```

Warm Up Problems:

FSWUP1. **PROJECTION (A MOST WONDERFUL PROCESS)**: Consider a vector represented as $|V_i\rangle = a_{i1}|B_1\rangle + a_{i2}|B_2\rangle + ... + a_{in}|B_n\rangle$ in terms of the basis set { …, $|B_i\rangle$, …, … }. Further, assume that the basis set is mutually orthogonal. $\Rightarrow$ $\langle B_j|B_i\rangle = \langle B_j|B_i\rangle\delta_{ij}$. The coefficient a_{ik} represents the amplitude of the $|B_k\rangle$ behavior in the vector $|V_i\rangle$.

a.) Use the inner product to develop a general expression for a_{ik} in terms of the ratio of inner products for vectors expanded in an orthogonal basis.

b.) Use the inner product projection procedure to project out the component A_y in the expression $\vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$ and where the inner product is are standard *dot*

product to find an expression for A_y. Compare with an application of the general procedure given an orthogonal basis. $a_{ik} = \dfrac{\langle B_k | V_i \rangle}{\langle B_k | B_k \rangle} \rightarrow A_y = \dfrac{\langle \hat{j} | \vec{A}_i \rangle}{\langle \hat{j} | \hat{j} \rangle}$

c.) Consider a function, $f(t) = c_{f0} + \sum_{m=1}^{\infty} a_{fm} \cos[\omega_m t] + \sum_{n=1}^{\infty} b_{fn} \sin[\omega_n t]$. Use the inner product to project out the coefficient a_{fk} in this expansion. Some of the orthogonality relations for the Fourier trig basis set are:

$$\frac{1}{T}\int_{-T/2}^{T/2} \sin[\omega_p t]\cos[\omega_m t]\,dt = 0; \frac{1}{T}\int_{-T/2}^{T/2} \sin[\omega_p t]\sin[\omega_m t]\,dt = \tfrac{1}{2}\delta_{pm};$$

$$\frac{1}{T}\int_{-T/2}^{T/2} \cos[\omega_p t]\cos[\omega_m t]\,dt = \tfrac{1}{2}\delta_{pm}$$

and the inner product is $\langle f | g \rangle = \dfrac{1}{T}\int_{-T/2}^{T/2} [f(t)]^* g(t)\,dt$. Show that the process develops the standard expression: $a_p = \dfrac{2}{T}\int_{-T/2}^{T/2} \cos[\omega_p t] f(t)\,dt$

Problems:

1.) A geometric proof of the identity cos(β – α) = cos(α) cos(β) + sin(α) sin(β) was presented in the trigonometry handout. Replace α by -α to develop the corresponding identity for cos(β+α). Use the identities for cos(β-α) and cos(β+α) to find expressions for cos(α) cos(β) and sin(α) sin(β) in terms of cos(β-α) and cos(β+α). Use the results to establish the (a) center and (b) rightmost of the following orthogonality relations. [$\omega_m = m\,(^{2\pi}/_{T})$]

$$\frac{1}{T}\int_{-T/2}^{T/2} \sin[\omega_p t]\cos[\omega_m t]\,dt = 0; \frac{1}{T}\int_{-T/2}^{T/2} \sin[\omega_p t]\sin[\omega_m t]\,dt = \tfrac{1}{2}\delta_{pm}; \frac{1}{T}\int_{-T/2}^{T/2} \cos[\omega_p t]\cos[\omega_m t]\,dt = \tfrac{1}{2}\delta_{pm}$$

Pay special attention to the factors $p - m$ and $p + m$. Which can be zero? Treat the cases $p = m$ and $p \neq m$ separately.

The leftmost relation is easily established using a symmetry argument. (c) Do so.

(d) Prove one more of the orthogonality relations for the Fourier expansion set:

$$\{1;\{\ldots,\cos[\omega_m t],\ldots\};\{\ldots,\sin[\omega_p t],\ldots\}\}.$$

It will involve 1 and either a sine or a cosine. If you feel that you have suffered to complete this problem, you have earned the right to use the orthogonality relations freely in the future. Note that a minimum of four evaluations is required.

Conclusion: Given the inner product: $\langle f|g\rangle = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,g(t)\,dt$, the set of functions $\{1;\{\ldots,\cos[\omega_m t],\ldots\};\{\ldots,\sin[\omega_p t],\ldots\}\}$ is mutually orthogonal if the frequencies are $\omega_m = m\,{}^{2\pi}/_{T}$. The constant 1 has a norm squared of 1 while all others have a norm squared of ½.

2.) Consider two functions of period T represented by their Fourier series:

$$f(t) = c_{f0}(1) + \sum_{m=1}^{\infty} a_{fm}\cos[\omega_m t] + \sum_{n=1}^{\infty} b_{fn}\sin[\omega_n t]$$

$$g(t) = c_{g0}(1) + \sum_{k=1}^{\infty} a_{gk}\cos[\omega_k t] + \sum_{p=1}^{\infty} b_{gp}\sin[\omega_p t]$$

Show that the two expressions for the inner product of periodic functions are equivalent. *Beware! The sum indices m and n have been used twice. Change to four distinct index symbols before substituting into the integral. As sine and cosine are real functions,* $[g(t)]^* = c_{g0}^*(1) + \sum_{m=1}^{\infty} a_{gm}^*\cos[\omega_m t] + \sum_{n=1}^{\infty} b_{gn}^*\sin[\omega_m t]$.

$$\langle g|f\rangle = \left(\tfrac{1}{T}\right)\int_{-T/2}^{T/2}[g(t)]^* f(t)dt = c_{g0}^* c_{f0} + \tfrac{1}{2}\sum_m a_{gm}^* a_{fm} + \tfrac{1}{2}\sum_p b_{gp}^* b_{fp}$$

Recall that: $\sum_{m=1}^{\infty} A_m \, \delta_{m\ell} = A_\ell$. Hint: Use the orthogonality relations. **SPECIAL NOTE: The summation indices are dummy labels. They may be relabeled at will. At minimum, rename the labels in a product to ensure that the labels are distinct.** Begin by rewriting the representations of *f(t)* and *g(t)* using distinct summation index labels throughout.

3.) The equation below is to be valid.

$$c_{f0} + \sum_{m=1}^{\infty} a_{fm} \cos[\omega_m t] + \sum_{n=1}^{\infty} b_{fn} \sin[\omega_n t] = c_{g0} + \sum_{k=1}^{\infty} a_{gk} \cos[\omega_k t] + \sum_{\ell=1}^{\infty} b_{g\ell} \sin[\omega_\ell t]$$

Show that this equality requires that:

$$c_{f0} = c_{g0}; \quad a_{fm} = a_{gm} \; for\ all\ m \geq 1; \quad b_{fn} = b_{gn} \; for\ all\ n \geq 1$$

Each term in a Fourier series represents an independent behavior. Two such series can only be equal if they are equal term by term.

4.) Consider the functions $f(t)$ and $g(t)$ with their corresponding Fourier representations:

$$f(t) = c_{f0}(1) + \sum_{m=1}^{\infty} a_{fm} \cos[\omega_m t] + \sum_{m=1}^{\infty} b_{fm} \sin[\omega_m t]$$

$$g(t) = c_{g0}(1) + \sum_{k=1}^{\infty} a_{gk} \cos[\omega_k t] + \sum_{p=1}^{\infty} b_{gp} \sin[\omega_p t]$$

What is the Fourier representation for the function $h(t) = A\,f(t) + B\,g(t)$ where A and B are constants? Begin with the standard relations for the a_n*'s* as an example:

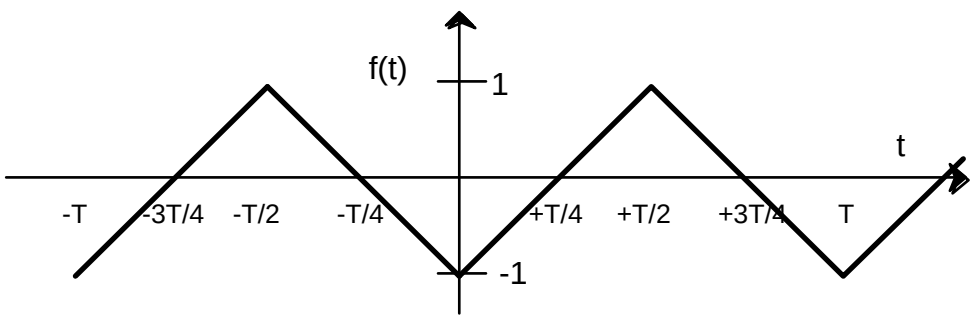
$$a_{fp} = \frac{2}{T} \int_{-T/2}^{T/2} \cos[\omega_p t]\, f(t)\, dt \,;\; a_{gp} = \frac{2}{T} \int_{-T/2}^{T/2} \cos[\omega_p t]\, g(t)\, dt \,;\; a_{hp} = \frac{2}{T} \int_{-T/2}^{T/2} \cos[\omega_p t]\, h(t)\, dt$$

Substitute $h(t) = A\,f(t) + B\,g(t)$. Conclude that Fourier series add component-wise or coefficient-wise in a fashion similar to vector addition.

5.) Consider the case of a function $f(t)$ with period 2π to be expanded over the

6.) Back to the violin string running from 0 to L. Fourier series can also be used to represent functions of spatial coordinates. The function $f(x)$ is to be periodic with period 2 L. (The string shape function is to be extended as a function that is anti-symmetric about $x = 0$.) What are the allowed spatial frequencies k_m for the Fourier series of $f(x)$? Give the expressions to calculate c_o, $\{\ldots a_m \ldots\}$ and $\{\ldots b_n \ldots\}$. Which are certainly zero? $f(x) = c_0 + \sum_{m=1}^{\infty} a_m \cos[k_m x] + \sum_{n=1}^{\infty} b_n \sin[k_n x]$

7.) Consider the (INVERTED) symmetric unit amplitude triangular wave.

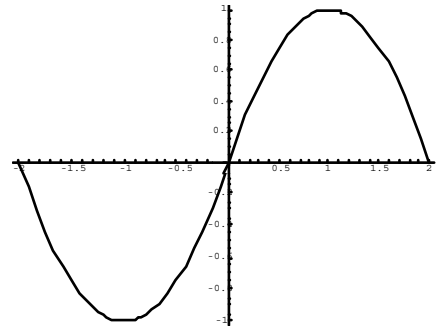


$$f(t) = \begin{cases} -1 + {}^{4t}\!/_{T} & for \quad 0 < t < {}^{T}\!/_{2} \\ -1 - {}^{4t}\!/_{T} & for \quad -{}^{T}\!/_{2} < t < 0 \end{cases}$$

Find the Fourier series for $f(t)$. Sketch the anti-symmetric triangular wave, one of the example functions treated above, and this symmetric triangular wave on the same plot. Compare the frequency content of the two triangular waves. Scale the new wave to have amplitude ${}^{T}/_{4}$. That is define g(t) = $({}^{T}/_{4})\, f(t)$. Give the Fourier series for $g(t)$. Prepare a sketch of ${}^{dg}/_{dt}$. What is the slope of $g(t)$ at t = ${}^{T}/_{4}$? Differentiate the Fourier series for $g(t)$ term by term. Comment on the form of the series after differentiation.

8.) Slope-matched parabolic sections. Consider the function of period 4 defined over the interval [-2,2] by the equations:

$$f(t) = \begin{cases} 1-(t-1)^2 = 2t - t^2 & for \quad 0 < t < 2 \\ -1+(t+1)^2 = 2t + t^2 & for \quad -2 < t < 0 \end{cases}$$



Compute df/dt. Is df/dt continuous at $x = 0$? Compute d^2f/dt^2. Is the second derivative continuous at $x = 0$? Compute the Fourier series for *f(t)*. How fast do the coefficients vanish with respect to *m* for large *m*? Note: The function is odd about zero so only sines appear. The function is even about $T/4$ so only sine terms that are odd multiples of the fundamental are to be included.

$$b_{2m+1} \to (4)\frac{2}{T}\int_0^{T/4} \sin\left[\omega_{2m+1} t\right] f(t)\, dt \to 2\int_0^1 \sin\left[(2m+1)(\pi/2)t\right]\left[2t - t^2\right] dt$$

Sketch the function and a few sines with even and odd multiples of the fundamental frequency. Argue that only the odd multiple sine waves will contribute. **ANSWER:** $b_p = 32/(p\,\pi)^3$ **for *p* odd; = 0 for *p* even.**

8 extension 1.) Continue problem 8. Differentiate the series term by term. Differentiate the series again term by term. Compare the result to series for the unit amplitude triangle and square waves. What amplitude do you expect for the first and second derivative of the matched parabola function defined in problem 8? The derivative of a function of an odd (even) function is an even (odd) function. How is this fact embodied in the results of differentiating the series expansion term by term?

8 extension 2.) Continue problem 8. The sum of the terms for m greater than M is bounded by $\pm\int_M^\infty 32/(\pi m)^3 dm \approx \pm 2.0641/M^2$. Argue that the series converges uniformly to the function $2x - x^2$ on the interval [0,2].

9.) Integrate the series for the unit amplitude anti-symmetric square wave term by term. Choose the lower limit as $t_i = -T/4$. Each integral term should be of the form: $b_p \int_{-T/4}^{t} \sin[p\,\omega_0 t']dt'$. Sketch the expected result. What is the average value of the function represented by the integrated series? Why was the lower limit chosen to be $t_i = -T/4$? Compare with problem 7. Note that t' is a dummy label for the integration variable. Comment. (*For a detailed discussion of dummy labels in integrals see the Differential Equations section where the solution of DEs by integration is treated.*)

10.) Use the series representations for the anti-symmetric square wave and triangular wave and the (Parseval) relation for the inner product to compute an infinite sum involving the cubes of the inverses of the odd integers. [See #2.]

11.) Develop the Parseval and Parseval Inner Product relations for the Fourier series using complex exponential basis functions.

12.) This problem illustrates an issue related to convergence. Sketch the anti-symmetric square wave. Sketch the derivative of the square wave. Differentiate the series representation for the square wave term by term, Does that series converge? The series for the square wave does not converge uniformly. Discuss the specific behavior of the convergence of the series representation of the square wave that fails to meet the standard for uniform convergence.

13.) The solution to the driven damped oscillator for a single cosine driving term was found to be

$$x_\omega(t) = R(\omega)\, f_0 \cos(\omega t - \theta) = \frac{f_0}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(2\gamma\omega\right)^2}} \cos(\omega t - \tan^{-1}\left[\frac{2\omega\gamma}{\omega_0^2 - \omega^2}\right])$$

for a driving force $F(t) = m f_o \cos[\omega t]$. A standard notation defines the Q or quality factor of the oscillator as $Q = m\omega_o / b$. Show that the equation for $x(t)$ can be expressed as:

$$x_\omega(t) = \frac{(F_o/k)}{\sqrt{\left(1-(\omega/\omega_o)^2\right)^2 + \left(\frac{1}{Q}\right)^2 (\omega/\omega_o)^2}} \cos\left(\omega t - \tan^{-1}\left[\frac{(\omega/\omega_o)}{Q\left[1-(\omega/\omega_o)^2\right]}\right]\right)$$

14. Assume that an oscillator has $k = 10^5\ {}^{k}/_{m}$; Q = 100; $\omega_o = 120\ \pi\ s^{-1}$. The driving force is a 20 Hz triangular wave with amplitude 10 N: (See prob. 13)

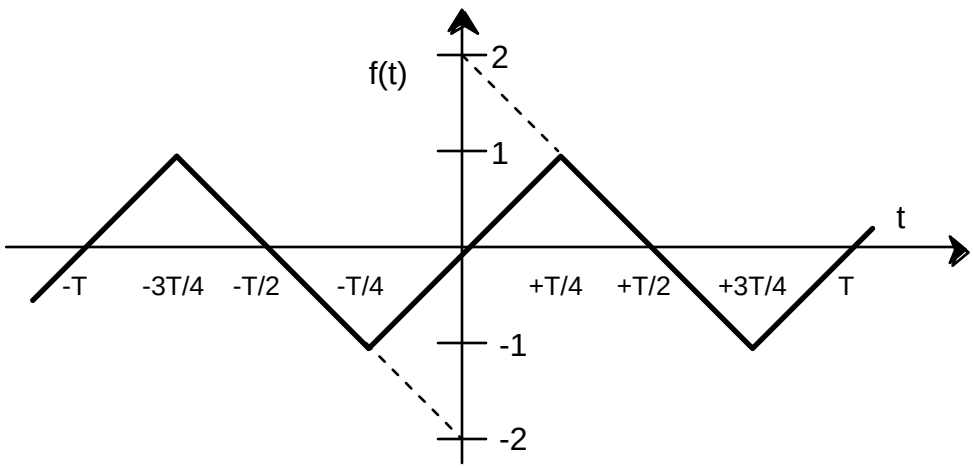
$$F(t) = \frac{80N}{\pi^2}\left[\sin\left(40\pi\ s^{-1} t\right) - \frac{1}{3^2}\sin\left(120\pi\ s^{-1} t\right) + \frac{1}{5^2}\sin\left(200\pi\ s^{-1} t\right) - ....\right]$$

Give the expression for $x(t)$ complete through the 100 Hz term. Make an insightful comment about the frequency of the dominant term in the response of the oscillator.

15.) The second example chosen was an anti-symmetric triangular wave. Noting that the function is odd about t = 0 and even about t = ${}^{T}/_{4}$, only the coefficients of the odd m sine terms are non-zero. Argue that these may be computed as

$b_m = \frac{8}{T}\int_0^{T/4} \sin\left[\omega_m t\right]\left(\frac{4t}{T}\right) dt$ for m odd. Compute the integrals to find b_m for m odd.

Collect everything and present the Fourier series for $f(t)$ defined just below.



$$f(t)=\begin{cases} 2-{}^{4t}\!/_{T} & for \quad {}^{T}\!/_{4}<t<{}^{T}\!/_{2} \\ {}^{4t}\!/_{T} & for \quad -{}^{T}\!/_{4}<t<{}^{T}\!/_{4} \\ -2-{}^{4t}\!/_{T} & for \quad -{}^{T}\!/_{2}<t<-{}^{T}\!/_{4} \end{cases}$$

16.) Find the Fourier series for $f(x)$, which has a period of 2π.

$$f(x)=\begin{cases} 0 & for \quad -\pi<x<0 \\ 1 & for \quad 0<x<{}^{\pi}\!/_{2} \\ 0 & for \quad {}^{\pi}\!/_{2}<x<\pi \end{cases}$$

After you compute the series, compare your result with the partial solution given below. Use the unit circle tool to evaluate trig functions for arguments that are a multiple of ½ π.

$$f(x)=\frac{1}{4}+\frac{1}{\pi}\left(\frac{\cos(x)}{1}-\frac{\cos(3x)}{3}+\frac{\cos(5x)}{5}-\ldots\right)+\frac{1}{\pi}\left(\frac{\sin(x)}{1}+\frac{\sin(3x)}{3}+\frac{\sin(5x)}{5}+\ldots\right)$$
$$+\frac{2}{\pi}\left(\frac{\sin(2x)}{2}+\frac{\sin(6x)}{6}+\frac{\sin(10x)}{10}+\ldots\right)$$

In the final form, the contents of each set of parentheses above should be replaced by an infinite sum with an index that assumes integer values. According to the rules for partial sums, the function might be better represented as:

$$f(x) = \frac{1}{4} + \frac{1}{\pi}\left(\frac{\cos(x)}{1} - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} - \ldots\right)$$
$$+ \frac{1}{\pi}\left(\frac{\sin(x)}{1} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \ldots\right) + \frac{2}{\pi}\left(\frac{\sin(2x)}{2} + \frac{\sin(6x)}{6} + \ldots\right)$$

What was the change? What requirement or standard is met by the second form that the first fails to meet by the change?

17.) To what value does the Fourier series found in problem 16 converge for $x = \pi$? Show that this leads to the result: $\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4}$ To what value does the series converge for $x = {}^{\pi}/_{2}$? Show that this leads to the same sum! What sum can be computed using Parseval's relation and the series?

18.) A function is said to be periodic with period T if $f(t) = f(t + T)$ for all t. For a period function $f(t)$ with period T, show that $f(-{}^{T}/_{2}) = f(+{}^{T}/_{2})$ and that $f(t) = f(t + 2\,T)$ for all t.

19.) Operations can be executed on the Fourier series for a function as well as on the function itself. Examples: differentiation and integration. Properly, the techniques are guaranteed valid only if the series converge uniformly. This 'requirement' is to be ignored. Start with the anti-symmetric triangular wave:

$$f(t) = \frac{8}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \sin\left[(2m+1)\frac{2\pi\, t}{T}\right]$$

Compute the series for the derivative of $f(t)$ by differentiating the series term by term. Sketch the derivative. The series represents a symmetric square wave with amplitude ${}^{4}/_{T}$. Compare the frequency content with that of the anti-symmetric square wave.

result: $$\frac{df}{dt}=\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)}\left(\frac{2\pi}{T}\right)\left(\cos\left[(2m+1)\frac{2\pi\,t}{T}\right]\right)$$

Factor out the amplitude. Compare this result directly with the anti-symmetric square wave shifted by T/4.

$$=\left(\frac{4}{T}\right)\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\left(\sin\left[(2m+1)\left(\frac{2\pi}{T}\right)\left(t+T/4\right)\right]\right)$$

$$=\left(\frac{4}{T}\right)\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\left(\sin\left[(2m+1)\left(\frac{2\pi\,t}{T}+\frac{\pi}{2}\right)\right]\right)$$

To finish use:

$$\sin\left[(2m+1)\left(\frac{2\pi\,t}{T}+\frac{\pi}{2}\right)\right]=$$
$$\cos\left[(2m+1)\frac{2\pi\,t}{T}\right]\sin\left[(2m+1)\frac{\pi}{2}\right]+\sin\left[(2m+1)\frac{2\pi\,t}{T}\right]\cos\left[(2m+1)\frac{\pi}{2}\right]$$

20.) Operations can be executed on the Fourier series for a function as well as on the function itself. Examples: differentiation and integration. Properly, the techniques are guaranteed valid only if the series converge uniformly. This 'requirement' is to be ignored. Start with the anti-symmetric triangular wave:

$$f(t)=\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\left[(2m+1)\frac{2\pi\,t}{T}\right]$$

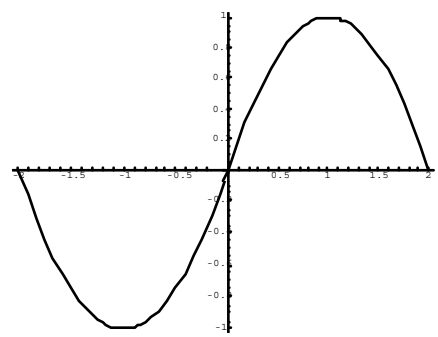
Next, *f(t)* is to be integrated. Sketch the integral of triangular wave *f(t)*.

$$g(t)=\int_0^t f(t')\,dt'=\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\int_0^t\sin\left[(2m+1)\frac{2\pi\,t'}{T}\right]dt'$$
$$=\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3}\left(T/2\pi\right)\left[1-\cos\left[(2m+1)\frac{2\pi\,t}{T}\right]\right]$$

Note that the coefficients of its Fourier series vanish as m^{-3} for large m. This dependence suggests that the function has a continuous first derivative. Does your sketch support this conclusion? Compare your result with problem 8 for the slope matched parabolas. Note that (see problem 10): $\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3} = \frac{\pi^3}{32}$. The series for the integrated function has a *non-zero* constant term. What does that tell you? How does the function for problem 8 differ?

21.) Consider the function *f(t)* that consists of two slope-matched parabolic sections. Consider expanding the function as defined over the interval $0 < t < 2$ as an even or odd function to complete the definition over the period $-2 < t < 2$. Would a sine of cosine series converge more rapidly? Assuming that the coefficients would decrease as m^{-n} for large m predict the value of n for a sine series expansion and for a cosine series expansion.

$$f(t) = \begin{cases} 1-(t-1)^2 = 2t - t^2 & for \quad 0 < t < 2 \\ -1+(t+1)^2 = 2t + t^2 & for \quad -2 < t < 0 \end{cases}$$



22. The Euler identity provides representations of the sine and cosine in terms of complex exponentials.

$$\sin(mx) = \frac{e^{imx} - e^{-imx}}{2i} \quad \text{and} \quad \cos(px) = \frac{e^{iox} + e^{-ipx}}{2}$$

Use these relations to transform the orthogonality integral for the sines and cosines to integrals of complex exponentials to establish:

$$\frac{1}{T}\int_{-T/2}^{T/2}\sin\left[\omega_p t\right]\cos\left[\omega_m t\right]dt = 0;\ \frac{1}{T}\int_{-T/2}^{T/2}\sin\left[\omega_p t\right]\sin\left[\omega_m t\right]dt = \tfrac{1}{2}\delta_{pm};\ \frac{1}{T}\int_{-T/2}^{T/2}\cos\left[\omega_p t\right]\cos\left[\omega_m t\right]dt = \tfrac{1}{2}\delta_{pm}$$

It is easiest to treat the $p \neq m$ cases and the $p=m$ cases separately. If you feel that you have suffered to complete this problem, you have earned the right to use the orthogonality relations freely in the future. ***Hints***: After substituting the complex exponentials, one needs calculate integrals of the form $\frac{1}{T}\int_{-T/2}^{T/2} e^{i\ell\omega_0 t}\,dt$ where $\omega_0 = 2\pi/T$.

Compute the integral for ℓ a non-zero integer. Do not forget the representation of sine in terms of the complex exponentials as you do this. Also, read the Tools of the Trade section of this handout. Give expression for ℓ in terms of m and n. Suppose that ℓ is zero; what is true about m and n? Set ℓ to zero and evaluate the integrand and then integrate.

23.) The Euler identity provides representations of the sine and cosine in terms of complex exponentials.

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Use these to establish the identity: $\cos(\beta - \alpha) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

Give the corresponding identity for $\cos(\beta+\alpha)$. Solve the identities for $\cos(\beta-\alpha)$ and $\cos(\beta+\alpha)$ to find expressions for $\cos(\alpha)\cos(\beta)$ and $\sin(\alpha)\sin(\beta)$ as sums of terms linear in trigonometric functions.

24.) Find the cosine series that represents $f(x) = \pi x - x^2$ over the interval $[0,\pi]$.

a.) As the function is expanded in a cosine series, its full period is 2π, and its extension across zero must be even. Sketch $f(x)$ for $-2\pi < x < 2\pi$.

b.) At what order derivative does the function have its first discontinuity? The coefficients are therefore expected to vanish no faster than what inverse power of the coefficient index?

c.) Give the equations used to compute the expansion coefficients c_0 and a_k for a general k. Be sure that you adjusted the standard Fourier series relations to account for the range of integration and the restriction to a cosine series. Compute them.

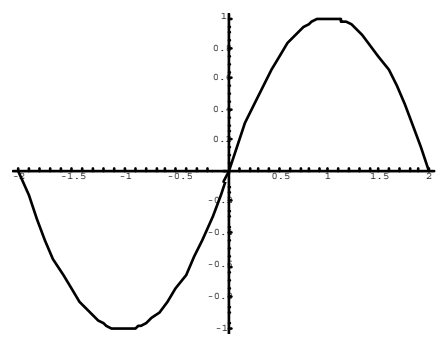
d.) Show that $f(x) = \pi^2/_4 - (x - {}^{\pi}/_2)^2$.

d.) Note that $f(x)$ is even about ${}^{\pi}/_2$. What does this tell you about a_k for k odd?

?Answer: $f(x) = \pi^2/_2 - \cos(2x) - {}^1/_4 \cos(4x) - {}^1/_9 \cos(6x) - {}^1/_{16} \cos(8x) - \ldots$

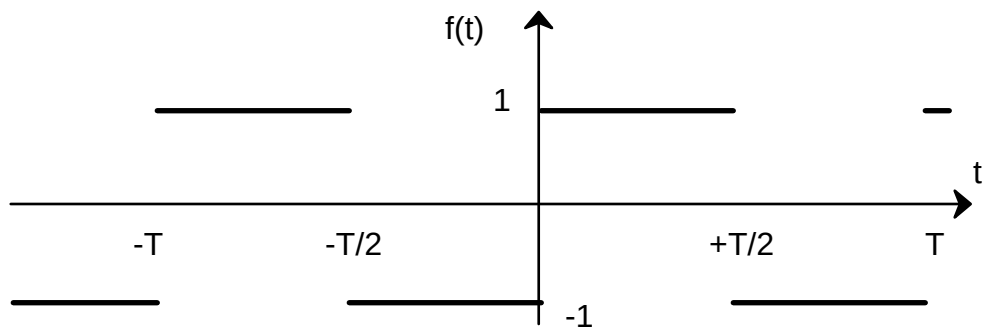
25.) Slope-matched parabolic sections. Consider the function of period 4 defined over the interval [-2,2] by the equations:

$$f(t) = \begin{cases} 1-(t-1)^2 = 2t - t^2 & for \quad 0 < t < 2 \\ -1+(t+1)^2 = 2t + t^2 & for \quad -2 < t < 0 \end{cases}$$



It has a Fourier expansion $f(t) = \sum_{m=0}^{\infty} \frac{32}{\pi^3 \, (2m+1)^3} \sin\left[(2m+1)\left(\frac{\pi}{2}\right)t\right]$. Consider the case $t = 1$. To what value should the series converge. Use $\sin\left[(2m+1)\left(\frac{\pi}{2}\right)\right] = (-1)^m$ and this result to find the sum $\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3}$.

26.) Start with the Fourier series for the anti-symmetric square wave.



The square wave *f(t)* has a period of *T* and is extended beyond the base period to display the discontinuities at each multiple of $^{T}/_{2}$. The function is represented as:

$$f(t) = \begin{cases} +1 & for \quad 0 < t < T/2 \\ -1 & for \quad -T/2 < t < 0 \end{cases} \qquad \text{[FS.30]}$$

FOURIER SERIES FOR THE ANTI-SYMMETRIC SQUARE WAVE:

$$b_p = \begin{cases} \left(4/\pi\, p \right) & for\ p\ odd \\ 0 & for\ p\ even \end{cases} \quad \text{or} \quad f(t) = \frac{4}{\pi} \sum_{m=0}^{\infty} \left(\frac{1}{2m+1} \right) \sin\left[(2m+1) \frac{2\pi\, t}{T} \right]$$

a.) Sketch $\int_{-T/4}^{t} f(t)\,dt$. It should be a triangular wave ranging between $-^{T}/_{4}$ and $^{T}/_{4}$.

b.) Compute $\int_{-T/4}^{t} \frac{4}{\pi} \sum_{m=0}^{\infty} \left(\frac{1}{2m+1} \right) \sin\left[(2m+1) \frac{2\pi\, t}{T} \right] dt$.

c.) Multiply your result by $-^{4}/_{T}$. Use part a.) to sketch the result.

d.) Shift the result. Replace *t* by $t - {}^{T}/_{4}$. Sketch the outcome expected.

******* change the wording ! ********

e.) Give the expressions for $\cos(\theta - m\, {}^{\pi}/_{2})$ for $m = 4\,n$, $4\,n + 1$, $4\,n + 2$ and $4\,n + 3$ where *n* is an integer.

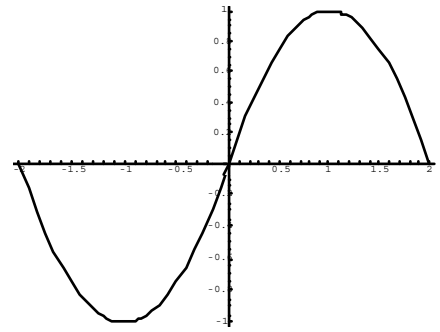
f.) Express the result of part d.) using sines rather than cosines.

g.) Compare the result to the Fourier series for the unit amplitude anti-symmetric triangular wave.

27.) Slope-matched parabolic sections. Consider the function of period 4 defined

over the interval [-2,2] by the equations:

$$f(t)=\begin{cases} 1-(t-1)^2=2t-t^2 & for \quad 0<t<2 \\ -1+(t+1)^2=2t+t^2 & for \quad -2<t<0 \end{cases}$$



It has a Fourier expansion $f(t)=\sum_{m=0}^{\infty}\frac{32}{\pi^3\,(2m+1)^3}\sin\left[(2m+1)\left(\frac{\pi}{2}\right)t\right]$. Use Parseval's relation to compute the sum $\sum_{m=0}^{\infty}\frac{1}{(2m+1)^6}$. Use the extended Parseval's relation (Equation [FS.17]) and Fourier series calculated in this handout to compute the sums $\sum_{m=0}^{\infty}\frac{1}{(2m+1)^4}$ and $\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^5}$. Before you use a pre-calculated series for a particular function, *modify* the function to make explicit that its period has been set to $T = 4$ to match the period of the slope-matched parabolas. Parseval's equality is for functions with the same period; that period, of course, can be T for both.
Answers: $\pi^6/960$; $\pi^4/96$; $5\pi^5/1536$.

28.) Develop the Fourier series for: $f(x)=x^2 \quad (-\pi<x\le\pi)$ assuming that the function is extended periodically with period 2π. Use the result to compute $\sum_{m=1}^{\infty}{(-1)^{m+1}}/{m^2}$. *Answer*: $x^2=\pi^2/3+4\sum_{m=1}^{\infty}(-1)^m\frac{\cos(mx)}{m^2} \quad (-\pi<x\le\pi)$; $\pi^2/12$

29.) Consider the Reimann zeta $\zeta(n)=\sum_{p=1}^{\infty}\frac{1}{p^n}$ and the sum $E(n)=\sum_{p=1}^{\infty}\frac{1}{(2p)^n}$. Show that $\zeta(n)-E(n)=\left(1-\frac{1}{2^n}\right)\zeta(n)$. What is $\zeta(n) - 2\,E(n)$? Use the result of the problem above to compute $\zeta(2)$. *Answer*: $\pi^2/6$

30.) Consider the function: $f(x) = x(\pi - x)$ for the interval $[0, \pi]$. Expand it in a cosine series and in a sine series. Check the extended functions for discontinuities and predict the fall-off in the magnitudes of the coefficients in the large index limit.

31.) Use the previous two problems to show that:

$$\sum_{n=1}^{\infty} \frac{1}{(n)^2} = \frac{\pi^2}{6} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n)^2} = \frac{\pi^2}{12} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32}$$

32.) Use the previous three problems to show that:

$$\sum_{n=1}^{\infty} \frac{1}{(n)^4} = \frac{\pi^4}{90} \qquad \sum_{n=1}^{\infty} \frac{1}{(n)^6} = \frac{\pi^6}{945}$$

Take the time to repeat the derivation of Parseval equalities to ensure that you use the correct normalization factors.

33.) Explain why one cannot use a combination of the sine and cosine series for $f(x) = x(\pi - x)$ found above to examine sum of terms like n^{-5}.

34.) **Blackbody Radiation**: The following integral rises in the study of blackbody radiation.

$$\int_0^{\infty} \frac{q^3\, dq}{e^q - 1} = \int_0^{\infty} q^3\, dq\, e^{-q}\left[1 - e^{-q}\right]^{-1} =$$

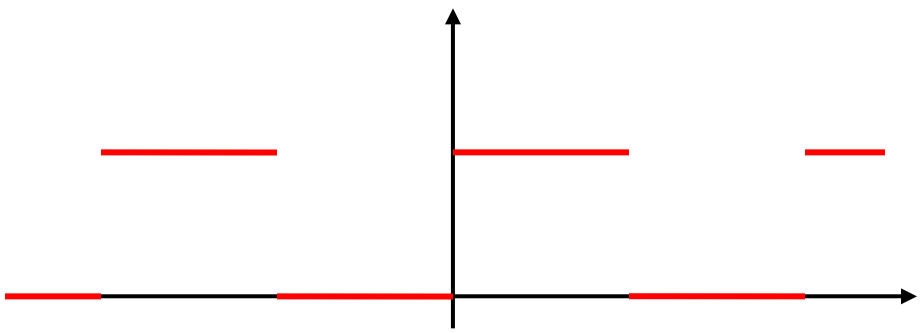
$$= \int_0^{\infty} q^3\, dq\, e^{-q}\left[1 + e^{-q} + e^{-2q} + e^{-3q} + \ldots\right] = \int_0^{\infty} q^3\, dq \sum_{n=1}^{\infty} e^{-nq}$$

Show that this integral is equal to $6!\sum_{n=1}^{\infty} n^{-4}$. This sum can be evaluated using Parseval's equality. Choose the Fourier cosine series for $x(x - \pi)$ for $0 < x < \pi$.

$$x(x-\pi) = \pi^2/6 - \sum_{m=1}^{\infty} \frac{\cos(2mx)}{m^2}$$

Show that $\int_0^{\pi} |x(x-\pi)|^2\, dx = \frac{\pi^4}{36} - (½)\sum_{m=1}^{\infty} m^{-4} \;\Rightarrow\; \sum_{m=1}^{\infty} m^{-4} = \frac{\pi^4}{90}$ leading to $\int_0^{\infty} \frac{q^3\, dq}{e^q - 1} = \frac{\pi^4}{15}$.

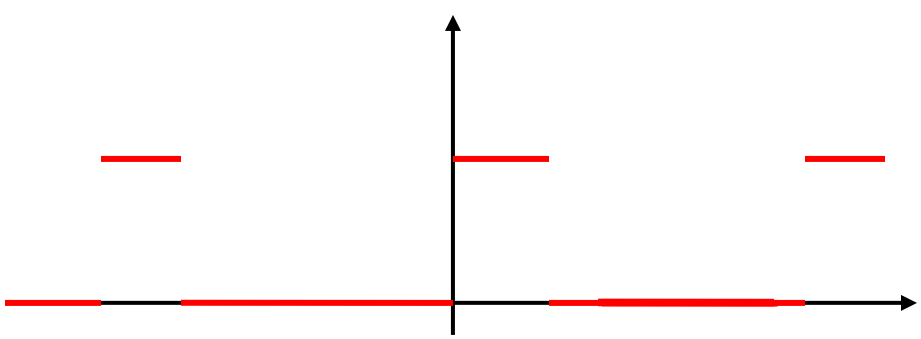
35.) Square waves come in several forms. Consider an odd square wave with period *T* stepping from 0 to +1. It is a square wave with a 50% duty cycle.



The square wave *f(t)* has a period of *T* and is extended beyond the base period to display the discontinuities at each multiple of $^{T}/_{2}$. The function is represented as:

$$f(t) = \begin{cases} +1 & for \quad 0 < t < T/2 \\ 0 & for \quad -T/2 < t < 0 \end{cases}$$

Next. consider an odd square wave with period *T* stepping from 0 to +1. It is a square wave with a 25% duty cycle.



The square wave *g(t)* has a period of *T* and is extended beyond the base period to display the discontinuities. The function is represented as:

$$g(t)=\begin{cases} +1 & for \quad 0<t<T/4 \\ 0 & for \quad -3T/4<t<0 \end{cases}$$

Compare the fundamental frequency for the expansion of $f(t)$ and the allowed spectrum of frequencies in the Fourier expansion of $f(t)$ with those for the expansion of $g(t)$.

36.) Consider the set of all complex-valued functions that are periodic with period T and that have a finite square integral. $\int_{-T/2}^{T/2} |f(t)|^2\, dt$ finite. The sum of two functions is:

$$\text{Sum}[f(t), g(t)] = f(t) + g(t)$$

The scalar multiple of a function is:

$$\text{ScalarMult}[d, f(t)] = d\, f(t)$$

For complex valued functions, complex numbers are the appropriate scalar field. (If the set is restricted to real-valued functions, then the real numbers should be chosen.) Show that the set defined with the prescribed operations is a vector space.

37.) A definition for the inner product of two functions that are periodic with period T and finite square integral has been proposed.

$$\langle g | f \rangle = \left(1/T\right) \int_{-T/2}^{T/2} \left(g(t)\right)^* f(t) dt$$

a.) What type of value does this operation return?

b.) Verify that the operation satisfies the axioms required of an inner product for the set of functions prescribed in the previous problem.

38.) Consider the set of all complex-valued functions that can be formed as linear combinations of the functions { 1, { .. , cos[$m\omega_o t$], … }, { … , sin[$n\omega_o t$], … }}

where $\omega_o = {}^{2\pi}/_{T}$ and $\omega_m = m\omega_o$ subject the restriction * below is a vector space with the operations on the representative members $g(t)$ and $f(t)$:

$$f(t) \;=\; c_{f0}(1) \;+\; \sum_{m=1}^{\infty} a_{fm}\cos[\omega_m t] \;+\; \sum_{m=1}^{\infty} b_{fm}\sin[\omega_m t]$$

$$g(t) \;=\; c_{g0}(1) \;+\; \sum_{m=1}^{\infty} a_{gm}\cos[\omega_m t] \;+\; \sum_{m=1}^{\infty} b_{gm}\sin[\omega_m t]$$

$$\text{Sum}[f(t), g(t)] = \left(c_{f0}+c_{g0}\right)(1) \;+\; \sum_{m=1}^{\infty}\left(a_{fm}+a_{gm}\right)\cos[\omega_m t] \;+\; \sum_{n=1}^{\infty}\left(b_{fn}+b_{gn}\right)\sin[\omega_m t]$$

The scalar multiple of a function is:

$$\text{ScalarMult}[d, f(t)] = d\,c_{f0}(1) \;+\; \sum_{m=1}^{\infty}\left(d\,a_{fm}\right)\cos[\omega_m t] \;+\; \sum_{n=1}^{\infty}\left(d\,b_{fn}\right)\sin[\omega_n t]$$

Retriction *: $\left|c_{f0}{}^{2}\right| \;+\; \tfrac{1}{2}\sum_{m=1}^{\infty}\left|a_{fm}\right|^2 \;+\; \tfrac{1}{2}\sum_{m=1}^{\infty}\left|b_{fm}\right|^2$ is finite.

Show that the set defined with the prescribed operations is a vector space. The scalars can be the real numbers or the complex numbers.

39.) A definition for the inner product of the vectors in the previous problem has been proposed.

$$\langle g(t)|f(t)\rangle \;=\; c_{g0}^{*}c_{f0}(1) \;+\; \tfrac{1}{2}\sum_{m=1}^{\infty} a_{gm}^{*}a_{fm} \;+\; \tfrac{1}{2}\sum_{n=1}^{\infty} b_{gn}^{*}b_{fn}\ \text{s}$$

a.) What type of value does this operation return?

b.) Verify that the operation satisfies the axioms required of an inner product for the set of functions prescribed in the previous problem.

40.) One period of a function is given below along with its Fourier series.

$$f(t) = \begin{cases} 0 & for \quad -1 < t < -\tfrac{1}{2} \\ 1 & for \quad -\tfrac{1}{2} < t < \tfrac{1}{2} \\ 0 & for \quad \tfrac{1}{2} < t < 1 \end{cases}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\cos(\pi t)}{1} - \frac{\cos(3\pi t)}{3} + \frac{\cos(5\pi t)}{5} + \ldots\right)$$

a.) What is the period of the function?

b.) What are the allowed frequencies ω_m in the expansion of a **function with this period**?

c.) Sketch the function for $-2 < t < +2$.

d.) To what value does the Fourier series converge at t = - $^1/_2$?

e.) To what value does the Fourier series converge at t = +1?

f.) The Fourier coefficient a_m for this function is computed as:

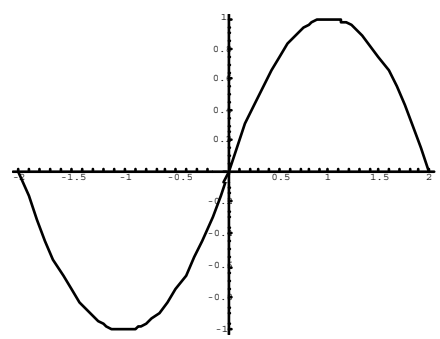
$$a_p = \frac{2}{T}\int_{-T/2}^{T/2} \cos\left[\omega_p t\right] f(t)\, dt \quad = \int_{-1/2}^{+1/2} \cos\left[\omega_p t\right][1]\, dt$$

Start with $\int_{-1/2}^{+1/2} \cos\left[\omega_p t\right][1]\, dt$ and show the steps to find a_p for all $p>0$ and hence generate (and display) the series for $f(t)$ written in a summation form.

h.) What property of the function $f(t)$ is represented by the constant $c_o = ½$?

41.) Slope-matched parabolic sections and beyond. Consider the function of period 4 defined over the interval [-2,2] by the equations:

$$f(t) = \begin{cases} t^1 (2-t)^1 & for \quad 0 < t < 2 \\ t^1 (2+t)^1 & for \quad -2 < t < 0 \end{cases}$$

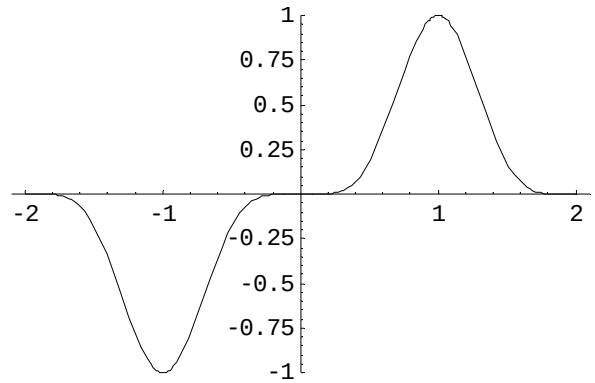


This function and its first derivative are continuous. However, $d^2 f / dt^2$ is discontinuous. It has maximum and minimum values of +1 and -1.

Consider the function:

$$g_n(t) = \begin{cases} t^n (2-t)^n & for \quad 0 < t < 2 \\ (-1)^{n+1} t^n (2+t)^n & for \quad -2 < t < 0 \end{cases}$$

Figure for $n = 6$.



a.) Show that this function and its first n derivatives are continuous and that $d^{n+1}g_n / dt^{n+1}$ is discontinuous. Confirm that $g_n(t)$ has maximum and minimum values of +1 and -1.

b.) It is expected that the Fourier expansion coefficients will vanish no faster than $1/m^N$ for large m. What is N?

c.) In the standard Fourier expansion series, which coefficients are sure to vanish? Explain.

d.) Compute the Fourier series for $g_2(t)$.

$$b_{2m+1} \to (4)\frac{2}{T}\int_0^{T/4} \sin\left[\omega_{2m+1} t\right] f(t)\, dt \to 2\int_0^1 \sin\left[(2m+1)(\pi/2)t\right]\left[t^n (2-t)^n\right] dt$$

ANSWER: $b_p = 32/(p\,\pi)^3$ **for *p* odd; = 0 for *p* even.**

```
termsFS = 21; FourierTrigSeries[
 - x^2 (1/2 + x)^2 + UnitStep[x] (x^2 (1/2 - x)^2 + x^2 (1/2 + x)^2), x, termsFS]
```

$$-\frac{(-12+\pi^2)\,\mathrm{Sin}[2\pi x]}{2\pi^5} + \frac{(4-3\pi^2)\,\mathrm{Sin}[6\pi x]}{162\pi^5} + \frac{(12-25\pi^2)\,\mathrm{Sin}[10\pi x]}{6250\pi^5} + \frac{(12-49\pi^2)\,\mathrm{Sin}[14\pi x]}{33614\pi^5} + \frac{(4-27\pi^2)\,\mathrm{Sin}[18\pi x]}{39366\pi^5} + \frac{(12-121\pi^2)\,\mathrm{Sin}[22\pi x]}{322102\pi^5} + \frac{(12-169\pi^2)\,\mathrm{Sin}[26\pi x]}{742586\pi^5} + \frac{(4-75\pi^2)\,\mathrm{Sin}[30\pi x]}{506250\pi^5} + \frac{(12-289\pi^2)\,\mathrm{Sin}[34\pi x]}{2839714\pi^5} + \frac{(12-361\pi^2)\,\mathrm{Sin}[38\pi x]}{4952198\pi^5} + \frac{(4-147\pi^2)\,\mathrm{Sin}[42\pi x]}{2722734\pi^5}$$

The coefficients for the terms beginning with 12 in the numerator are of the form:

$\frac{(12-m^2\pi^2)}{2m^5}$. Note that these terms approach zero as m^{-3} for large m. This is consistent with the statement that the coefficients vanish ***no faster than*** $m^{-[n+1]}$ where the nth derivative has the first one that is discontinuous.

42.) A standard trigonometric Fourier series for a function $f(x)$ with period L has the form: $f(x) \;=\; c_0\,(1) \;+\; \sum_{m=1}^{\infty} a_m \cos[mk_0x] \;+\; \sum_{m=1}^{\infty} b_m \sin[mk_0x]$ where $k_o = {}^{2\pi}/_{L}$.

a.) Show that this can be cast in the form:

$$f(x) \;=\; c_0\, e^{i(0)} \;+\; \sum_{m=1}^{\infty} \tfrac{1}{2}(a_m - ib_m)\, e^{imk_0x} + \tfrac{1}{2}(a_m + ib_m)\, e^{-imk_0x} = \sum_{m=-\infty}^{\infty} \alpha_m\, e^{imk_0x}$$

This result justifies the form of the complex Fourier series, $\sum_{m=-\infty}^{\infty} \alpha_m\, e^{imk_0x}$.

b.) Show that $(L^{-1})\int_{-L/2}^{L/2} (e^{imk_0x})^*\, e^{ink_0x}\, dx = \delta_{mn}$.

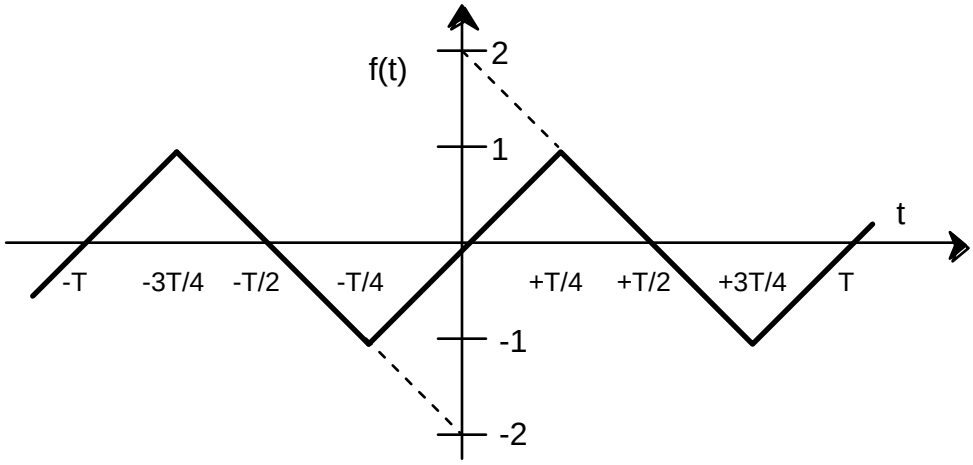
c.) Pre-multiply $f(x) = \sum_{m=-\infty}^{\infty} \alpha_m\, e^{imk_0x}$ by $(e^{ipk_0x})^*$ and use the orthogonality relation developed in part b to project out the coefficient α_p.

43.)
$$\sin(x)\sin(y) = \tfrac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$
$$\cos(x)\cos(y) = \tfrac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$
$$\sin(x)\cos(y) = \tfrac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$$

Give the Maclaurin series for $\sin(x)$ and $\cos(x)$ correct to order x^2. Use these expansions to verify the three trig identities for products of sines and cosines to second order for small arguments.

44.) Show that the series for the anti-symmetric triangle wave converges uniformly to the triangle wave by showing that the choice of $N > {}^{1}/_{4\varepsilon}$ is adequate to ensure

that $|f(t) - s_N(t)| < \varepsilon$ for all $t \in [-\frac{1}{2}T, \frac{1}{2}T]$. Assume that $|f(t) - s_\infty(t)| \equiv 0$ for all $t \in [-\frac{1}{2}T, \frac{1}{2}T]$.



base period: $-{}^{T}//_{2} < t < {}^{T}//_{2}$

$$f(t) = \begin{cases} 2 - {}^{4t}/_{T} & for \quad {}^{T}/_{4} < t < {}^{T}/_{2} \\ {}^{4t}/_{T} & for \quad -{}^{T}/_{4} < t < {}^{T}/_{4} \\ -2 - {}^{4t}/_{T} & for \quad -{}^{T}/_{2} < t < -{}^{T}/_{4} \end{cases}$$

$$f(t) = \frac{8}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \sin\left[(2m+1)\frac{2\pi t}{T}\right]$$

45.) Consider the discontinuous sawtooth $f(t) = \frac{1}{2}t$ in the base period $[-\pi, \pi]$.

Show that the series representation is: $f(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nt)$. What series sum can be computed by examining $f(\frac{1}{2}\pi)$? Identify the series and compute the sum that follows from the Parseval relation for this series.

Note that $\int t \sin(mt)\, dt = -\frac{\partial}{\partial m} \int \cos(mt)\, dt$.

46.) Look at the previous problem. Define $g(t) = \int_0^t f(t')\,dt'$ over the same base period. Compute the series directly and by integrating the series from the previous problem term by term.

Lessons Learned

Problem 1. Orthgonality relations are extremely important. The relations are so important that one should sweat blood for hours proving them. The expressions for the sine and cosine in terms of complex exponentials are useful. Learn them; love them.

Problem 2. **A Parseval identity is a representation of an inner product as the sum of the products of the corresponding expansion coefficients of the vectors.** The representation of the inner product is not unique. Inner products return a scalar value that gauges the degree to which two entities share common behavior. For vectors we have two procedures to gauge the

$$\vec{A}\cdot\vec{B} = A\,B\cos\theta = A_x B_x + A_y B_y + A_z B_z$$

degree to which the two are directed in the same direction. One can have two sets of basis behaviors that completely characterize the members of the set under study. Consider a 3D Cartesian coordinate system (unprimed) and a second system (primed) with axes that are rotated relative to those of the first system. Each vector has a representation in each system.

$$\vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k} = A_x^{/}\hat{i}^{/} + A_y^{/}\,\hat{j}^{/} + A_z^{/}\,\hat{k}^{/}$$
$$\vec{B} = B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k} = B_x^{/}\hat{i}^{/} + B_y^{/}\,\hat{j}^{/} + B_z^{/}\,\hat{k}^{/}$$

The inner product can be computed in either representation

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z = A_x^{/} B_x^{/} + A_y^{/} B_y^{/} + A_z^{/} B_z^{/}$$

Either set of basis behaviors, $\left\{\hat{i},\hat{j},\hat{k}\right\}$ or $\left\{\hat{i}^{/},\hat{j}^{/},\hat{k}^{/}\right\}$ is adequate to describe all the 3D vectors. Such sets are said to be *complete*. Problem two reveals that there are two

equally valid methods to compute the inner product of two functions that are periodic with period T. And, in particular, the set { 1, …, cos($m\ \omega_0\ t$), …. , …, sin($n\ \omega_0\ t$), ….} is a complete set to characterize those same functions give that $\omega_0 = {}^{2}\pi/_{T}$ and that m and n are the counting integers 1, 2, 3, … .

Problem 3. If the basis is chosen to be orthogonal and complete, every entity in the set of interest can be represented as a sum of scalar multiples of members of that basis. To find the scalar multiple for a particular basis member, compute the inner product of the member with the entity being represented. To find the x component of a vector $\vec{A}$, one computes the inner product of the *x character* basis member $\hat{i}$ with the vector $\vec{A}$ and normalizes by dividing by the inner product of the basis member with itself. $A_x = \dfrac{\hat{i}\cdot\vec{A}}{\hat{i}\cdot\hat{i}} = \dfrac{\langle \hat{i} | \vec{A} \rangle}{\langle \hat{i} | \hat{i} \rangle}$. Verify that

$$a_{f\,p} = \frac{2}{T}\int_{-T/2}^{T/2} \cos\left[\omega_p t\right] f(t)\,dt = \frac{\dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} \cos\left[\omega_p t\right] f(t)\,dt}{\dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} \cos\left[\omega_p t\right]\cos\left[\omega_p t\right] dt} = \frac{\left\langle \cos\left[\omega_p t\right] \middle| f(t) \right\rangle}{\left\langle \cos\left[\omega_p t\right] \middle| \cos\left[\omega_p t\right] \right\rangle}$$

Two expansions in terms of the same set of orthogonal behaviors can only be equal if the coefficients are equal behavior by behavior. $\vec{A} = \vec{B} \Rightarrow A_x = B_x, \ldots.$

Problem 4. The Fourier series expansion is a linear operation. The various Fourier coefficients are added and multiplied coefficient by coefficient just as vectors are added component by component. $h(t) = A\,f(t) + B\,g(t)$ leads to:

$a_{hp} = A\ a_{fp} + B\ a_{gp}$. just as $\vec{C} = a\,\vec{A} + b\,\vec{B}$ leads to $C_x = a\ A_x + b\ B_x$.

Even though the result seems obvious, one should anchor the result by computing the coefficients for $h(t)$ in the canonical fashion.

$$c_{h0} = \frac{1}{T}\int_{-T/2}^{T/2} h(t)\,dt \quad ; \quad a_{hp} = \frac{2}{T}\int_{-T/2}^{T/2} \cos\left[\omega_p t\right] h(t)\,dt \quad ; \quad b_{hp} = \frac{2}{T}\int_{-T/2}^{T/2} \sin\left[\omega_p t\right] h(t)\,dt$$

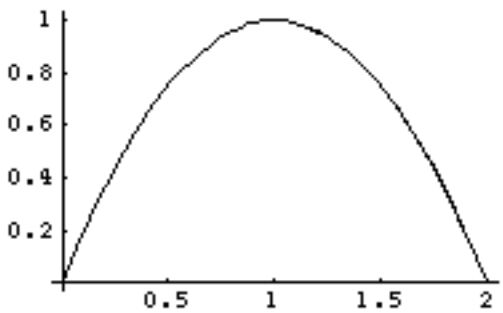
Hence,

$$a_{hp} = \frac{2}{T}\int_{-T/2}^{T/2} \cos\left[\omega_p t\right]\left[A\,f(t) + B\,g(t)\right] dt$$

$$= A\left\{\frac{2}{T}\int_{-T/2}^{T/2} \cos\left[\omega_p t\right] f(t)\, dt\right\} + B\left\{\frac{2}{T}\int_{-T/2}^{T/2} \cos\left[\omega_p t\right] g(t)\, dt\right\} = A\,a_{fp} + B\,a_{gp}$$

Problem Discussion

Problem 8 with matched slope parabolas:



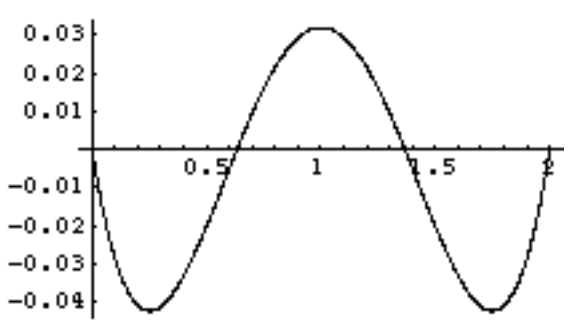
Plot of the function

Partial Sum S_n = (32/(Pi^3)) Sum[Sin[*m* Pi t/2]/(*m*^3),{*m*,1,*n*,2}]

The sum index begins with $m = 1$ and sums terms through index n incrementing by 2 to include only the odd terms.

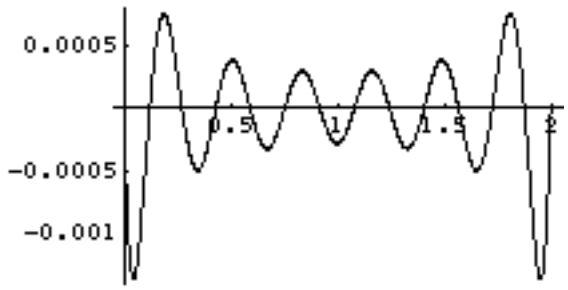
NOTE: As 2 t - t^2 is subtracted from the partial sum of the Fourier series, the **error term** is plotted.

```
Plot[(32/(Pi^3)) Sum[ Sin[ m Pi t/2]/(m^3),{m,1,1,2}] -( 2 t - t^2), {t,0,2},
  PlotRange -> All, PlotPoints -> 1024]
```

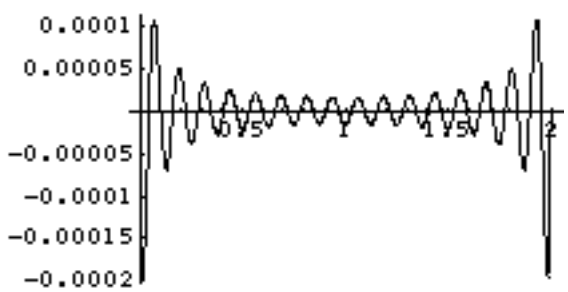


Plot of the ERROR with $m_{max} = 1$

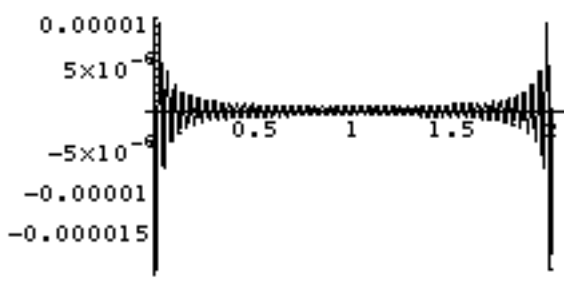
```
Plot[(32/(Pi^3)) Sum[ Sin[ m Pi t/2]/(m^3),{m,1,11,2}] -( 2 t - t^2), {t,0,2},
  PlotRange -> All, PlotPoints -> 1024]
```



Plot of the ERROR with $m_{max} = 11$



Plot of the ERROR with $m_{max} = 31$

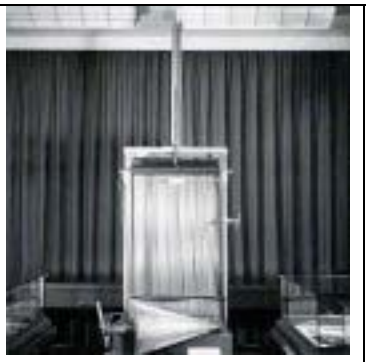


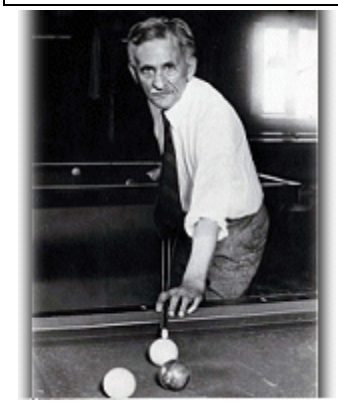
Plot of the ERROR with $m_{max} = 101$

What behavior of the function occurs at $t = 0$ and at $t = 2$ that causes the convergence to be slower in the regions of those points.

Gibbs overshoot phenomenon and consistent partial sums:

The Gibbs phenomenon was discovered by Albert Michelson[1] while he was summing a Fourier series using coefficients for a periodic function that he measured using an optical-mechanical system. Noting the overshoot, he brought it to the attention of J. Willard Gibbs who computed its magnitude. The overshoot phenomenon demonstrates the importance of completing partial sums consistently to an upper frequency.

Albert Michelson (pre-discovery photo)	The Gibbs phenomenon was discovered by Albert Michelson while he was summing a Fourier series using coefficients for a periodic function that he measured using an optical-mechanical system. Michelson was a U S Naval Academy midshipman and instructor. One of his speed-of -light measurements was made at the academy.	 Michelson-Stratton harmonic analyzer



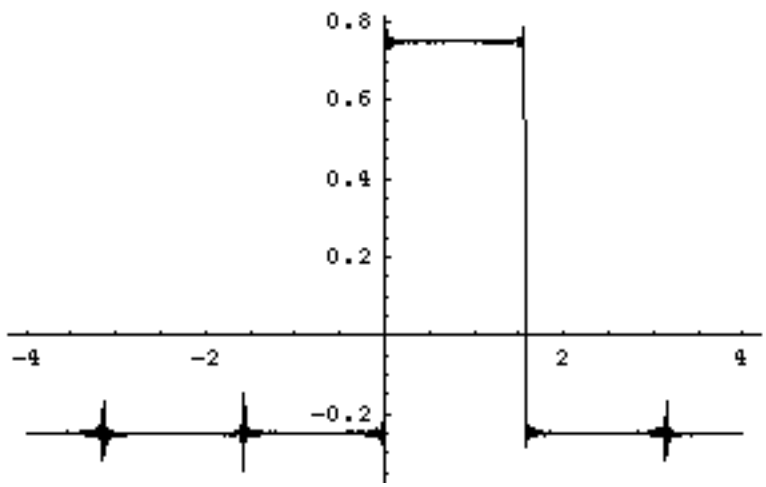
Consider the Fourier series for the function *f(x)* which has period 2π.

$$f(x) = \begin{cases} -\frac{1}{4} & for \quad -\pi < x < 0 \\ \frac{3}{4} & for \quad 0 < x < \pi/2 \\ -\frac{1}{4} & for \quad \pi/2 < x < \pi \end{cases}$$

[1] McQuarrie, page 735

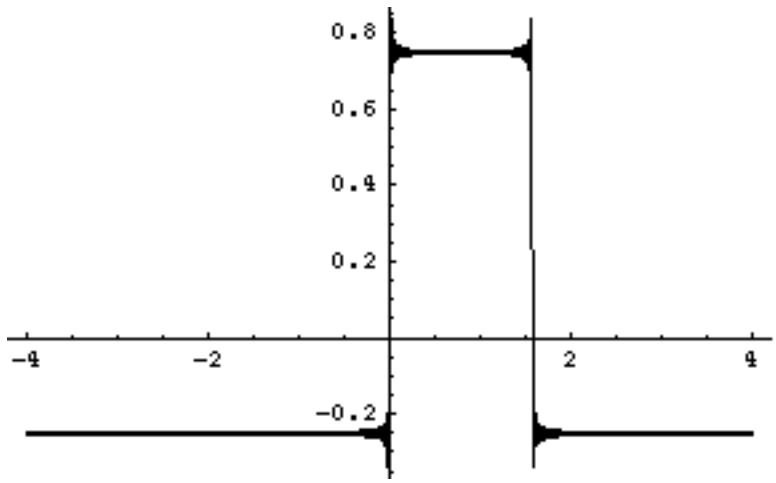
Examine an ***incorrect*** partial sum. Note the first two sum are through a frequency 201 while the last term in the sum reaches a frequency of 402. The result is the ***improper*** appearance of Gibbs (overshoot) ringing at points that alias the positions of the discontinuities.

```
Plot[(1/Pi) *
    Sum[  ( Sin[( 2 m + 1) x]/( 2 m + 1)+
     (-1)^m  Cos[( 2 m + 1) x]/( 2 m + 1)
    +2 * Sin[(4 m + 2) x]/(4 m + 2)), {m,0,100}],
    {x,-4,4}]
```

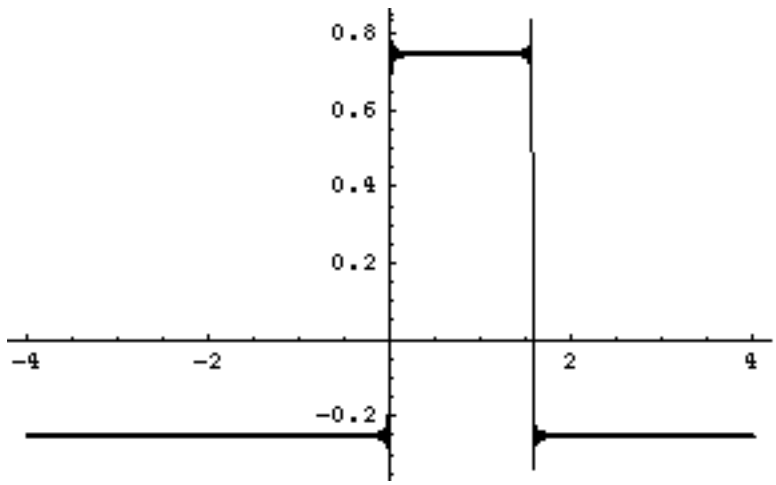


A proper partial sum contains all terms of frequency up to the maximum and none others. The sum below contains all term with frequencies up to 202 and none higher. Note that the series now appears to converge uniformly in regions in which the function is continuous. The *ringing behaviors* at $\pm\pi$ and at $-\pi/2$ are ***artifacts*** of an improper partial sum.

```
Plot[(1/Pi) *
 (Sum[ ( Sin[( 2 m + 1) x]/( 2 m + 1)+ (-1)^m  Cos[( 2 m + 1) x]/( 2 m + 1)
    ), {m,0,100}] +  Sum[2 * Sin[(4 m + 2) x]/(4 m + 2), {m,0,50}]),
    {x,-4,4}, PlotPoints -> 512, PlotRange -> All]
```



The sum is repeated keeping all terms up to ω = 402.



The Gibbs phenomenon occurs only at the discontinuities and, as the frequency limits was doubled, the domain over which it occurs is halved. ***Correct partial sums include all terms with frequency up to the maximum chosen and none higher.***

Linear Transformations and the Various Fourier Series:

Read only if you have studied Vector Spaces and Linear Transformations.

Consider the relations between: $f(t) = c_0\,(1) + \sum_{m=1}^{\infty} a_m \cos[m\omega_0 t] \;+\; \sum_{m=1}^{\infty} b_m \sin[m\omega_0 t]$

where $\omega_0 = 2\pi/\mathrm{T}$ and: $f(t) \;=\; \sum_{m=-\infty}^{\infty} d_m \exp\left[i\, m \left(2\pi/T\right) t \right]$. These represent the vector $f(t)$, *an element in a function space*, using two choices of orthogonal basis set vectors (*functions*). Following the conventions discovered for partial sums, all expansions are to be order in increasing frequency order.

Traditional Basis Set:

$$\{\ 1, \cos(\omega_o t), \sin(\omega_o t), \cos(2\omega_o t), \sin(2\omega_o t), \cos(3\omega_o t), \sin(3\omega_o t), \ldots\ \} =$$

$$\{T_1(t), \quad T_2(t), \quad T_3(t), \quad T_4(t), \quad T_5(t), \quad T_6(t), \quad T_7(t), \ldots\ \}$$

Complex Basis Set:

$$\{\ 1, e^{i\omega_o t}, e^{-i\omega_o t}, e^{i2\omega_o t}, e^{-i2\omega_o t}, , e^{i3\omega_o t}, e^{-i3\omega_o t}, , e^{i4\omega_o t}, e^{-i4\omega_o t}, \ldots\ \} =$$

$$\{C_1(t), \quad C_2(t), \quad C_3(t), \quad C_4(t), \quad C_5(t), \quad C_6(t), \quad C_7(t), \ldots\ \}$$

The function $f(t)$ is represented by column vectors in each scheme.

$$f(t) \rightarrow \begin{bmatrix} c_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \\ \vdots \end{bmatrix}_{TRADITIONAL} \qquad\qquad f(t) \rightarrow \begin{bmatrix} d_0 \\ d_1 \\ d_{-1} \\ d_2 \\ d_{-2} \\ d_3 \\ d_{-3} \\ \vdots \end{bmatrix}_{COMPLEX}$$

The transformation matrix from the traditional to the complex basis has matrix elements $\{i,j\} = \langle C_i | T_j \rangle$.

References:

1. K. F. Riley, M. P. Hobson and S. J. Bence, ***Mathematical Methods for Physics and Engineering***, 2nd Ed., Cambridge, Cambridge UK (2002).
2. J.W. Brown and R. V Churchill, ***Fourier Series and Boundary Value***

Problems, 6th Edition, McGraw-Hill, Boston (2001).

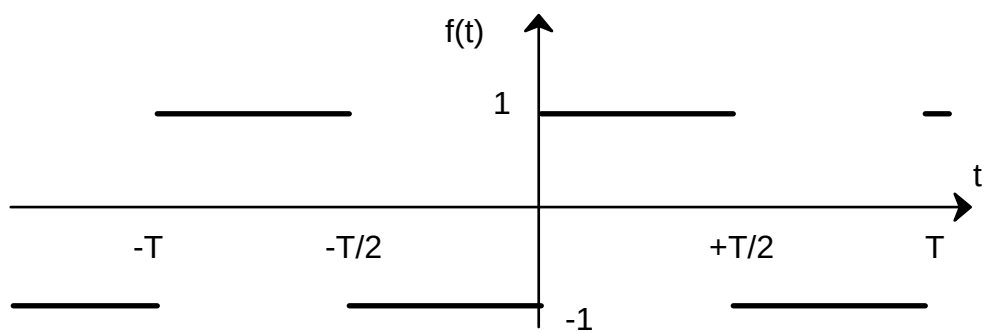
3. H. F. Weinberger, ***Partial Differential Equations***, Blaisdell, Watham, MA (1965).

4. Discovery exercises are inspired by or directly borrowed from the Physics Education Group at Oregon State headed by Corinne A. Manogue.

5. Donald A. McQuarrie, ***Mathematical Methods for Scientists and Engineers***, University Science Books, Sausalito, CA (2003).

Fourier Series Collected:

1.) An odd **square wave** with period *T* stepping from -1 to +1 is presented.



The square wave *f(t)* has a period of *T* and is extended beyond the base period to display the discontinuities at each multiple of $^{T}/_{2}$. The function is represented as:

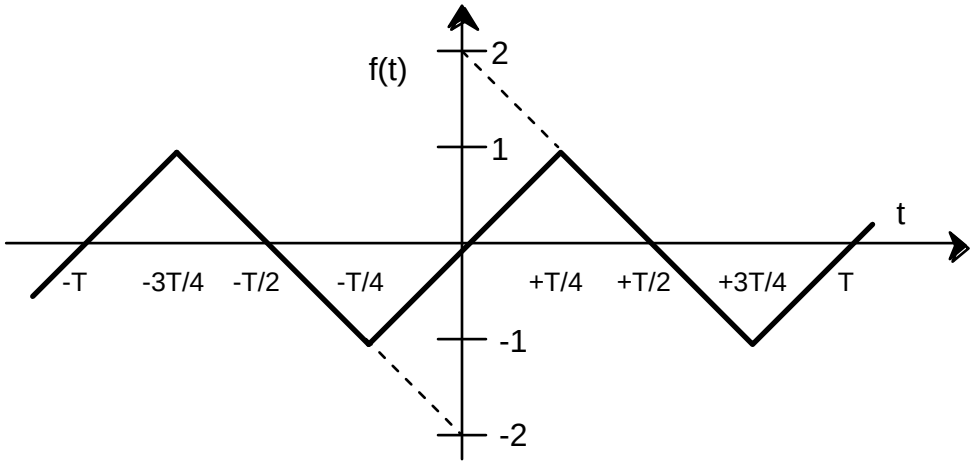
$$f(t) = \begin{cases} +1 & for \quad 0 < t < T/2 \\ -1 & for \quad -T/2 < t < 0 \end{cases} \qquad \text{[FS.31]}$$

FOURIER SERIES FOR THE ANTI-SYMMETRIC SQUARE WAVE:

$$b_p = \begin{cases} \left(4/\pi\, p \right) & for\ p\ odd \\ 0 & for\ p\ even \end{cases} \quad \text{or} \quad f(t) = \frac{4}{\pi} \sum_{m=0}^{\infty} \left(\frac{1}{2m+1} \right) \sin\left[(2m+1) \frac{2\pi\, t}{T} \right]$$

2.) The second example chosen is a triangular wave. This function is smoother than the square wave as it has no discontinuities. It does have periodic

discontinuities in its first derivative. *Be alert to the differences associated with expanding a smoother function.*



base period: - $T//_2 < t < T//_2$

$$f(t) = \begin{cases} 2 - 4t/T & for \quad T/4 < t < T/2 \\ 4t/T & for \quad -T/4 < t < T/4 \\ -2 - 4t/T & for \quad -T/2 < t < -T/4 \end{cases} \qquad \text{[FS.32]}$$

Once again, the function is odd with average value zero.

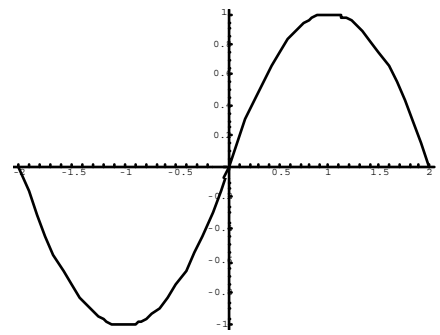
$$f(t) \quad = \quad \sum_{m=1}^{\infty} b_m \sin[\omega_m t] = \quad \sum_{m=1}^{\infty} b_m \sin\left[m\frac{2\pi\, t}{T}\right] \text{ where}$$

FOURIER SERIES FOR THE ANTI-SYMMETRIC TRIANGULAR WAVE:

$$f(t) = \frac{8}{\pi^2}\left[\sin\left[m\frac{2\pi\, t}{T}\right] - \frac{1}{3^2}\sin\left[m\frac{2\pi\, t}{T}\right] + \frac{1}{5^2}\sin\left[m\frac{2\pi\, t}{T}\right] + \ldots\right]$$
$$= \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\left[(2m+1)\frac{2\pi\, t}{T}\right] \qquad \text{[FS.33]}$$

3.) Slope-matched parabolic sections. Consider the function of period 4 defined over the interval [-2,2] by the equations:

$$f(t)=\begin{cases} 1-(t-1)^2=2t-t^2 & for \quad 0<t<2 \\ -1+(t+1)^2=2t+t^2 & for \quad -2<t<0 \end{cases}$$



It has a Fourier expansion $f(t)=\sum_{m=0}^{\infty}\frac{32}{\pi^3\ (2m+1)^3}\sin\left[(2m+1)\left(\frac{\pi}{2}\right)t\right]$.

4.) Asymmetric 25% square pulse which has a period of 2π.

$$f(x)=\begin{cases} 0 & for \quad -\pi<x<0 \\ 1 & for \quad 0<x<\pi/2 \\ 0 & for \quad \pi/2<x<\pi \end{cases}$$

In the final form, the contents of each set of parentheses should be replaced by an infinite sum with an index that assumes integer values. Use the unit circle tool to evaluate trig functions for a set of even spaced arguments.

$$f(x)=\frac{1}{4}+\frac{1}{\pi}\left(\frac{\cos(x)}{1}-\frac{\cos(3x)}{3}+\frac{\cos(5x)}{5}+\ldots\right)+\frac{1}{\pi}\left(\frac{\sin(x)}{1}+\frac{\sin(3x)}{3}+\frac{\sin(5x)}{5}+\ldots\right)$$
$$+\frac{2}{\pi}\left(\frac{\sin(2x)}{2}+\frac{\sin(6x)}{6}+\frac{\sin(10x)}{10}+\ldots\right)$$

Fourier series for: $f(x)=x^2 \quad (-\pi<x\le\pi)$ assuming that the function is extended periodically with period 2π. Use the result to compute $\sum_{m=1}^{\infty}(-1)^{m+1}/m^2$. *Answer*:

$$x^2=\pi^2/3+4\sum_{m=1}^{\infty}(-1)^m\ \frac{\cos(mx)}{m^2} \qquad (-\pi<x\le\pi);\ \pi^2/12$$