Integral Transforms – Fourier and Laplace

ROUGH DRAFT - - INCOMPLETE

Concepts of primary interest:

- **Fourier Series**
- **Fourier Transforms**
- **Laplace Transforms**
- **Additional Transforms**

Additional handouts

- **Fourier Transform Examples**
- **Laplace Transform Examples**
- **Differential Equation Solution Examples**

Sample calculations:

- **Fourier Transform of a Rectangular Pulse**
- **Fourier Transform of a Gaussian**
- **Expectation Values**
- **Exercising the Translation and Linear Phase Properties**
- **Group velocity and the Fourier transform**

Applications

- **Mega-App: Fraunhofer Diffraction**

Additional Integral Transforms

Fourier Bessel or Hankel Transform

$$g(k) = \int_0^\infty f(x)\, J_m(kx)\, x\, dx$$

$$f(x) = \int_0^\infty g(k)\, J_m(kx)\, k\, dk$$

Mellin Transform

$$\phi(z) = \int_0^\infty t^{z-1}\, f(t)\, dt$$

$$f(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} t^{-z}\, \phi(z)\, dz$$

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Hilbert Transform

$$g(y) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{f(x)dx}{x-y}$$

$$f(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{g(y)dy}{y-x}$$

Tools of the Trade

http://mathworld.wolfram.com/

http://scienceworld.wolfram.com/

http://www.efunda.com/math/ math_home/math.cfm

http://www.exampleproblems.com/wiki/index.php?title=Main_Page

Proposed Edits:

add Schiff minimum uncertainty problem

rework SC4

review all problems; DE

rework the signal bandwidth problem

Introduction to Integral Transforms

The Fourier transform is the perhaps the most important integral transform for physics applications. Given a function of time, the Fourier transform decomposes that function into its pure frequency components, the sinusoids (sinωt, cosωt, $e^{\pm i\omega t}$). When driven by a pure frequency, linear physical systems respond at the *same frequency* with amplitudes that are frequency dependent. As a result, Fourier decomposition and superposition are attractive tools to analyze the systems. Applications include the

solution of some differential equations, wave packet studies in quantum mechanics and the prediction of far-field diffraction patterns in optics. It is crucial to the application of the Fourier transform technique that an inverse transform exists that it recovers the time-dependent function from its frequency component representation.

Laplace transforms are based on Fourier transforms and provide a technique to solve some inhomogeneous differential equations. The Laplace transform has a reverse transform, but it is rarely used directly. Rather a table of transforms is generated, and the inverse (or reverse) is accomplished by finding matching pieces in that table of forward transforms. The Laplace transforms often take the form of a rational function with a polynomial in the denominator. A study of the singularities of these forms provides resonant response information for mechanical and electronic systems.

Fourier Transforms: The Fourier Series - Extended

Joseph Fourier: French mathematician who discovered that any periodic motion can be written as a superposition of sinusoidal and cosinusoidal vibrations. He developed a mathematical theory of heat in *Théorie Analytique de la Chaleur (Analytic Theory of Heat),* (1822), discussing it in terms of differential equations. Fourier was a friend and advisor of Napoleon. Fourier believed that his health would be improved by wrapping himself up in blankets, and in this state he tripped down the stairs in his house and killed himself. The paper of Galois that he had taken home to read shortly before his death was never recovered.

Eric W. Weisstein @http://scienceworld.wolfram.com/biography/Fourier.html, a Wolfram site

The development begins by motivating the spatial Fourier transform as an extension of a spatial Fourier series. Fourier series are appropriate for periodic functions with a finite period L. The generalization of Fourier series to forms appropriate for more general functions defined from $-\infty$ to $+\infty$ is not as painful as it first appears, and the

process illustrates the transition from a sum to an integral, a good thing to understand. The functions to be expanded are restricted to piecewise continuous square integrable functions with a finite number of discontinuities. Functions meeting these criteria are *well-behaved* functions. Everything that follows is restricted to *well-behaved* cases. (Less restrictive developments may be found elsewhere.)

Exercise: A standard trigonometric Fourier series for a function $f(x)$ with period L has the form**:**

$$f(x) \;=\; c_0\,(1) \;+\; \sum_{m=1}^{\infty} a_m \cos\left[m k_0 x\right] \;+\; \sum_{m=1}^{\infty} b_m \sin\left[m k_0 x\right] \quad \text{where } k_0 = {}^{2\pi}/_{L}.$$

Show that this can be cast in the form:

$$f(x) \;=\; c_0\, e^{i(0)} \;+\; \sum_{m=1}^{\infty} \tfrac{1}{2}(a_m - i b_m)\, e^{i m k_0 x} + \tfrac{1}{2}(a_m + i b_m)\, e^{-i m k_0 x} = \sum_{m=-\infty}^{\infty} \alpha_m\, e^{i m k_0 x}$$

This result justifies the form of the complex Fourier series used below.

The development that follows is intended to provide a motivation for Fourier transforms based on your knowledge of Fourier series. It is not a mathematical proof, and several terms are used loosely (particularly those in quotes). The complex form of the Fourier series is the starting point. **Notation Alert**: $\exp[i\theta] \leftrightarrow e^{i\theta}$

$$f(x) \;=\; \sum_{m=-\infty}^{\infty} \alpha_m \exp\left[i\, m \left({}^{2\pi}\!/_{L}\right) x\right] \qquad \text{[IT.1]}$$

A typical coefficient α_p (the amplitude of the $\exp\left[i\, p \left({}^{2\pi}\!/_{L}\right) x\right]$ behavior) is projected out of the sum by multiplying both sides by the complex conjugate of $\exp\left[i\, p \left({}^{2\pi}\!/_{L}\right) x\right]$ and then integrating over one period as required by the inner product. **Some call this procedure Fourier's trick; we prefer to call it projection.**

$$\int_{-L/2}^{L/2} \exp\left[-i\, p \left({}^{2\pi}\!/_{L}\right) x\right] f(x)\, dx \;=\; \sum_{m=-\infty}^{\infty} \alpha_m \int_{-L/2}^{L/2} \exp\left[-i\, p \left({}^{2\pi}\!/_{L}\right) x\right] \exp\left[i\, m \left({}^{2\pi}\!/_{L}\right) x\right] dx = L\, \alpha_p$$

or $$\alpha_p = \left(1/L\right) \int_{-L/2}^{L/2} \exp\left[-i\, p \left(2\pi/L\right) x\right] f(x)\, dx\,. \qquad \text{[IT.2]}$$

The orthogonality relation for the Fourier series basis set of complex exponentials has been used: $\int_{-L/2}^{L/2} \exp\left[-i\, p \left(2\pi/L\right) x\right] \exp\left[i\, m \left(2\pi/L\right) x\right] dx = L\,\delta_{pm}$.

> If you have studied vector spaces, note that this relation is consistent with and inner product: $\langle f(x) | g(x) \rangle = \frac{1}{L} \int_{-L/2}^{L/2} \left[f(x) \right]^* g(x)\, dx$.

Exercise: $\int_{-L/2}^{L/2} \exp\left[-i\, p \left(2\pi/L\right) x\right] \exp\left[i\, m \left(2\pi/L\right) x\right] dx = \int_{-L/2}^{L/2} \exp\left[i\,(m-p) \left(2\pi/L\right) x\right] dx$

Show that this integral vanishes for integers m and p if $m \neq p$. Show that it has the value L if $m = p$.

The trick is to define $k = {}^{2\pi m}/_L$ in $f(x) \;=\; \sum_{m=-\infty}^{\infty} \alpha_m \exp\left[i\, m \left(2\pi/L\right) x\right]$ and to multiply and divide by ${}^{2\pi}/_L$ to yield:

$$f(x) \;=\; \sum_{m=-\infty}^{\infty} \alpha_m \left(L/2\pi\right) \exp\left[i\, m \left(2\pi/L\right) x\right] \left(2\pi/L\right) = \sum_{m=-\infty}^{\infty} \alpha_m \left(L/2\pi\right) \exp\left[i\, k_m x\right] \Delta k$$

Where $\Delta k = {}^{2\pi}/_L$, the change in k as the index m increments by one between terms in the sum. The factor Δk must appear explicitly in order to convert the sum into an integral. In the lines above, the equation was multiplied and divided by $\Delta k = {}^{2\pi}/_L$ to identify $g(k)$ in the form $\Sigma\, g(k)\, \Delta k$ that becomes the integral $\int g(k)\, dk$. We conclude that: $g(k) = ({}^{L}/_{2\pi})\, \alpha(k) \exp[im({}^{2\pi}/_L)x] = ({}^{L}/_{2\pi})\, \alpha\,(k) \exp[ikx]$. In the limit $L \to \infty$, Δk becomes the infinitesimal dk in the integral, and k effectively becomes a continuous rather than a discrete variable [$\alpha_m \to \alpha(k)$; $k_m \to k$], and the sum of a great many small contributions becomes an integral. (See Converting Sums to Integrals in the Tools of the Trade section for a discussion of identifying and factoring out the infinitesimal.)

$$f(x) \;=\; \left(\tfrac{1}{2\pi}\right)\int_{-\infty}^{\infty}[\alpha(k)\,L]\,e^{ikx}dk = \left(\tfrac{1}{2\pi}\right)\int_{-\infty}^{\infty}\left[\,\tilde{f}(k)\,\right]e^{ikx}dk$$

The equation for $\alpha_p\,L$ evolves into that for $\alpha(k)$ as $L \to \infty$.

$$\tilde{f}(k) = \alpha(k)\,L = \int_{-L/2}^{L/2} e^{-ikx}\;f(x)\,dxx \to \int_{-\infty}^{\infty} e^{-ikx}\;f(x)\,dx$$

The function $\tilde{f}(k)$ is the Fourier transform of $f(x)$ which is the amplitude to find wiggling at the spatial frequency k in the function $f(x)$.

Notation Alert: The twiddle applied over a function's symbol denotes the Fourier transform of that function. The Fourier transform of $f(x)$ is to be represented as $\tilde{f}(k)$.

Sadly, there is no universal convention defining the Fourier transform, and factors of 2π are shuttled from place to place in different treatments. Some hold that the balanced definition is the only true definition:

$$f(x) \;=\; \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\,\tilde{f}(k)\,\right]e^{ikx}dk \qquad \tilde{f}(k) \;=\; \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\,f(x)\,]\,e^{-ikx}dx \qquad \text{[IT.3]}$$

In truth, the inverse of 2π must appear, but it can be split up in any fashion.

$$f(x) \;=\; \left(\tfrac{1}{2\pi}\right)^{1-S}\int_{-\infty}^{\infty}\left[\,\tilde{f}(k)\,\right]e^{ikx}dk \qquad \tilde{f}(k) \;=\; \left(\tfrac{1}{2\pi}\right)^{S}\int_{-\infty}^{\infty}[\,f(x)\,]\,e^{-ikx}dx$$

The common choices are $S = 0$, $S = 1$ and $S = ½$. The balanced form: $S = ½$ is adopted in this note set. Quantum mechanics texts often adopt $S = 0$.

$$f(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\,\tilde{f}(k)\,\right]e^{ikx}dk \;\;;\;\; \tilde{f}(k) \;=\; \int_{-\infty}^{\infty}[\,f(x)\,]\,e^{-ikx}dx \qquad \text{[IT.4]}$$

OUR CONVENTION for the Fourier transform:

$$\boxed{f(x) \;=\; \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\,\tilde{f}(k)\,\right]e^{ikx}dk \;\;;\; \tilde{f}(k) \;=\; \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\,f(x)\,]\,e^{-ikx}dx \qquad \text{[IT.3]}}$$

Somewhat surprisingly, the temporal transform pair interchanges the signs of the exponents.

$$f(t) \;=\; \left(1\big/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\,\tilde{f}(\omega)\,\right]e^{-i\omega t}d\omega \qquad \tilde{f}(\omega) \;=\; \left(1\big/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\,f(t)\,\right]e^{i\omega t}dt \qquad \text{[IT.5]}$$

This sign convention is a consequence of the form chosen for a plane wave ($e^{i[kx-\omega t]}$) in quantum mechanics. The Fourier transform $\tilde{f}(\vec{k},\omega)$ is amplitude of the $e^{i[kx-\omega t]}$ plane wave character in the function $f(\vec{r},t)$.

The Fourier transform has an inverse process that is almost identical to the Fourier transform itself. Fourier transforming and inverting are *balanced* processes.

Combining the Fourier transform and its inverse identifies a representation of the Dirac delta:

$$f(t) = \left(1\big/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\,\tilde{f}(\omega)\,\right]e^{-i\omega t}d\omega = \left(1\big/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\left(1\big/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\,f(t')\,\right]e^{i\omega t'}dt'\right]e^{-i\omega t}d\omega$$

$$f(t) \;=\; \int_{-\infty}^{\infty}\left[\,f(t')\,\right]\left\{\left(1\big/2\pi\right)\int_{-\infty}^{\infty}\left[e^{i\omega(t'-t)}\right]d\omega\right\}dt'$$

$$\Rightarrow \quad \delta(t-t') \;=\; \left(1\big/2\pi\right)\int_{-\infty}^{\infty}\left[e^{i\omega(t'-t)}\right]d\omega \qquad \text{[IT.6]}$$

Compare the middle equation above with the defining property of the Dirac delta.

$$\int_{a}^{b} f(x)\;\delta(x-x_0)\;dx = \begin{cases} f(x_0) \;\; if \;\; x_0 \in (a,b) \\ 0 \quad if \quad x_0 \notin [a,b] \end{cases}$$

The Fourier transform has a rich list of properties. Equation [IT.6] indicates that each $e^{-i\omega t}$ is an independent (wiggling) behavior; that is: [IT.6] is an orthogonality relation. Suffice it to say that analogs of the convergence properties, inner products and Parseval relations found for the Fourier series exist and much more.

Sample Calculation FT1: Fourier Transform of a Rectangular Pulse

Consider the rectangular pulse with unit area: $f(t)=\begin{cases} 1/2a & for\,|t|<a \\ 0 & for\,|t|>a \end{cases}$

$$\tilde{f}(\omega) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[f(t)\right]e^{i\omega t}dt = \left(\frac{1}{2a\sqrt{2\pi}}\right)\int_{-a}^{a} e^{i\omega t}dt$$

$$=\left(\frac{1}{2a\sqrt{2\pi}}\right)\left[\frac{e^{i\omega t}}{i\omega}\right]\Bigg|_{-a}^{a} = \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{\sin(\omega a)}{\omega a}\right) = \left(\frac{1}{\sqrt{2\pi}}\right)\mathrm{sinc}(\omega a)$$

Note that the sinc ("*sink*") function sinc(x) is defined to be $\sin(x)/x$.

Sample Calculation FT2: Fourier Transform of a Gaussian

Consider the Gaussian $f(t)=\left(\frac{1}{a^{1/2}\pi^{1/4}}\right)e^{-\left(t^2/2a^2\right)}$

$$\tilde{f}(\omega) = \left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty}\left[f(t)\right]e^{i\omega t}dt$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty}\left[\left(\frac{1}{a^{1/2}\pi^{1/4}}\right)e^{-\left(t^2/2a^2\right)}\right]e^{i\omega t}dt = \left(\frac{a^{1/2}}{\pi^{1/4}}\right)e^{-(\omega a)^2/2}$$

The transform of the Gaussian follows from the tabulated integral $\int_{-\infty}^{\infty} e^{-u^2}du=\sqrt{\pi}$ after a change of variable. The trick is completing the square in the exponent. Choosing $u=\left[\frac{t}{\sqrt{2}\,a}-\frac{ia\omega}{\sqrt{2}}\right]$, $u^2=\left[\frac{t^2}{2a^2}-i\omega t-\frac{a^2\omega^2}{2}\right]$. The integral $\int_{-\infty}^{\infty} e^{-(0.5)(t/a)^2}e^{i\omega t}\,dt$ becomes: $\int_{-\infty}^{\infty} e^{-(u)^2}e^{-(\omega a)^2/2}\,\sqrt{2}\,a\,du$. You should be prepared to use this completing-the-square trick and perhaps even to extend it. Also used: $\int_{-\infty}^{\infty} u^{2m}e^{-u^2}du = 2G(2m)=\Gamma(m+1/2)=\left(m-1/2\right)\left(m-3/2\right)\ldots\left(1/2\right)\sqrt{\pi}$.

One observation is that the Fourier transform of a Gaussian is another Gaussian. *There are a few other functions that have this property*. Examples are the Hermite-Gaussian

and the Gaussian-Laguerre functions used to describe the transverse amplitude variations of laser beams.[1]

Uncertainty: Following conventions adopted in Quantum Mechanics, the uncertainties in t and in ω are to be computed for the Gaussian example above. The average uncertainty squared is the difference of the average of the square and the square of the average.

$$(\Delta t)^2 = \left\langle (t - \langle t \rangle)^2 \right\rangle = \left\langle t^2 \right\rangle - \langle t \rangle^2 \text{ and } \left(\Delta \omega^2\right) = \left\langle (\omega - \langle \omega \rangle)^2 \right\rangle = \left\langle \omega^2 \right\rangle - \langle \omega \rangle^2$$

$$\langle t \rangle = \frac{\int_{-\infty}^{\infty} e^{-t^2/2a^2}\,(t)\,e^{-t^2/2a^2}\,dt}{\int_{-\infty}^{\infty} e^{-t^2/2a^2}\;e^{-t^2/2a^2}\,dt} \quad ; \quad \left\langle t^2 \right\rangle = \frac{\int_{-\infty}^{\infty} e^{-t^2/2a^2}\,(t^2)\,e^{-t^2/2a^2}\,dt}{\int_{-\infty}^{\infty} e^{-t^2/2a^2}\;e^{-t^2/2a^2}\,dt}$$

$$\langle \omega \rangle = \frac{\int_{-\infty}^{\infty} e^{-\omega^2 a^2/2}\,(\omega)\,e^{-\omega^2 a^2/2}\,d\omega}{\int_{-\infty}^{\infty} e^{-\omega^2 a^2/2}\;e^{-\omega^2 a^2/2}\,d\omega} \quad ; \quad \left\langle \omega^2 \right\rangle = \frac{\int_{-\infty}^{\infty} e^{-\omega^2 a^2/2}\,(\omega^2)\,e^{-\omega^2 a^2/2}\,d\omega}{\int_{-\infty}^{\infty} e^{-\omega^2 a^2/2}\;e^{-\omega^2 a^2/2}\,d\omega}$$

It follows that $\Delta t = a/\sqrt{2}$ and that $\Delta\omega = 1/\sqrt{2}\,a$ yielding the uncertainty product $\Delta\omega\,\Delta t = ½$. A Gaussian transform pair has the smallest possible uncertainty product[2] and that the general result is $\Delta\omega\,\Delta t \geq ½$.

Sample Calculation FT3:

$$\left\langle t^2 \right\rangle = \frac{\int_{-\infty}^{\infty} e^{-t^2/2a^2}\,(t^2)\,e^{-t^2/2a^2}\,dt}{\int_{-\infty}^{\infty} e^{-t^2/2a^2}\;e^{-t^2/2a^2}\,dt} = \frac{\int_{-\infty}^{\infty} e^{-t^2/a^2}\,(t^2)\,dt}{\int_{-\infty}^{\infty} e^{-t^2/a^2}\;dt} = \frac{a^3 \int_{-\infty}^{\infty} e^{-u^2}\,u^2\,du}{a\;\int_{-\infty}^{\infty} e^{-u^2}\;dt}$$

$$= \frac{a^3[2G(2)]}{a\;\;[2G(0)]} = \frac{a^3[\Gamma(3/2)]}{a\;\;[\Gamma(1/2)]} = \frac{a^3[½\,\Gamma(½)]}{a\;\;[\Gamma(½)]} = \frac{a^2}{2}$$

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[1] Ed Montgomery (centre.edu) notes that hyperbolic secant meets the test as does any function plus its properly scaled Fourier transform. See also: Self-Fourier functions and self-Fourier operators, Theodoros P. Horikis and Matthew S. McCallum, JOSA A, Vol. 23, Issue 4, pp. 829-834 (2006)

[2] L. I. Schiff, **Quantum Mechanics**, 3rd Ed., page 60, McGraw-Hill (1968).

The functions $G(n)$ and $\Gamma(n)$ are discussed in the Definite Integrals section of these handouts. Note that $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi} = 2\ G(0) = \Gamma(½)$.

Quantum Mechanics and Expectation Values: Expectations values are computed in quantum by sandwiching the operator for the quantity of interest between the complex conjugate of the wavefunction and the wavefunction and integrating over the full range. If the wavefunctions have been normalized, the process is represented as:

$$\langle O \rangle = \int_{-\infty}^{\infty} \psi^*(x)\ \hat{O}\ \psi(x)\, dx$$

In the case that the wavefunctions have not been normalized, the procedure must by supplemented by dividing by the normalization integral. Suppose that you know a multiple of $\psi(x)$, but not $\psi(x)$ itself. That is: you know $u(x) = c\ \psi(x)$, but c is not known.

$$\langle O \rangle = \frac{\int_{-\infty}^{\infty} \psi^*(x)\ \hat{O}\ \psi(x)\, dx}{\int_{-\infty}^{\infty} \psi^*(x)\ \ \psi(x)\, dx} = \frac{\int_{-\infty}^{\infty} c^*\, u^*(x)\ \hat{O}\ c\, u(x)\, dx}{\int_{-\infty}^{\infty} c^*\, u^*(x)\ c\, u(x)\, dx} = \frac{\int_{-\infty}^{\infty} u^*(x)\ \hat{O}\ u(x)\, dx}{\int_{-\infty}^{\infty} u^*(x)\ u(x)\, dx}$$

You can use un-normalized wavefunctions if you divide by the normalization integral on the fly. In many cases, the normalization constants have complicated representations that are difficult and tedious to evaluate. In these cases, division by the normalization integral $\int_{-\infty}^{\infty} u^*(x)\ u(x)\, dx$ is actually the preferred method. Study the sample calculation above as an example.

The transform of the Gaussian demonstrates an ***important, general property*** of Fourier transforms. If the base function is tightly localized, its Fourier transform is *broad* (it contains significant high frequency components). It takes a broad range of frequencies to interfere constructive at one point and destructively at a nearby point. A function that has rapid variations has high frequency components. A function that

varies only slowly can have a *narrow* transform (one will mostly low frequency components). Making a small corresponds to an $f(t)$ that varies rapidly and that is tightly localized. Hence its transform in ω-space is broad for small a. These observations are summarized in the *uncertainty relation*.

$$\Delta\omega\, \Delta t \geq \tfrac{1}{2}$$

This uncertainty relation is fundamental whenever a wave representation is used. Consider a function with two wave components with frequencies ω and (ω + Δω) that are in phase at a time t and that are to be out of phase by $t + \Delta t$. A relative phase change of π is required or the wave to shift from being in-phase to being out-of-phase.

$$(\omega + \Delta\omega)\, t - \omega\, t = m\, 2\, \pi \qquad \text{and} \qquad (\omega + \Delta\omega)\,(t + \Delta t) - \omega\,(t + \Delta t) = m\, 2\, \pi + \pi$$

$$(\omega + \Delta\omega)\,(\Delta t) - \omega\,(\Delta t) = \pi \qquad \Rightarrow \qquad \boxed{\Delta\omega\, \Delta t = \pi}$$

The details are slightly different, but not the idea. In a wave description, localization is achieved by have wave components with frequencies split by Δω that slip from being in phase to be out of phase in the localization span of Δt. If the localization region size Δt is to be made smaller, then the frequency spread Δω must be larger. The quantum mechanics minimum product of ½ differs from the π found above because quantum prescribes very specific procedures to compute Δω and Δt.

Information may be encoded onto a high frequency carrier wave. If audio information up to 40 *kHz* is to be encoded on a 100.3 *MHz* carrier wave then the final signal wave has frequencies smeared for about ±40 *kHz* around 100.3 *MHz*. The 40 *kHz* encoded signal has variations as fast as 25 μs and obeys the $\Delta\omega\, \Delta t \geq \tfrac{1}{2}$ relation. (*The more formal statement is that a system with bandwidth f must support risetimes* (Δt's) *as fast as* $1/(\pi f)$. A high definition television picture has more pixels per frame and hence contains information that varies more rapidly than the information necessary for the

standard NTSC broadcasts. The result is that $\Delta t_{HDTV} < \Delta t_{NTSC}$ so that $\Delta\omega_{HDTV} > \Delta\omega_{NTSC}$. HDTV broadcasts require more bandwidth than do NTSC broadcasts. (Digital encoding offers some economies as compared to analog.) A popular infrared laser (the YAG laser) emits near infrared light with a wavelength around 1.064 μm with a gain width of about 30 GHz allowing it to generate output pulses as short as 30 ps. To get shorter pulses, a laser with a greater bandwidth must be used. A narrow gain bandwidth laser is suitable only for long pulses or CW operation. On the flip side, if the emission of a laser has a bandwidth of Δω, then that emission has temporal variations that occur in as little time as $\Delta\omega^{-1}$.

Exercise: Use $u = \left[\frac{t}{\sqrt{2}\,a} - \frac{ia\omega}{\sqrt{2}}\right]$ and complete the evaluation of the Fourier transform of the Gaussian.

Exercise: We are interested in integrals of the form: $\int_{\infty}^{\infty} e^{-(0.5)(t/a)^2} e^{i\omega t} dt$. Consider $\int_{\infty}^{\infty} e^{-[a^2t^2+bt+c]} dt$. Show that $a^2 t^2 + bt + c = [\, at + (b/2a)]^2 + [c - (b/2a)^2]$. Show that:

$$\int_{\infty}^{\infty} e^{-[a^2t^2+bt+c]} dt = a^{-1} e^{-\left[c-\left(b/2a\right)^2\right]} \int_{\infty}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{a} e^{-\left[c-\left(b/2a\right)^2\right]}$$

Exercise: Compute the product of the full width of $e^{-0.5(t/a)^2}$ between its e^{-2} of maximum value points and the full width of its transform between the e^{-2} points of the transform. Based on you result, propose a value for the product $\Delta\omega\, \Delta t$. These identifications of the full widths Δω and Δt are somewhat arbitrary. Compare your result with that found using the quantum mechanics conventions above.

Exercise: Use the changes of variable: $r^2 = u^2 + v^2$ and $z = r^2$ to compute $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$ as the square root of $\int_{-\infty}^{\infty} e^{-u^2} du \int_{-\infty}^{\infty} e^{-v^2} dv = \int_0^{2\pi} d\phi \int_0^{\infty} e^{-r^2} r\, dr$.

Mathematica 5.2 Syntax	` is to the left of the 1 key
<<Calculus`FourierTransform`	loads the Fourier package
UnitStep[x]	= 0 for x < 0; = 1 for x > 1
FourierTransform[*expr*(*t*), *t* , *w*)]	----- Fourier transform of *expr*(*t*).
InverseFourierTransform[*expr*(*w*), *w* , *t*)]	----- inverse transform of *expr*(*w*).
Mathematica 6 Syntax	
<<Calculus`FourierTransform`	not required; Fourier transform library is preloaded
`<<FourierSeries``	New load command needed to load the Fourier series library

Some Properties of the Fourier Transform:

These properties are to be discussed in the spatial domain. In this case *k* is the spatial frequency that might be given in radians per meter. In photography, the more common frequency specification is line pairs per millimeter. You should restate each of the properties in temporal (time-frequency) terminology.

$$f(x) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\tilde{f}(k)\right]e^{ikx}dk \qquad \tilde{f}(k) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[f(x)\right]e^{-ikx}dx$$

A. **Relation to Dirac Delta**:

$$f(x) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\tilde{f}(k)\right]e^{ikx}dk \qquad f(x) = \left(1/2\pi\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[f(x')\right]e^{-ikx'}dx'\,e^{ikx}dk$$

$$f(x) = \left(1/2\pi\right)\int_{-\infty}^{\infty} f(x')\left\{\int_{-\infty}^{\infty}\left[e^{ik(x-x')}\right]dk\right\}dx' \Rightarrow \delta(x-x') = \left(1/2\pi\right)\int_{-\infty}^{\infty} e^{ik(x-x')}\,dk$$

The functions $\phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$ are orthogonal with respect to the inner product $\int_{-\infty}^{\infty} (g(x))^* \, f(x) \, dx$, and they are complete basis if all k from negative infinity to positive infinity are included in the set. The statement that the set is a complete basis means that all well-behaved functions can be *faithfully* represented as a linear combination of members of the set.

$$f(x) \;=\; \int_{-\infty}^{\infty} \tilde{f}(k) \left[\left(\frac{1}{\sqrt{2\pi}} \right) e^{ikx} \right] dk$$

The linear combination becomes an integral. The Fourier transform is the function representing the expansion coefficients in that linear combination of the Fourier basis functions.

It also follows that: $\delta(k-\ell) = \left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} e^{i(k-\ell)x} \, dx$ by a change of variables.

The representations of the Dirac delta below should be added to you library of useful facts.

$$\delta(k-\ell) = \left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} e^{i(k-\ell)x} \, dx \quad ; \quad \delta(x-x') = \left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} e^{ik(x-x')} \, dk$$

They can be used to establish the Parseval Equalities which are property C below.

B. Symmetry Property for Real Functions. $\tilde{f}(-k) = \tilde{f}(k)^*$

$$\tilde{f}(k) \;=\; \left(\frac{1}{\sqrt{2\pi}} \right) \int_{-\infty}^{\infty} \left[f(x) \right] e^{-ikx} dx \Rightarrow \tilde{f}(-k) \;=\; \left(\frac{1}{\sqrt{2\pi}} \right) \int_{-\infty}^{\infty} \left[f(x) \right] e^{+ikx} dx$$

$$\tilde{f}(k)^* \;=\; \left(\left(\frac{1}{\sqrt{2\pi}} \right) \int_{-\infty}^{\infty} \left[f(x) \right] e^{-ikx} dx \right)^* \;=\; \left(\frac{1}{\sqrt{2\pi}} \right) \int_{-\infty}^{\infty} \left[f(x) \right]^* e^{+ikx} dx$$

$$= \left(\frac{1}{\sqrt{2\pi}} \right) \int_{-\infty}^{\infty} \left[f(x) \right]^* e^{+ikx} dx \quad \text{as } \left[f(x) \right]^* = f(x) \text{ and } dx^* = dx$$

The last step follows because $f(x)$ and dx are real. Hence: $\tilde{f}(-k) = \tilde{f}(k)^*$ for real

functions $f(x)$. The symmetry property for real functions is important. The symmetry property for purely imaginary functions is somewhat less useful. $\tilde{f}(-k) = -\tilde{f}(k)^*$ for pure imaginary functions $f(x)$.

C. Plancherel's theorem, a generalized Parseval's relation:

By our convention, a relation between an inner product of two entities and the sum of the product of their corresponding expansion coefficients for an expansion in an orthogonal basis is a Parseval's relation. Here the inner product is $\int_{-\infty}^{\infty} (g(x))^* \; f(x)\, dx$, the expansion set is the functions e^{ikx} and the sum is an integral, a sum over a continuous parameter. The Parseval relation for the Fourier transform expansion is called Plancherel's theorem. Our first Parseval's relation was the one for ordinary vectors: $\vec{A} \cdot \vec{B} = AB\cos\theta = A_x B_x + A_y B_y + A_z B_z$.

Given: $f(x) = \left(1 \big/ \sqrt{2\pi} \right) \int_{-\infty}^{\infty} \left[\tilde{f}(k) \right] e^{ikx} dk \qquad \tilde{f}(k) = \left(1 \big/ \sqrt{2\pi} \right) \int_{-\infty}^{\infty} \left[f(x) \right] e^{-ikx} dx$

and: $g(x) = \left(1 \big/ \sqrt{2\pi} \right) \int_{-\infty}^{\infty} \left[\tilde{g}(\ell) \right] e^{i\ell x} d\ell \qquad \tilde{g}(\ell) = \left(1 \big/ \sqrt{2\pi} \right) \int_{-\infty}^{\infty} \left[g(x) \right] e^{-i\ell x} dx$.

Parseval Equality: $\int_{-\infty}^{\infty} (g(x))^* \; f(x)\, dx = \int_{-\infty}^{\infty} (\tilde{g}(k))^* \; \tilde{f}(k)\, dk$

Using: $f(x) = (2\pi)^{S-1} \int_{-\infty}^{\infty} \left[\tilde{f}(k) \right] e^{ikx} dk \qquad \tilde{f}(k) = (2\pi)^{-S} \int_{-\infty}^{\infty} \left[f(x) \right] e^{-ikx} dx$

General Parseval Equality:

$$\int_{-\infty}^{\infty} (g(x))^* \; f(x)\, dx = (2\pi)^{2S-1} \int_{-\infty}^{\infty} (\tilde{g}(k))^* \; \tilde{f}(k)\, dk$$

This equality states that the inner product of two functions can be computed directly

using the definition $\int_{-\infty}^{\infty} (g(x))^* \; f(x)\; dx$ or alternatively in terms of the expansion coefficients for those functions in terms of a complete basis set. The application of Fourier methods to the diffraction of light provides a concrete interpretation of this equality.

Parseval's equality followed by replacing both functions in the inner product with their Fourier transforms representations. Use distinct frequency variable labels; the label used for *f*(*x*) should be distinct from that used in the Fourier representation of *g*(*x*). The factors are re-ordered, and the spatial integral is executed first to generate a frequency delta function.

D. Linear Phase Shift Translates the Transform:

$$g(x) = f(x)\; e^{ik_0 x} \Rightarrow \tilde{g}(k) = \tilde{f}(k - k_0)$$

If the original function *f(x)* is multiplied by a linearly varying phase e^{iax}, the Fourier Transform is translated in *k*-space by *a* in the +*k* sense. This property is *nice* as a formal property, and it has a cool realization in the diffraction pattern of a blazed grating.

If the original function is translated, the transform is multiplied by a linear phase factor.

$$g(x) = f(x - b) \Rightarrow \tilde{g}(k) = \tilde{f}(k)\; e^{-ikb}\,.$$

This paired behavior between uniform translations and multiplication by a linearly varying phase is expected because the Fourier transform and its inverse are *almost* identical.

The analogous results for the temporal transforms are:

$$g(t) = f(t)\; e^{i\omega_0 t} \Rightarrow \tilde{g}(\omega) = \tilde{f}(\omega + \omega_0) \text{ and } g(t) = f(t - b) \Rightarrow \tilde{g}(\omega) = \tilde{f}(\omega) e^{i\omega b}$$

E. Convolution: $f \circ g(x) = \int_{-\infty}^{\infty} f(x')\, g(x-x')\, dx' = \int_{-\infty}^{\infty} g(x')\, f(x-x')\, dx'$

Please note that other sources place a different symbol between the functions to designate a convolution. A star is a popular alternative to the floating circle used here.

In a sense, a convolution represents smearing of function by another. Each point value of the function $f(x)$ is spread or blurred over the width of the function $g(x)$, and then everything is summed to get the result.

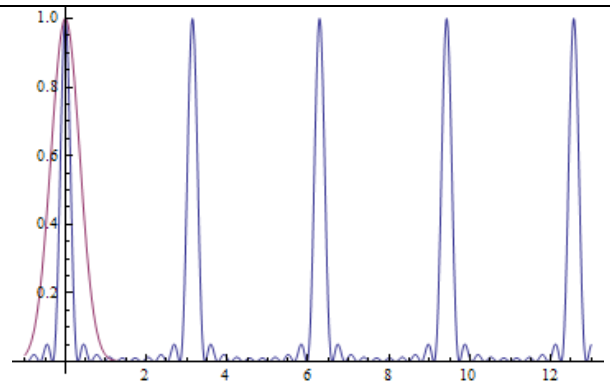
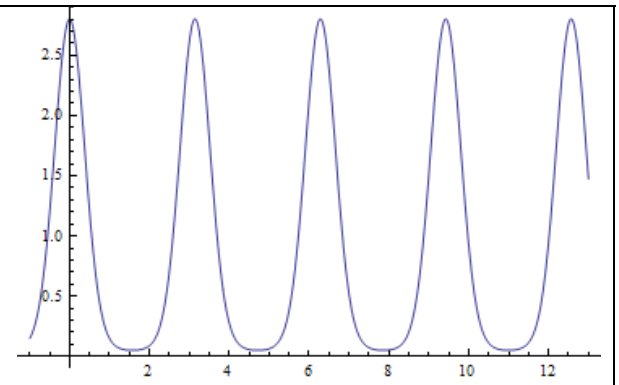
The Fourier transform of a convolution of two functions is the product of their Fourier transforms.

$$\{\widetilde{f \circ g}(k)\} = \sqrt{2\pi}\, \tilde{f}(k)\, \tilde{g}(k)$$

Convolution process is best understood by studying an example. The smearing function is chosen to be the Gaussian Exp[- 4 x^2] and the other function is to be $[{}^{\sin(10\,x)}/_{(10\,\sin(x))}]^2$, the intensity function describing the diffraction pattern for ten equally spaced narrow slits. Both functions are plotted in the left panel below. The convolution represents taking each point value of the ten slit pattern and smearing it with the Gaussian. Point by point the slit function is Gaussian smeared, and the result is summed with the Gaussian smears of all the previous points to build up the convolution. Stare at the right panel image until you believe it represents the point by point smearing and summing of the slit pattern. Stare at the right panel again. Convince yourself that it also represents the Gaussian smeared point by point using the ten slit pattern as the smearing function. The function *f* smeared using *g* is identical to the function *g* smeared using *f* as is reflected by the two representations of the convolution. The representations can be shown to be equal by using a change of integration variable.

$$f \circ g(x) = \int_{-\infty}^{\infty} f(x')\, g(x-x')\, dx' = \int_{-\infty}^{\infty} g(x')\, f(x-x')\, dx'$$

Plots of the Gaussian smear Exp[- 4 x^2] and the ten slit diffraction pattern [sin(10 x)/(10 sin(x))]^2.	**Plots of the convolution of the Gaussian smear Exp[- 4 x^2] and the ten slit diffraction pattern.**

	
Plot[{(Sin[10 x]/(10 Sin[x]))^2, Exp[-4 x^2]}, {x,-1,13},PlotRange -> All]	**Plot[NIntegrate[10 Exp[-4 u^2] (Sin[10 (x-u)]/ (10 Sin[x - u]))^2,{u,-2,2}],{x,-1,13}, PlotRange→All]**

Far-field (*i.e.* Fraunhofer) diffraction amplitudes in optics are *mathematically* the Fourier transform of the function representing the transmitted amplitude at the aperture. For example, a ten-slit pattern of identical finite width slits is the convolution of the finite slit with the array, the ten narrow slits. Therefore the diffraction pattern for ten finite-width slits is the product of the pattern for the single finite-width slit and the pattern for ten narrow slits. More is it to be made of this point later. For now, believe that convolutions and Fourier transforms have some fantastic applications.

Summary: The Fourier transform of a convolution of two functions if the product of their Fourier transforms: $\{f \circ \tilde{g}(k)\} = \sqrt{2\pi}\,\tilde{f}(k)\,\tilde{g}(k)$. (Assumes $S = ½$)

Autocorrelation integrals have a similar property. (*See auto-coherence in optics.*)

$$A(x) = \int_{-\infty}^{\infty} f(x')^* \; f(x'+x)\,dx'$$

Note that an autocorrelation is similar to the inner product of a function with itself. It differs in that the function at x is compared to the function at $x + x'$ rather than for

the same argument value. The inner product gauges the degree to which the two functions wiggle in the same pattern. The auto-correlation gauges the degree to which a function's local wiggle pattern persists as the argument changes. The Fourier transform of a functions autocorrelation is the product of that function's Fourier transform with its complex conjugate.

$$\{\widetilde{A(x)}\} = \{\int_{-\infty}^{\infty} f(x')^* \ \widetilde{f(x'+x)}\, dx'\} = \sqrt{2\pi}\, \tilde{f}(k)\ \tilde{f}(k)^* = \sqrt{2\pi}\, |\ \tilde{f}(k)|^2$$

Auto- and cross-correlations are treated in the problem section.

F. Scaling: If the original function is spread linearly by a factor M, its Fourier transform is narrowed by that same factor (spread by a factor of M^{-1}).

Suppose that the function $f(x)$ is one for $|x| < 1$ and equal to zero otherwise; then $f(x/3)$ is equal to one for $|x| < 3$. Dividing the argument of a function by M has the effect of spreading that function by a factor of M along the abscissa without changing its amplitude (range along the ordinate).

$$\left\{ \widetilde{f(x/M)} \right\} = M\ \tilde{f}(Mk)$$

Assume $M > 1$. $f(x/M)$ is wider than $f(x)$ and $\tilde{f}(Mk)$ is narrower than $\tilde{f}(k)$.

An example of this scaling is provided by the Gaussian and its transform.

$$f(x) = e^{-(x/a)^2} \Rightarrow \tilde{f}(k) = \left(a/\sqrt{2}\right) e^{-(ka)^2/4}$$

Simply replace a by M. A standard application to single slit diffraction is the observation that the diffraction pattern of the slit gets broader as the slit gets narrower.

G. Linear Operation: The Fourier transform of a linear combination of functions is that same linear combination of their Fourier transforms.

$$\widetilde{\{a\, f(x) + b\, g(x)\}} = a\, \tilde{f}(k) + b\, \tilde{g}(k)$$

H. Large *k* Behavior: In the limit of large k, the magnitude of the Fourier transform of a well-behaved function vanishes ***no faster*** than $|k|^{-n}$ if the function and its derivatives have their first discontinuity in order n-1. The rectangular pulse is discontinuous (its 0th order derivative is discontinuous), and its transform vanishes as $|k|^{-1}$ as $|k|$ becomes large. A discontinuous function has its first discontinuity for the derivative of order zero. (n - 1 = 0) The transform vanishes as: $|k|^{-1}$. The Gaussian is continuous and has continuous derivatives through infinite order. The transform of a Gaussian vanishes faster than any inverse power of $|k|$ for large $|k|$. The property discussed in this paragraph should be considered in terms of functions over the domain of all complex numbers. That is: the analytic properties of the functions as functions of a complex variable must be considered.

I. Uncertainty: $\Delta\omega\, \Delta t \geq ½$ or $\Delta k\, \Delta x \geq ½$ The Fourier transform of a 'narrow' function is has a minimum width that increases as the width of the function increases. Rapid variations in a function require that there be high frequencies to accurately represent those variations.

J. Derivative Property: The Fourier transform of the derivative of a function is ik times the Fourier transform of the function if both are well-defined.

$$\tilde{f}(k) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\, f(x)\,\right] e^{-ikx} dx; \quad \widetilde{\frac{df}{dx}}(k) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\frac{df}{dx}\right] e^{-ikx} dx$$

$$\widetilde{\frac{df}{dx}}(k) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\frac{df}{dx}\right] e^{-ikx} dx = \left(1/\sqrt{2\pi}\right)\left[\, f(x)\, e^{-ikx}\right]_{-\infty}^{\infty} + ik\left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx$$

$$\text{or } \widetilde{\frac{df}{dx}}(k) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\frac{df}{dx}\right]e^{-ikx}dx = +ik\left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty} f(x)\,e^{-ikx}dx = +ik\,\tilde{f}(k)$$

If the function and its derivatives in a differential equation are replaced by their Fourier representations, the differential equation becomes and algebraic equation to be satisfied by the Fourier transform. The inverse Fourier transform of the solution to that equation is then the solution to the differential equation.

K. Symmetric and Anti-symmetric functions: Separate the function *f*(*x*) into its even and odd components: $f(x) = f_{odd}(x) + f_{even}(x)$. Using the definition, it follows that:

$$\tilde{f}(k) \;=\; \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\,f(x)\,\right]e^{-ikx}dx = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty} f(x)\left[\cos(kx) + i\sin(kx)\,\right]dx$$

$$\text{for } f(x) \text{ even:}\quad \tilde{f}(k) \;=\; 2\left\{\left(1/\sqrt{2\pi}\right)\int_{0}^{\infty} f(x)\cos(kx)\,dx\right\}$$

$$\text{for } f(x) \text{ odd:}\quad \tilde{f}(k) \;=\; 2i\left\{\left(1/\sqrt{2\pi}\right)\int_{0}^{\infty} f(x)\sin(kx)\,dx\right\}$$

The forms in braces are cosine and sine transforms. They are not to be considered further.

Fourier methods appear difficult and are extremely mysterious on first encounter. Pay the price. The rewards for mastering Fourier methods are enormous and cool. In the time domain, the Fourier transform identifies the frequency content of a function of time. Modern SONAR and passive acoustic monitoring systems depend on examining the received signal transformed into frequency space. Many systems are identified by their tonal patterns, distinct frequency combinations, in their acoustic emissions. In quantum mechanics, the spatial Fourier transform of the wave function reveals its

plane-wave or momentum content. In optics, the spatial Fourier transform of the wave amplitude at an aperture predicts the Fraunhofer diffraction pattern of that aperture. Scaling up to radar wavelengths, the spatial Fourier transform of an antenna predicts the far-field radiation pattern of that antenna. This result also applies to hydrophone arrays in acoustics. There are problems that appear to defy solution in the time domain that yield results freely when transformed to the (Fourier) frequency domain.

Sample Calculation FT4: The translation and linear phase properties are to be exercised to develop the Fourier transform of $g(t)=\left(\frac{1}{a^{1/2}\pi^{1/4}}\right)e^{-\left((t-t_0)^2/2a^2\right)}e^{-i\omega_0 t}$ from the earlier result that $f(t)=\left(\frac{1}{a^{1/2}\pi^{1/4}}\right)e^{-\left(t^2/2a^2\right)}$ has the transform $\tilde{f}(\omega)=\left(\frac{a^{1/2}}{\pi^{1/4}}\right)e^{-\left(a^2\omega^2/2\right)}$.

CAUTION: This problem was changed in the middle of the calculation. It needs to be repeated as it is probable that one or more signs are incorrect. (Report errors to tank@usna.edu.)

The temporal relations are:

$$g(t)=f(t)\,e^{i\omega_0 t}\Rightarrow\tilde{g}(\omega)=\tilde{f}(\omega+\omega_0)\ \text{ and }\ g(t)=f(t-b)\Rightarrow\tilde{g}(\omega)=\tilde{f}(\omega)e^{i\omega b}.$$

Start with $h(t)=\left(\frac{1}{a^{1/2}\pi^{1/4}}\right)e^{-\left((t)^2/2a^2\right)}e^{-i\omega_0 t}$ and apply $g(t)=f(t)\,e^{i\omega_0 t}\Rightarrow\tilde{g}(\omega)=\tilde{f}(\omega+\omega_0)$.

$$\tilde{h}(\omega)=\left(\frac{a^{1/2}}{\pi^{1/4}}\right)e^{-\left(a^2(\omega-\omega_0)^2/2\right)}$$

Next, the translation property $g(t)=f(t-b)\Rightarrow\tilde{g}(\omega)=\tilde{f}(\omega)e^{i\omega b}$ is applied with $b=t_o$.

That yields the Fourier transform of $G(t)=e^{+i\omega_0 t_0}g(t)=\left(\frac{1}{a^{1/2}\pi^{1/4}}\right)e^{-\left((t-t_0)^2/2a^2\right)}e^{-i\omega_0(t-t_0)}$.

$$\tilde{G}(\omega)=\left(\frac{a^{1/2}}{\pi^{1/4}}\right)e^{-\left(a^2(\omega+\omega_0)^2/2\right)}e^{i\omega t_0}$$

Finally, the linearity property is invoked. $\widetilde{\{a\,f(x)+b\,g(x)\}}=a\,\tilde{f}(k)+b\,\tilde{g}(k)$

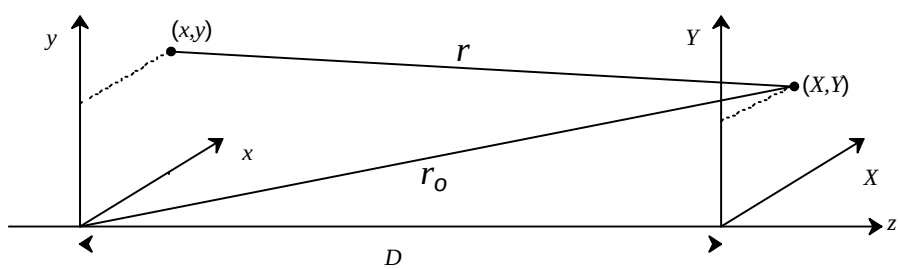
$$\tilde{g}(\omega) = e^{-i\omega_0 t_0}\tilde{G}(\omega) = e^{-i\omega_0 t_0}\left(\frac{a^{1/2}}{\pi^{1/4}}\right)e^{-\left(a^2(\omega+\omega_0)^2/2\right)}e^{i\omega t_0} = \left(\frac{a^{1/2}}{\pi^{1/4}}\right)e^{-\left(a^2(\omega+\omega_0)^2/2\right)}e^{i(\omega-\omega_0)t_0}$$

Thus $g(t) = \left(\frac{1}{a^{1/2}\pi^{1/4}}\right)e^{-\left((t-t_0)^2/2a^2\right)}e^{-i\omega_0 t} \rightarrow \tilde{g}(\omega) = \left(\frac{a^{1/2}}{\pi^{1/4}}\right)e^{-\left(a^2(\omega+\omega_0)^2/2\right)}e^{i(\omega-\omega_0)t_0}$.

Mega-Application: Fourier Transforms Applied to Fraunhofer Diffraction

(*** The following is a placeholder for a future development. It needs **major revisions**!)

In the Huygens's construction, each point on an optical wavefront is a source point for an expanding spherical wave biased toward 'forward' propagation. Subsequent wave fronts are predicted by finding surfaces on which these waves add in phase. One approximate mathematical model for this procedure is a scalar approximation, the Fresnel-Kirchhoff integral.



Aperture Plane Diffraction Plane

The coordinates (x, y) are for the aperture plane and (X, Y) are for the diffraction plane. The field amplitude in the diffraction plane is $U_P(X, Y)$.

$$U_P(X,Y) = -\frac{ik}{4\pi}\left(\frac{e^{i(kr_0-\omega t)}}{r_0}\right)\int(2)\,A(x,y)\,e^{-i\Delta(x,y)}dx\,dy$$

The Grand Aperture Function: $A(x, y) = U_A(x, y)S(x, y)\ t(x, y)\ e^{i\phi(x,y)}$

where:

$U_A(x, y)$: The incident amplitude at the aperture.

$S(x, y)$: The shape function; 1 if (x,y) open, 0 if closed.

$t(x, y)$: The fractional amplitude transmission coefficient at (x,y).

$\phi(x,y)$: The phase shift at the point (x,y) due to the aperture.

The factor $\frac{e^{i(kr_0-\omega t)}}{r_0}$ represents a spherical wave, the factor (2) is the obliquity factor (the bias toward the forward direction) that is approximately two in the forward direction, $k = 2\pi/\lambda$, and $\Delta(x, y)$ is the path length difference between points in the aperture to the point of interest in the diffraction plane.

More precisely $\Delta(x, y) = r - r_o = \sqrt{D^2+(X-x)^2+(Y-y)^2}$ and $r_o = \sqrt{D^2+X^2+Y^2}$. The binomial theorem yields a few terms in the expansion:

$$\Delta(x, y) = r - r_o = \sqrt{D^2+(X-x)^2+(Y-y)^2} \approx -\left(X/r_0\right)x-\left(Y/r_0\right)y+\left(x^2+y^2\right)/(2r_0)+\ldots$$

For *small D*, the diffraction pattern is complicated, and it changes shape as D increases. For larger D, the terms in $\Delta(x, y)$ that are quadratic and higher in x and y becomes small compared to ¼ λ. In this Fraunhofer limit, the curvature of the wavefront is *negligible* and the diffraction pattern spreads geometrically. The *pattern* is fixed, but its transverse dimensions grow in direct proportion to D for increasing D. In this geometric or Fraunhofer limit:

$$U_P(X,Y) = -\frac{ik}{4\pi}(2)\frac{e^{i(kr_0-\omega t)}}{r_0}\int A(x,y)\,e^{-i\left[k\frac{X}{r_0}x+k\frac{Y}{r_0}y\right]}dx\,dy\,.$$

The amplitude in the diffraction plane is just some constants and a phase factor times the Fourier transform of the aperture function $A(x, y)$. $U_P(X, Y) \sim \tilde{A}\left[kX/r_0, kY/r_0\right]$. The

overall phase factor is not an issue as it is the intensity of the light rather than its amplitude that is directly observable.[1]

$$I_P(X, Y) \sim |U_P(X, Y)|^2 \sim |\ \tilde{A}\left[{}^{kX}\!/_{r_0}, {}^{kY}\!/_{r_0}\right]\ |^2$$

As the pattern is spreading geometrically, $\tilde{A}\left[{}^{kX}\!/_{r_0}, {}^{kY}\!/_{r_0}\right]$ can be interpreted as the amplitude diffracted in the direction specified by ${}^{X}\!/_{r_0}$ and ${}^{Y}\!/_{r_0}$. This identification can be made more concrete by recalling that a plane wave is focused to a 'point' in the focal plane of a lens. In the canonical configuration, that aperture is the focal length *f* before the lens and the patterns are observed on the focal plane *f* after the lens. In this case, the relative phases of amplitude at point on the focal plane are corrected and are those computed using the 2D Fourier transform.

A.) **Relation to Dirac Delta**: For an incident plane wave, the amplitude at the aperture is

$$U_A(x, y) = E_0\, e^{i\vec{k}\cdot\vec{r}} = E_0\, e^{i[k_x x + k_y y + k_z z]} \quad \text{for all } x, y.$$

which has a diffraction pattern proportional to:

$$U_P(x, y) \approx E_0\, \delta(\tfrac{kX}{D} - k_x)\ \ \delta(\tfrac{kY}{D} - k_y) \quad \text{for all } x, y.$$

This result is more transparent if one thinks about the pattern in the focal plane of an ideal lens. (Set $D = f$) An incident plane wave is focused to a point on the focal plane of the lens. In fact, the wave amplitude at each point on the focal plane is the amplitude of the corresponding plane-wave component of the light incident on the lens. **The 2-D Fourier transform is the decomposition of the light into plane-wave components, and each of these components maps to a point on the focal plane of the lens**. Without the lens, the delta function means that each plane wave component of the light leaving the aperture is observed in the far-field traveling with

[1] Optical frequencies are of order 5 x 10^{14} Hz while the fastest detectors have a response time of a picosecond. Optical

its unique precisely defined direction. (We have been discussing the behavior of a plane wave with infinite transverse extent. A finite "plane wave" is a sum of many infinite plane waves. Hence a finite plane wave with finite transverse extent focuses to a smeared spot. See uncertainty.)

B.) Symmetry Property for Real Functions. $U_P(X,Y)^* = U_P(-X,-Y)$

An aperture function is real if it does not introduce phase shifts ($\phi(x,y) = 0$) and the incident wave U_A has the same phase everywhere across the aperture (for example in the case of a normally incident plane wave). For real aperture functions, the diffraction intensity pattern has inversion symmetry $I_P(X,Y)^* \sim |U_P(X,Y)^*|^2 = |U_P(-X,-Y)|^2 \sim I_P(-X,-Y)$. The Fraunhofer intensity patterns from **real** apertures are expected to have all the symmetries of the aperture *plus inversion symmetry*.

C.) Parseval's Equality: Conservation of Energy. The integral of $I_A(x, y)$ the intensity at the aperture over the aperture plane is equal to the integral of $I_P(X, Y)$,the intensity in the diffraction plane, over the area of the diffraction plane. It is equivalent to:

$$\int_{Aperture} \left|A(x,y)\right|^2 dx\, dy \equiv \int_{\substack{Diffraction \\ plane}} \left|U_P(X,Y)\right|^2 dX\, dY$$

D.) Linear Phase Shift Translates Theorem: Multiplication of the amplitude at the aperture by a linearly varying phase translates the diffraction pattern as expected from geometric optics.

$$U_A \rightarrow U_A\, e^{i\,k\,x\,\sin\theta} \Rightarrow U_P \rightarrow U_P(X - D\sin\theta)$$

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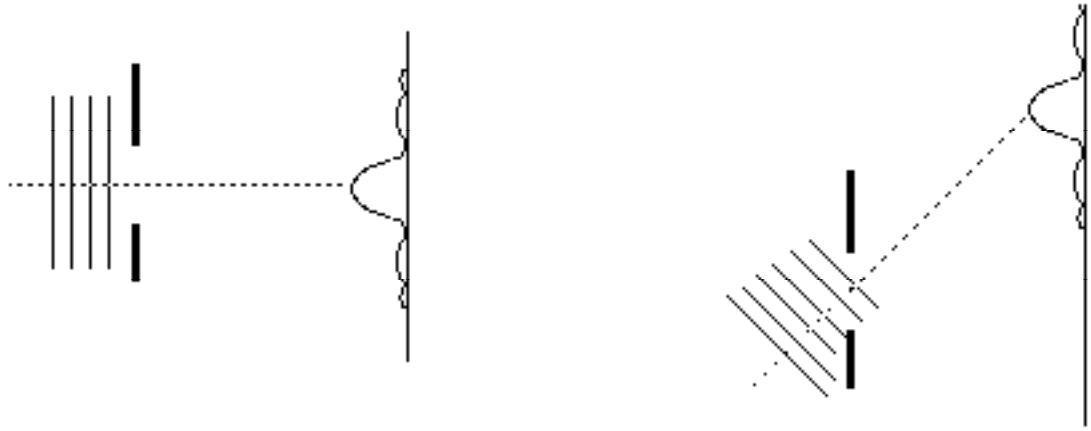
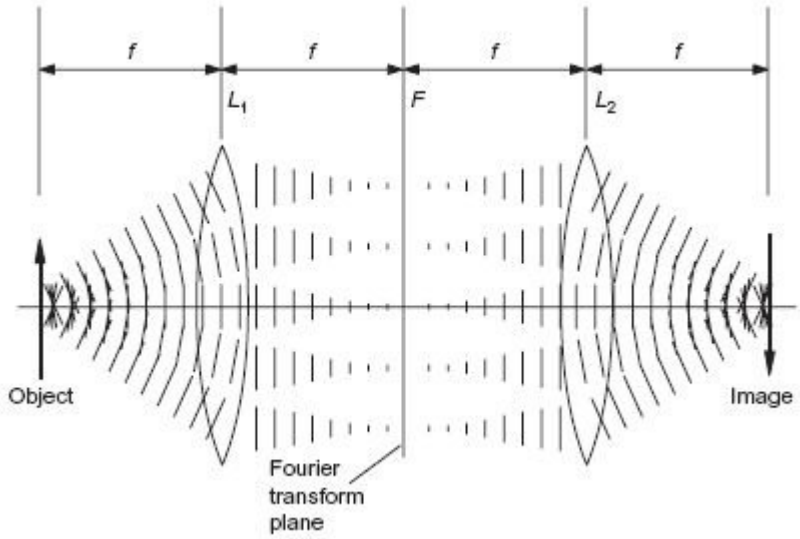
detectors cannot follow the amplitude of the waves; they must square law (intensity) detect the wave.



Diffraction Pattern of an Aperture



4f Configuration for Fourier Optics

Figure from Answers.com; Fourier Optics

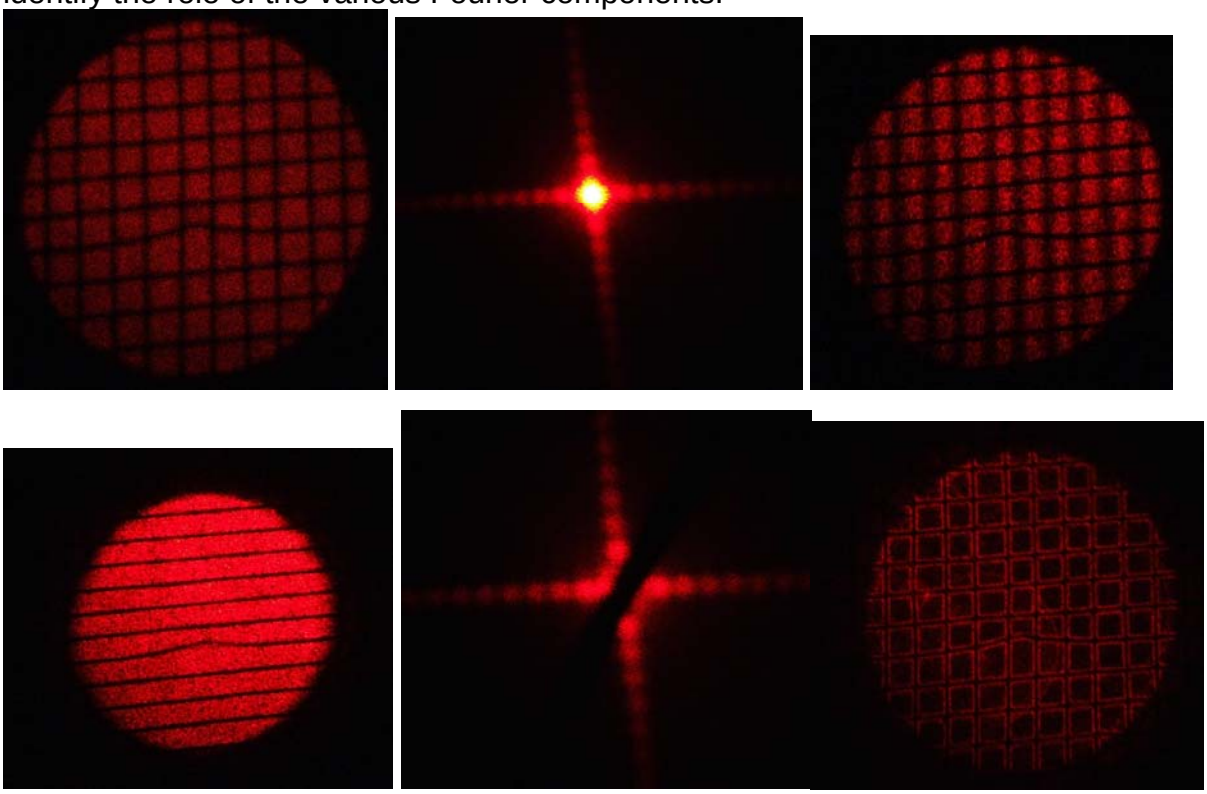
The first lens forms the Fourier transform in its focal plane, and the second lens Fourier transforms the wave amplitude a second time which differs by a sign from the inverse transform. The result is an inverted image. The re-inverted image is studied as various components of the transform are blocked in the transform plane to identify the contribution that the components make to the image.

At the object side, the image depicts the object's wave as the sum of plane waves. Each plane wave represents one Fourier component of the object's eave amplitude in the object plane. The action of a lens is to focus each incident plane wave to a '*point*' in its focal plane. Waves emanating from each point in the focal plane strike the second lens which increases their convergence to the extent that they leave the lens as plane waves. In geometric optics rays from a point in the focal plane form a bungle of parallel rays after the lens, and the ray from the point and through the lens center in not deflected which defines the direction of the bundle of parallel rays. The slope of each

ray bundle is the negative of the slope of its originator parallel ray bundle as it left the object. The overall result is that the wave in the final focal plane is inverted relative to the object.

The object plan is one focal length before the first lens; the lenses are separated by the sum of the focal lengths of the lens; and the final image is formed one focal length after the final lens. When both lenses have the same focal lens, the configuration is identified as a 4*f* system.

They open arrangement allows on to manipulate the pattern be blocking (or perhaps phase shifting) point in the Fourier transform plane. The second lens then re-transforms the wave to displace an image missing the blocked Fourier components. One can then identify the role of the various Fourier components.



The upper left image is of a small piece of screen formed using a 4f lens system. The next image is of the diffraction pattern of that screen in the transform plane. An adjustable slit was used to block most of the horizontal arms of the diffraction pattern thereby blocking the Fourier components required for rapid variations in the horizontal direction yielding the image on the upper right. The images of the vertical screen wires blur. When the slit was narrowed further to pass only the vertical flare, the reconstructed image became that on the lower left which contains no information about horizontal variations. The slit was replaced by a wire to block just the central (low frequency) part of the transform. The result was the lower right image which contains information about the high frequency (rapid change) character, but without the low frequency average character.

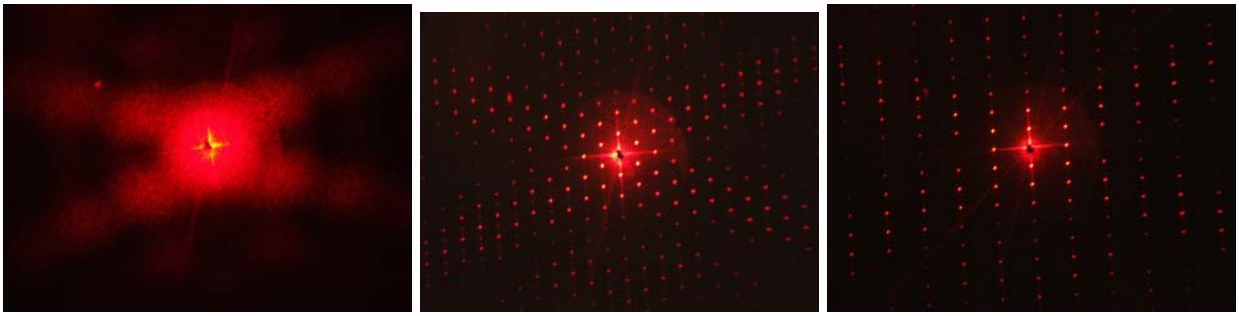
The linear phase factor can be realized by using an incident plane wave with non-normal incidence. It can also be achieved by placing a wedge prism over the aperture. The blazing of a grating effectively provides a linear phase factor that translates (or directs) the diffracted light into a particular diffraction order. Without blazing, the zero order diffraction is the most intense. Unfortunately, there is no dispersion (wavelength separation) in this order. Proper blazing can concentrate the diffracted energy in the higher orders with proportionately higher wavelength discrimination.

D′.) Translation Becomes a Linear Phase Shift Theorem: In machine vision, a burr on a needle may be more easily identified as a fault by examining the Fourier transform image. If the needle is misplaced, machine recognition could be difficult, but the Fourier view has only a linear phase which does not appear in the intensity (magnitude squared of the Fourier transform).

E.) Convolution: An aperture of identical sub-apertures can be represented as the convolution of the sub-aperture function centered on the origin with an array function which is the sum of delta functions that locate the centers of each sub-aperture in the full aperture. Consider $g(x)$ (= 1 if $|x| < ½\, a$; 0 otherwise) and its convolution with $f(\mathrm{x}) = \delta(x-b) + \delta(x) + \delta(x+b)$. Sketch $g(x)$ and the convolution of $g(x)$ with $f(x)$. By the theorem, the diffraction amplitude due to the full aperture is the amplitude due to the centered sub-aperture times the amplitude that would be due to an array of point openings arranged according to the array function. Intensities follow by squaring amplitudes. Hence the diffraction pattern of an array of identical sub-apertures is the pattern due to the array modulated (multiplied) by the pattern of

the sub-aperture. The sub-aperture is necessarily smaller than the full aperture so its diffraction pattern is large compared to the array pattern. The slowly varying aperture pattern modulates the more rapidly varying array pattern. What does this say about the diffraction pattern of N identical slits of width a equally spaced along a line with separation b?

The convolution theorem may be used in the reverse direction as well. Because the Fourier transform of a Fourier transform is the essentially the origin function, we can consider the aperture function and the Fraunhofer diffraction pattern to be Fourier transforms of one another. The grand aperture function is in the form of a product. In a simple case $A(x,y) = U_A(x,y)\, S(x,y)$. The expected diffraction is the convolution of the diffraction pattern (Fourier transform) of $U_A(x,y)$ considering a fully open aperture and the Fourier transform of the shape function. For example, consider U_A to be an infinite plane wave that may not be normally incident. This incident wave would transform to a delta function at some point X^*,Y^* on the focal plane. Let the shape function be a circular opening. The aperture transforms to an Airy disk/ring pattern centered about the intersection of the optical axis of the transform lens with the focal plane. As the radius of the circular opening is decreased, the linear dimensions of the Airy pattern increase by the same factor. Thus the diffraction pattern is the convolution of a centered Airy pattern with a delta function at X^*,Y^* which just translates the Airy disk to the new center position X^*,Y^*. The effect of the limiting circular opening is to spread (technical term is fuzz out) the point focus of the plane wave into Airy pattern. Decreasing the size of the opening will increase the spreading. In the case of a more complicated incident wave, the pattern that could be represented as the sum of delta functions, and closing down a circular aperture would cause the focal plane pattern to spread point by point causing the loss of sharpness and detail. If a rectangular limiting opening was used, the spreading would be of the form $\mathrm{sinc}^2(\alpha X)\ \mathrm{sinc}^2(\beta Y)$ rather than an Airy disk.



Convolution of an A with hexagonal and rectangular arrays. The *aperture shape* is an **A**. The black blocking letter on a transmitting slide gives a pattern equivalent to an **A** opening on a blocking background plus a bright central peak according to Babinet's principle. The leftmost image is for a random array of **A**s. The central image is for a hexagonal array of **A**s., and the rightmost image is for a rectangular array of **A**s.

Exercise: Discuss the convolution property and Babinet's principle in the context of the images above.

F.) Scaling: If an aperture is uniformly scaled down by a factor M in one linear direction, then the diffraction pattern will spread uniformly in that same dimension by the factor M. Narrow slits have wide diffraction patterns! Note: It is permissible to scale x and y independently.

G.) Linear Operation: $\Rightarrow$ Superposition. The aperture can be partitioned into several parts. The net diffracted amplitude will be the sum of the amplitudes due to the individual parts. The amplitude must be squared to find the intensity, and interference is expected among the contributions from the various segments. Babinet's Principle of complimentary screens is a special case of linearity. An aperture that consists of small openings that transmit the incident radiation is complimentary to an aperture that that transmits the radiation except for that in the areas that are open in the first aperture where it totally blocks the radiation. The sums

of the diffracted amplitudes from the two correspond to transmitting the complete incident wave which would have diffracted energy only in the forward direction. In the off-axis direction, the diffracted amplitudes of the two apertures (screens) sum to zero. Hence their squares (intensities) are identical except in the forward direction.

H. Large *k* Behavior: An aperture with a hard edge, a transmission coefficient that drop discontinuously to zero, leads to a grand aperture function $A(x, y)$ that is discontinuous and, as a result, leads to a Fourier transform that vanishes only slowly as k becomes large. Large k means that the energy is being diffracted far from the center or at large angles - usually a waste. Apodizing is a procedure in which the transition from transmitting to blocking is smoothed thereby eliminating or reducing the discontinuities in the derivatives of the aperture function $A(x,y)$ and thereby decreasing the energy diffracted out of the central region of diffraction pattern.

I. Smearing and Uncertainty: The spatial uncertainty relations are $\Delta k_x \, \Delta x \geq ½$ and $\Delta k_y \, \Delta y \geq ½$. If the x extent if the aperture is limited to Δx, then there is a spread in the k_x content of the Fourier transform of $\Delta k_x \approx {}^{1}/_{2\,\Delta x} = {}^{k\Delta Y}/_{D}$, and the Fourier pattern will be spread in angle by ${}^{\Delta X}/_{D} = {}^{1}/_{(2\,k\,\Delta x)}$. Equivalently $\Delta X \approx D({}^{\lambda}/_{\Delta x})$. A narrow aperture leads to a broad diffraction pattern. In the same manner, a lens focuses a plane wave to a spot with a radius of about $f\,({}^{\lambda}/_{d})$ or the focal length times the wavelength divided by the lens diameter. The ratio of the focal length to the lens diameter is called the *f*-number *f/#* of the lens. The smallest focal spot that can be formed by a lens is about its *f/# times* λ.

Group velocity and the Fourier transform:

Consider a wave pulse $g(x) = f(x)\, e^{ik_o x}$ traveling in the $+x$ direction at time $t = 0$ that is an envelope function $f(x)$ times the plane wave $e^{ik_o x}$. The Fourier transform $\tilde{g}(k)$ of the function $g(x) = f(x)\, e^{ik_o x}$ is $\tilde{f}(k - k_0)$.

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

$$\tilde{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ik_0 x}\, e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(k-k_0)x}\, dx = \tilde{f}(k - k_0)$$

The Fourier transform expands $f(x)$ as a sum of pure spatial frequency components

$$\left(\frac{1}{\sqrt{2\pi}} e^{ikx} \right)$$

At a time t, a component such as the one above will have developed into

$$\left(\frac{1}{\sqrt{2\pi}} e^{i[kx - \omega_k t]} \right)$$

where $\omega_k = \omega(k)$, the value of ω as dictated by the Schrödinger equation. One can invert the transform after adding the time dependence to find $g(x, t)$:[1]

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}(k)\, e^{+ikx}\, dx; \quad g(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}(k)\, e^{+i(kx - \omega_k t)}\, dx$$

ASSUME that the envelope function $g(x)$ varies slowly over a distance $\lambda_o = {}^{2\pi}/_{k_o}$. The function $g(x)$ can be represented in terms of spatial frequencies near k_o. Expand $\omega(k)$.

$$\omega(k) = \omega_0 + \left.\frac{d\omega}{dk}\right|_{k_0} (k - k_0) + \frac{1}{2} \left.\frac{d^2\omega}{dk^2}\right|_{k_0} (k - k_0)^2 + \ldots$$

Next assume that the first two terms are adequate to faithfully represent $\omega(k)$.

$$\omega(k) = \omega_0 + v_G (k - k_0) \quad \text{where} \quad \omega(k_0) = \omega_0 \quad \text{and} \quad v_G = \left.\frac{d\omega}{dk}\right|_{k_0}$$

Recalling the inverse transform,

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[1] This is analogous to expanding a QM wave function using the spatial energy eigenfunctions, $\psi(x,0) = \sum a_n u_n(x)$, and then adding the time dependence term-by-term to find: $\psi(x,t) = \sum a_n u_n(x) e^{-i\omega_n t}$.

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}(k) e^{ikx}\, dk$$

and re-summing the time developed components, we find the shape and position of the wave for time *t*.

$$g(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}(k) e^{i\left(kx - \omega_0 t - v_G t[k-k_0]\right)}\, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k-k_0) e^{i\left(kx - \omega_0 t - v_G t[k-k_0]\right)}\, dk$$

$$g(x,t) = \frac{1}{\sqrt{2\pi}} e^{i\left(k_0 x - \omega_0 t\right)} \int_{-\infty}^{\infty} \tilde{f}(k-k_0) e^{i(k-k_0)(x - v_G t)}\, dk$$

With the change of variable $\ell = k - k_o$,

$$g(x,t) = \frac{1}{\sqrt{2\pi}} e^{i\left(k_0 x - \omega_0 t\right)} \int_{-\infty}^{\infty} \tilde{f}(\ell) e^{i\ell(x - v_G t)}\, d\ell = f(x - v_G t) e^{i\left(k_0 x - \omega_0 t\right)}$$

$$g(x,t) = f(x - v_G t) e^{i\left(k_0 x - \omega_0 t\right)}$$

The result is the time-dependent representative plane wave modulated by an envelope function with fixed shape and width that translates at speed v_G.

1.) The pulse envelope translates at the group velocity (*or group speed*, $\left.\frac{d\omega}{dk}\right|_{k_0}$) v_G with its envelope shape undistorted.

2.) The factor $e^{i\left(k_0 x - \omega_0 t\right)}$ shows that the wave crests inside the envelope propagate at the *phase velocity*, $\left.\frac{\omega}{k}\right|_{k_0}$.

In quantum mechanics, a free particle has energy $E = \frac{\hbar^2 k^2}{2m}$ and frequency $\omega_k = \frac{\hbar k^2}{2m}$. The phase velocity is $\frac{\omega_k}{k} = \frac{\hbar k}{2m} = \frac{p}{2m}$ or half the classical particle velocity. The probability lump translates at the group velocity $\frac{d\omega_k}{dk} = \frac{\hbar k}{m} = \frac{p}{m}$ which agrees with the classical particle velocity.

For an animated example, go to:
http://usna.edu/Users/physics/tank/TankMovies/MovieGuide.htm
Select: PlaneWaveAndPackets and PacketSpreadCompareNoFrmLbl.gif

As you view the animation, use your finger tip to follow one wave crest. Notice that the wave packet translates faster than does any one of the wave crests.

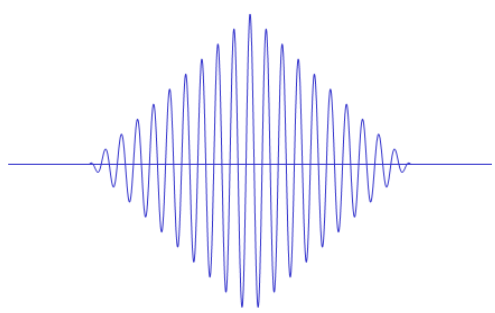
Conclusion: For a wave packet, the group velocity is analogous to the classical velocity of a particle described by the wave packet.

Some pulses require a broad range of frequencies for their representation. In such cases, the term $\frac{1}{2}\frac{d^2\omega}{dk^2}\Big|_{k_0}(k-k_0)^2$ must be included, and it leads to distortions of the pulse shape. The distortions expected most often are spreading and the degradation of sharp features.

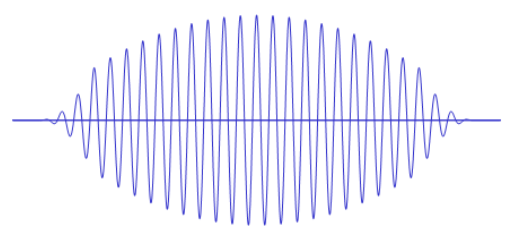
*** recompute the first figure below.

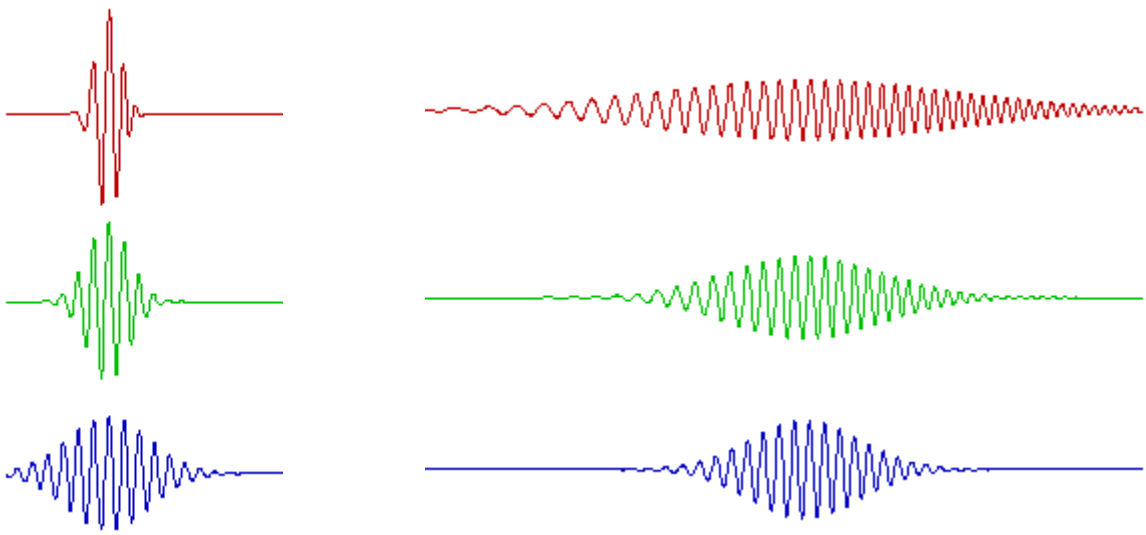
Wave packet example requiring quadratic terms ⇒ pulse distortion

Initial pulse with sharp features



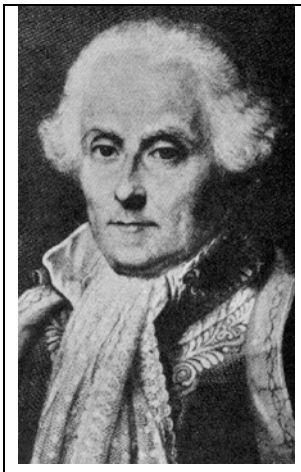
Later time, spread, less sharp





For cases in which the Taylor's series expansion of $\omega(k)$ does not converge rapidly, the pulse shapes will always distort, and the concept of a group velocity $^{d\omega}/_{dk}$ is of no value. If one finds that $^{d\omega}/_{dk} > c$, the group velocity (first order expansion) approximation is failing rather than Special Relativity.

The Laplace Transform



Pierre Laplace: French physicist and mathematician who put the final capstone on mathematical astronomy by summarizing and extending the work of his predecessors in his five volume *Mécanique Céleste (Celestial Mechanics)* (1799-1825). This work was important because it translated the geometrical study of mechanics used by Newton to one based on calculus, known as physical mechanics. He studied the Laplace transform, although Heaviside developed the techniques fully. He proposed that the solar system had formed from a rotating solar nebula with rings breaking off and forming the planets. Laplace believed the universe to be completely deterministic.

Eric W. Weisstein http://scienceworld.wolfram.com/biography/Laplace.html, a Wolfram site

Laplace transforms are based on Fourier transforms and provide a technique to solve

some inhomogeneous differential equations. The Laplace transform has the *Bromwich* (*a.k.a. Fourier-Mellin*) *integral* as its inverse transform, but it is not used during a first exposure to Laplace transforms. Rather a table of transforms is generated and the inverse (or reverse) is accomplished by finding matching pieces in that table of forward transforms. That is: Laplace transforms are to be considered as *operational* mathematics. Learn the rules, turn the crank, find the result, and avoid thinking about the details. Postpone the studying the relationship of the Laplace transform to the Fourier transform and the computation of inverse transforms using the contour integration of complex analysis until your second encounter with Laplace transforms.

The Laplace transforms sometimes take the form of a rational function with a polynomial in the denominator. A study of the singularities of these forms provides resonant response information to sinusoidal driving terms for mechanical and electronic systems

In our operational approach, a few Laplace transforms are to be computed, several theorems about the properties of the transforms are to be *stated* and perhaps two sample solutions of differential equations are to be presented. To apply Laplace transform techniques successfully, you must have an extensive table of transforms, exposure to a larger set of sample solutions and practice executing the technique. Regard this introduction only as a basis to recognize when the techniques might be effective. Study the treatment in one or more engineering mathematics texts if you need to employ Laplace transforms. The inversion by matching step, in particular, requires skill, familiarity and luck.

The *Unit Step* function vanishes for a negative argument and is equal to one for a positive argument. It has several optional names including the Heaviside function and several symbolic representations including $u(t)$ and $\theta(t)$.

www.geocities.com/neveyaakov/electro_science/heaviside.html]	**Oliver W. Heaviside** was English electrical engineer who adapted complex numbers to the study of electrical circuits. He developed techniques for applying Laplace transforms to the solution of differential equations. In addition, he reformulated Maxwell's field equations in terms of electric and magnetic forces and energy flux. In 1902 Heaviside correctly predicted the existence of the ionosphere, an electrically conducting layer in the atmosphere, by means of which radio signals are transmitted around the earth's curvature.

In his text, Wylie uses the Fourier transform of the unit step function to motivate the Laplace transform as follows.

$$u(t) = \begin{cases} 0 & for\ t < 0 \\ 1 & for\ t < 0 \end{cases} \qquad \tilde{u}(\omega) = \frac{1}{\sqrt{2\pi}} \left(\left. \frac{\cos(\omega t) - i\sin(\omega t)}{-i\omega} \right|_0^\infty \right)$$

The function $u(t)$ is not square integrable, and the Fourier transform is not defined. If one regulates the behavior by adding a decaying exponential *convergence* factor e^{-at}, the behavior improves.

$$U_a(t) = \begin{cases} 0 & for\ t < 0 \\ e^{-at} & for\ t < 0 \end{cases} \qquad \tilde{U}_a(\omega) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a + i\omega} \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{a - i\omega}{a^2 + \omega^2} \right)$$

In the general case, for each function $f(t)$, the auxiliary function $F(t)$ is considered.

$$F(t) = \begin{cases} 0 & for\ t < 0 \\ f(t)\ e^{-at} & for\ t < 0 \end{cases}$$

Applying the Fourier transform prescription **<u>with $S = 0$</u>**,

$$g(\omega)=\int_0^\infty F(t')e^{+i\omega t'}\,dt'=\int_0^\infty f(t')\,e^{-at'}\,e^{+i\omega t'}\,dt'=\int_0^\infty f(t')\,e^{-(a-i\omega)t'}\,dt'$$

$$f(t)=\tfrac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{+(a-i\omega)t}\,d\omega$$

Using the change of variable: $s = a - i\omega$, it follows that:

$g(s)=\int_0^\infty f(t)e^{-st}\,dt$ The Laplace Transform	$f(t)=\tfrac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} g(s)e^{st}\,ds$ Bromwich Integral

The evaluation of the inverse transform requires the full power of complex variables and complex integrations along paths. Rather than computing the inverses, inverses are to be found by matching pieces found in tables of forward transforms.

Table LT1: Laplace Transforms

($f(t) \Rightarrow f(t)\,u(t)$ where $u(t)$ is the Heaviside function)

$f(t)$, $t>0$	*method*	$\mathcal{L}[f(t)]=g(s)$
1 or $u(t)$	$g(s)=\int_0^\infty e^{-st}\,dt=\left.\frac{e^{-st}}{-s}\right\|_0^\infty$	$\frac{1}{s}$
t^n	$g(s)=\int_0^\infty t^n e^{-st}\,dt=n\int_0^\infty t^{n-1}e^{-st}\,dt$	$\frac{n!}{s^{n+1}}$
e^{-at}	$g(s)=\int_0^\infty e^{-(s+a)t}\,dt=\left.\frac{e^{-(s+a)t}}{-(s+a)}\right\|_0^\infty$ or apply shift theorem to $u(t)$	$\frac{1}{(s+a)}$
$e^{i\omega t}$	$g(s)=\int_0^\infty e^{-(s-i\omega)t}\,dt=\left.\frac{e^{-(s-i\omega)t}}{-(s-i\omega)}\right\|_0^\infty$	$\frac{1}{(s-i\omega)}$
$\cos(\omega t)$	$\cos(\omega t)=\left(\tfrac{1}{2}\right)\left[e^{i\omega t}+e^{-i\omega t}\right]\Rightarrow g(s)=\left(\tfrac{1}{2}\right)\left[\frac{1}{(s-i\omega)}+\frac{1}{(s+i\omega)}\right]$	$\frac{s}{(s^2+\omega^2)}$
$\sin(\omega t)$	$\sin(\omega t)=\left(\tfrac{1}{2i}\right)\left[e^{i\omega t}-e^{-i\omega t}\right]\Rightarrow g(s)=\left(\tfrac{1}{2i}\right)\left[\frac{1}{(s-i\omega)}-\frac{1}{(s+i\omega)}\right]$	$\frac{\omega}{(s^2+\omega^2)}$
$\cosh(bt)$	$\cosh(bt)=\left(\tfrac{1}{2}\right)\left[e^{bt}+e^{-bt}\right]\Rightarrow g(s)=\left(\tfrac{1}{2}\right)\left[\frac{1}{(s-b)}+\frac{1}{(s+b)}\right]$	$\frac{s}{(s^2+b^2)}$
$\sinh(bt)$	$\sinh(bt)=\left(\tfrac{1}{2}\right)\left[e^{bt}-e^{-bt}\right]\Rightarrow g(s)=\left(\tfrac{1}{2}\right)\left[\frac{1}{(s-b)}-\frac{1}{(s+b)}\right]$	$\frac{b}{(s^2+b^2)}$
$\delta(t-t_0)$	$g(s)=\int_0^\infty \delta(t-t_0)\,e^{-st}\,dt=e^{-t_0 s}$	$e^{-t_0 s}$

Mathematica Syntax	UnitStep[x] = $u(x)$
LaplaceTransform[*expr*(t), t , s)]	----- Laplace transform
of *expr*(t).	
InverseLaplaceTransform[*expr*(s), s , t)]	----- inverse transform of
expr(s).	

Properties of Laplace Transforms:

Linearity: The Laplace transform of a linear combination of functions is that same linear combination of the Laplace transforms of the functions.

$$\mathcal{L}[a\, f_1(t) + b\, f_2(t)] = a\, \mathcal{L}[\, f_1(t)] + b\, \mathcal{L}[\, f_2(t)]$$

This property follows from the linearity of the integration. Linearity should always be noted when applicable, and, in the case of Laplace transforms, it is crucial in the matching to find an inverse process.

The *well-behaved* criteria for functions to be Laplace transformed that they be piecewise regular functions bounded by e^{Mt} for all $t > T$ for some M and T. In some cases continuity through some order of the derivatives is needed.

Transform of the Derivative: $\mathcal{L}[f'(t)\,] = s\, \mathcal{L}[\, f(t)] - f(\,0^+)$

The Laplace transform of the derivative of a function is s times the Laplace transform of the function minus the limiting value of the function as its argument approaches zero from positive values. This property follows from the definition and integration by parts.

$$g(s) = \int_0^\infty f'(t)\; e^{-st}\, dt = f(t)\; e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)\; e^{-st}\, dt$$

That is: The process of taking a derivative is replaced by the algebraic operations of multiplication and addition. The solution of differential equations is replaced by the solution of algebraic equations followed by transform inversions.

The derivative relation can be used recursively to yield:

$$\mathcal{L}[f^{[n]}(t)] = s^n \mathcal{L}[f(t)] - s^{n-1} f(0^+) - s^{n-2} f^{[1]}(0^+) - s^{n-3} f^{[2]}(0^+) - \ldots - f^{[n-1]}(0^+)$$

Transform of an Integral: $\mathcal{L}[\int_a^t f(t')\, dt'] = s^{-1} \mathcal{L}[f(t)] + s^{-1} \int_a^0 f(t')\, dt'$

Integration of the function is equivalent to division by the independent variable plus a boundary term. The proof of this property is postponed to the problem section.

The Shifting Property #1: $\mathcal{L}[e^{-at} f(t)] = \mathcal{L}[f(t)]_{s+a} = g(s+a)$ where $g(s) = \mathcal{L}[f(t)]$

$$g_a(s) = \int_0^\infty \left\{e^{-at} f(t)\right\} e^{-st}\, dt = \int_0^\infty \left\{f(t)\right\} e^{-(s+a)t}\, dt = g(s+a)$$

Shifting Property #2: $\mathcal{L}[f(t-a)\, u(t-a)] = e^{-as} \mathcal{L}[f(t)] = e^{-as} g(s)$ where $g(s) = \mathcal{L}[f(t)]$

The proof follows from the definition and a change of variable. Note that the unit step function ensures that the integration runs from zero to infinity.

Convolution Property: $f \otimes g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau \Rightarrow \mathcal{L}[f \otimes g(t)] = \mathcal{L}[f(t)]\, \mathcal{L}[g(t)]$

Application LT1: Solution of an Inhomogeneous Differential Equation:

A simple harmonic oscillator is initially at rest at its equilibrium position. At $t = 0$, a constant unit force is applied. Find the response of the oscillator for $t > 0$. [$m = 1$; $k = 4$; $F_o = 1$]

$$\frac{d^2 y}{dt^2} + 4\,y = u(t) \;\rightarrow\; y^{[2]} + 4\,y = u(t)$$

Using the linearity property, the differential equation is transformed into an algebraic equation for the Laplace transform of the response $y(t)$.

$$\mathcal{L}[\,y^{[2]}(t)] + 4\,\mathcal{L}[\,y(t)] = \mathcal{L}[\,u(t)]$$

The final transform $\mathcal{L}[\,u(t)]$ is found in the table and is s^{-1}. Using the derivative property and the initial conditions $y(0) = 0$ and $y^{[1]}(0) = 0$, the DE becomes:

$$s^2\,\mathcal{L}[\,y(t)] - s\,y(\,0^+) - y^{[1]}(\,0^+) + 4\,\mathcal{L}[\,y(t)] = (s^2 + 4)\,\mathcal{L}[\,y(t)] = \mathcal{L}[\,u(t)] = s^{-1}$$

Solving, $\mathcal{L}[\,y(t)] = s^{-1}\,(s^2 + 4)^{-1}$ or:

$$y(t) = \mathcal{L}^{-1}[s^{-1}\,(s^2 + 4)^{-1}]$$

An approach to inverting the transform is to be presented to illustrate the use of the integral property. A more common alternative is presented at the end of Application LT3.

Now the magic of *inversion by matching* begins. First the piece $(s^2 + 4)^{-1}$ is matched.

$$\mathcal{L}^{-1}[(s^2 + 4)^{-1}] = (^1/_2)\,\sin(\,2\,t\,)$$

The factor s^{-1} appeared in the integral property:

$$\mathcal{L}[\int_a^t f(t')\;dt'\,] = s^{-1}\,\mathcal{L}[\,f(t)] + s^{-1}\;\int_a^0 f(t')\;dt'$$

$$s^{-1}\,\mathcal{L}[\,f(t)] = s^{-1}\;\int_a^0 f(t')\;dt' - \mathcal{L}[\int_a^t f(t')\;dt'\,]$$

$$s^{-1}\,\mathcal{L}[(^1/_2)\,\sin(\,2\,t\,))] = [s^{-1}\,(s^2 + 4)^{-1}] \text{ so, setting } a = 0:$$

$$y(t) = \left(\tfrac{1}{2}\right)\int_0^t \sin(2t')\;dt' = \left(\tfrac{1}{4}\right)[1 - \cos(2t)] \qquad y'(t) = y^{[1]}(t) = \left(\tfrac{1}{2}\right)\sin(2t)$$

The oscillator executes simple harmonic motion about its new equilibrium position $y = +\,^1/_4$. It was at rest at time zero and so has a limiting velocity as time approaches zero from positive values of zero because the force applied and hence the mass's acceleration are finite. As the acceleration is defined, the velocity is a continuous function of time.

Application LT2: Solution of an Inhomogeneous Differential Equation:

A simple harmonic oscillator is initially at rest at its equilibrium position. For $t > 0$, a decaying force e^{-rt} is applied. Find the response of the oscillator for $t > 0$. [$m = 1$; $k = 4$; $F_o = 1$]

$$\frac{d^2 y}{dt^2} + 4\,y = F(t) \;\rightarrow\; y^{[2]} + 4\,y = F(t) = e^{-rt}$$

First, the transform of the decaying exponential is located in the table. $\mathcal{L}[e^{-rt}] = \frac{1}{s+r}$, a result that follows from the transform of $u(t)$ and shift property #1.

$$s^2\mathcal{L}[\,y(t)] - s\,y(\,0^+) - y^{[1]}(\,0^+) + 4\,\mathcal{L}[\,y(t)] = (s^2 + 4)\mathcal{L}[\,y(t)] = \mathcal{L}[\,F(t)] = \frac{1}{s+r}$$

$$\mathcal{L}[\,y(t)] = (s + r)^{-1}\,(s^2 + 4)^{-1}$$

The plan is to shift out of this problem.

$$\mathcal{L}\,[y(t)] = \left(\frac{1}{s+r}\right)\left(\frac{1}{s^2+4}\right) \Rightarrow \mathcal{L}\,[e^{-rt}\,y(t)] = \left(\frac{1}{s}\right)\left(\frac{1}{(s-r)^2+4}\right) = \left(\frac{1}{s}\right)\mathcal{L}\left[\left(\tfrac{1}{2}\right)e^{rt}\,\sin(2t)\right]$$

$$e^{-rt}\,y(t) = \int_o^t \left(\tfrac{1}{2}\right) e^{rt'}\,\sin(2t')\,dt' = \frac{2 + e^{+rt}\left(r\sin(2t) - 2\cos(2t)\right)}{8 + 2r^2}$$

The integral is straight forward after sin(2 t) is expressed in terms of $e^{i\omega t}$ and $e^{-i\omega t}$. It is treated in two problems in the Integration/Definite Integrals handout.

$$y(t) = \frac{2\,e^{-rt} + r\sin(2t) - 2\cos(2t)}{8 + 2r^2}$$

The solution found in application LT1 is easily understood and can be found without Laplace transforms. Could you have found the solution to application LT2 by another method?

Use the Mathematica code below to verify that $y(t)$ is a solution to the equation and that $y(0)$ and $y'(0)$ are zero. Set $r = 0.25$ and Plot[y[t],{t,0,50}]. Discuss the behavior. Change r and repeat.

```
Mathematica Verification
Integrate[Exp[r t] Sin[ 2 t]/2,{t,0,T}]
```

```
y[t_] = (2 Exp[- r t] -2 Cos[2 t] + r Sin[2 t])/(8+2 r^2)
dy[t_] = D[y[t],t]
ddy[t_] = D[D[y[t],t],t]
FullSimplify[ddy[t] + 4 y[t]]
r = 0.25; Plot[y[t],{t,0,50}]
```

Application LT3: Driven second Order ODE with constant coefficients:

$$y^{[2]}(t) + b\, y^{[1]}(t) + c\, y(t) = d\, F(t)$$

$$\{s^2\, \mathcal{L}[y(t)] + s\, y(0) + y^{[1]}(0)\} + b\, \{ s\, \mathcal{L}[y(t)] + y(0) \} + c\, \mathcal{L}[y(t)] = d\, \mathcal{L}[F(t)]$$

$$\{s^2 + b\, s + c\}\, \mathcal{L}[y(t)] = d\, \mathcal{L}[F(t)] + s\, y(0) + (b\, y(0) + y^{[1]}(0))$$

$$\mathcal{L}[y(t)] = [d\, \mathcal{L}[F(t)] + s\, y(0) + (b\, y(0) + y^{[1]}(0))] * [s^2 + b\, s + c\,]^{-1}$$

Consider a particular example.

$$\frac{d^2 y}{dt^2} - 3\frac{dy}{dt} + 2\, y = 2\, e^{-t}$$

$$b = -3; \quad c = 2; \; d = 2; \; y(0) = 2; \; y^{[1]}(0) = 1; \quad F(t) = e^{-t}; \; \mathcal{L}[F(t)] = (s+1)^{-1}$$

$$\mathcal{L}\left[y(t)\right] = \frac{(s+1)^{-1}}{s^2 - 3s + 2} = \frac{1}{(s+1)(s-2)(s-1)} = \frac{A}{(s+1)} + \frac{B}{(s-2)} + \frac{C}{(s-1)}$$

Review partial fractions (RHB, p.18); they can be useful in untangling Laplace transforms.

The requirements are:

$$A\,(s^2 - 3\,s + 2) + B\,(s^2 - 1) + C\,(s^2 - s - 2) = 2\,s^2 - 3\,s - 3 \quad \text{or}$$

$$A + B + C = 2; \qquad -3\,A - C = -3; \qquad 2\,A - B - 2\,C = -3$$

Solving, it follows that: $A = {}^1/_3$; $B = -{}^1/_3$; $C = 2$

From the tables, $\mathcal{L}^{-1}[(s+1)^{-1}] = e^{-t}$; $\mathcal{L}^{-1}[(s-2)^{-1}] = e^{2t}$; $\mathcal{L}^{-1}[(s-1)^{-1}] = e^{t}$. Hence,

$$\boxed{y(t) = {}^1/_3\, e^{-t} - {}^1/_3\, e^{2t} + 2\, e^{t}\,.}$$

Returning to Application: LT2: $\frac{d^2 y}{dt^2} + 4\, y = e^{-rt}$ with homogeneous initial conditions.

$b = 0;\quad c = 4;\ d = 1;\ y(0) = 0;\ y^{[1]}(0) = 0;\quad F(t) = e^{-r\,t};\ \mathcal{L}[F(t)] = (s + r)^{-1}$

$$\mathcal{L}\left[y(t)\right] = \frac{(s+r)^{-1}}{s^2+4} = \frac{1}{(s+r)(s-2i)(s+2i)} = \frac{A}{(s+r)} + \frac{B}{(s-2i)} + \frac{C}{(s+2i)}$$

The requirements are:

$$A\,(s^2 + 4) + B\,(s^2 + [r + 2i]\,s + 2\,r\,i) + C\,(s^2 + [r - 2i]\,s - 2\,r\,i) = 1 \quad \text{or}$$

$$A + B + C = 0; \qquad [r + 2i]\,B + [r - 2i]\,C = 0; \qquad 4\,A + 2\,r\,i\,B - 2\,i\,r\,C = 1$$

After some effort, $A = \dfrac{2}{8+2r^2}$; $B = \dfrac{r+2i}{2i\left(8+2r^2\right)}$; $C = \dfrac{-r-2i}{2i\left(8+2r^2\right)}$

$$\mathcal{L}^{-1}[(s - r)^{-1}] = e^{rt}\ ;\ \mathcal{L}^{-1}[(s - 2i)^{-1}] = e^{+i2t};\ \mathcal{L}^{-1}[(s + 2i)^{-1}] = e^{-i2t}$$

$$y(t) = \frac{2}{8+2r^2}e^{-rt} + \frac{r+2i}{2i\left(8+2r^2\right)}e^{+2it} + \frac{-r-2i}{2i\left(8+2r^2\right)}e^{-2it}$$

$$y(t) = \frac{2e^{-rt} + r\sin(2t) - 2\cos(2t)}{8+2r^2}$$

There are multiple paths that lead to the answer. Inverting Laplace transforms by manipulating and matching is an art that requires practice and luck. Prepare by working through the details of a long list of examples.

Additional Integral Transforms

Fourier Bessel or Hankel Transform

$$g(k) = \int_0^\infty f(x)\,J_m(kx)\,x\,dx$$

$$f(x) = \int_0^\infty g(k)\,J_m(kx)\,k\,dk$$

Mellin Transform

$$\phi(z) = \int_0^\infty t^{z-1}\,f(t)\,dt$$

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} t^{-z}\,\phi(z)\,dz$$

Hilbert Transform

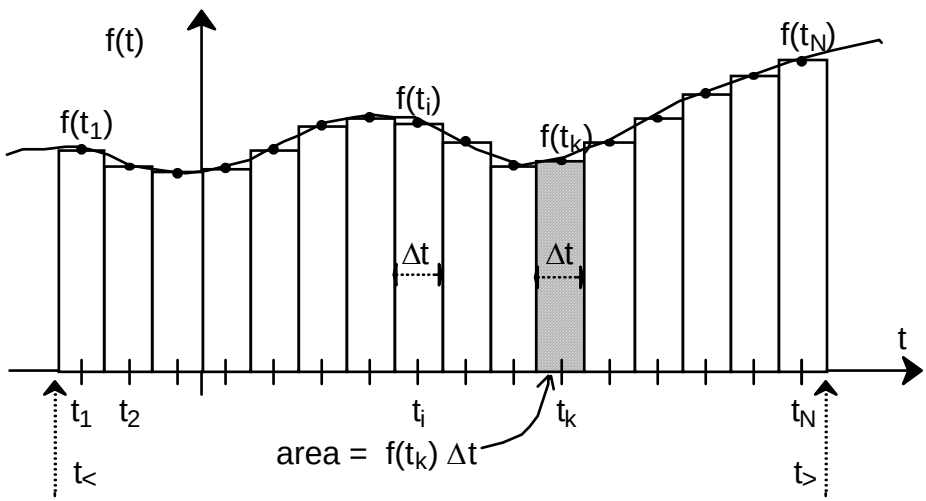
$$g(y) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{f(x)dx}{x-y}$$

$$f(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{g(y)dy}{y-x}$$

Tools of the Trade:

Converting Sums to Integrals

It is said that an integral is a sum of little pieces, but some precision is required before the statement becomes useful. Beginning with a function *f(t)* and a sequence of values for $t = \{t_1, t_2, t_3, \ldots, t_N\}$, the sum $\sum_{i=1}^{i=N} f(t_i)$ does not represent the integral $\int_{t_<}^{t_>} f(t)\,dt$ even if a great many closely spaced values of t are used. Nothing has been included in the sum to represent *dt*. One requires $\sum_{i=1}^{i=N} f(t_i)\,\Delta t_i$ where $\Delta t_i = \left(\frac{1}{2}\right)[t_{i+1} - t_{i-1}]$ is the average interval between sequential values of *t* values at t_i. For well-behaved cases, the expression $\sum_{i=1}^{i=N} f(t_i)\,\Delta t_i$ approaches the Riemann sum definition of an integral as the t-axis is chopped up more and more finely. As illustrated below, in the limit Δt goes to zero, the sum $\sum_{i=1}^{i=N} f(t_i)\,\Delta t_i$ approaches the area under the curve between $t_<$ and $t_>$. That is: it represents $\int_{t_<}^{t_>} f(t)\,dt$ provided the sequence of sums converges, and life is good. The theory of integration is not the topic of this passage. The goal is simply to remind you that the Δt must be factored out of each term that is being summed in order to identify the integrand.



Problems

1.) The convolution of two functions is: $f \circ g(x) = \int_{-\infty}^{\infty} f(x')\, g(x-x')\, dx'$.

Show that $\int_{-\infty}^{\infty} f(x')\, g(x-x')\, dx' = \int_{-\infty}^{\infty} g(x')\, f(x-x')\, dx'$.

2.) Parseval's equality, $\int_{-\infty}^{\infty} (g(x))^* \, f(x)\, dx = \int_{-\infty}^{\infty} (\tilde{g}(k))^* \, \tilde{f}(k)\, dk$, follows by replacing both functions in the inner product with their Fourier transform representations using $\tilde{g}(\ell)$ and $\tilde{f}(k)$, and then interchanging the orders of integration to complete the *x* integration first. Show the steps in this development. (It is assumed that *k* and ℓ were chosen as the *distinct* Fourier dummy variable labels for the functions *f* and *g*. Property A of the Fourier transform provides the relation between the *x* integral and the Dirac delta.)

$$f(x) \;=\; \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\tilde{f}(k)\right]e^{ikx}dk \quad ; \quad g(x) \;=\; \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\left[\tilde{g}(\ell)\right]e^{i\ell x}d\ell$$

3.) Show that the Fourier transform of the convolution of two functions is the product of

their Fourier transforms. $\widetilde{f \circ g}(k) = [2\pi]^{\alpha}\ \tilde{f}(k)\,\tilde{g}(k)$ where α may be ½ but can be other values depending on the precise definition chosen for the convolution and the apportionment of the 2π in the definition of the Fourier transform and its inverse.

4.) Compute the Fourier transform of the continuous piecewise smooth function:

$$f(x) = \begin{cases} x+1 & for\ -1 < x < 0 \\ 1-x & for\ \ 0 < x < 1 \\ 0 & for\ \ \ |x| > 1 \end{cases}$$

Sketch the function. What is the lowest order in which a derivative of this function is discontinuous? What does property H predict about the Fourier transform of this function? $\tilde{f}(k) = \sqrt{\frac{2}{\pi}}\ \frac{1-\cos(k)}{k^2}$

$$Answer:\ \sqrt{2/\pi}\left(\frac{1-\cos(k)}{k^2}\right) \text{for } S = ½;\ \text{How can } (1 - \cos[k]) \text{ be rewritten?}$$

$$\text{The } S = 0 \text{ choice answer is } \frac{2\left(1-\cos(k)\right)}{k^2} = \frac{4}{k^2}\sin^2\left(k/2\right)$$

5.) The Fourier transform of the somewhat smooth function below is $\frac{n!}{\sqrt{2\pi}}\ \ \frac{1}{(1+ik)^{n+1}}$:

$$f(x) = \begin{cases} 0 & for\ x < 0 \\ x^n\ e^{-x} & for\ x > 0 \end{cases}$$

Sketch the function. What is the lowest order in which a derivative of this function is discontinuous? What does property H predict about the Fourier transform of this function? Compute the Fourier transform for the case $n = 1$.

6.) Find the Fourier transform of the continuous piecewise smooth function:

$f(x) = e^{-a|x|}; real\ a > 0$.

Sketch the function. What is the lowest order in which a derivative of this function is discontinuous? What does the property H predict about the Fourier transform of this function?

$$Answer: \quad \frac{2a}{\sqrt{2\pi}\ (a^2+k^2)}$$

7.) Compute the Fourier Transform of $f(t) = \frac{1}{\sqrt{\tau\sqrt{\pi}}} e^{-(t)^2/2\tau^2}\, e^{-i\omega_0 t}$. Verify that the product of the temporal width of the function τ and the spectral width of the transform Δω is of order 1. The technique of choice is to complete the square in the exponent and use change of variable.

$$-[t^2/2\tau^2 + ibt] = -\left[t/\sqrt{2}\tau + ib\,\tau/\sqrt{2}\right]^2 + ???$$

Compare with problem 20.

8.) An amplitude modulated signal is of the form $f(t) = A(t)\cos(\Omega t)$ where Ω is the carrier frequency (Ω >> ω) and $A(t)$ is an amplitude modulation function that carries the information in the signal. Show that: $\tilde{f}(\omega) = (1/2)\left[\tilde{A}(\Omega+\omega) + \tilde{A}(-\Omega+\omega)\right]$. Suppose that the signal $A(t)$ has frequency components spread from - ω_{signal} to + ω_{signal}. The point is that if you wish to encode information with frequency spread ± ω_{signal} on a carrier wave at frequency Ω, you need the bandwidth from Ω - ω_{signal} to Ω + ω_{signal} to carry the signal. For $A(t)$ corresponding to rapid variations, $\tilde{A}(\omega)$ must include amplitudes for high frequencies meaning that a large bandwidth is required to transmit the signal. We normally describe the required band of frequencies as: Ω - ω_{signal} to Ω + ω_{signal}.

9.) Compute the Fourier transform of the Dirac delta function $\delta(t - t_0)$. Discuss its behavior for large |ω| in the context of property H.

10.) Compute the Laplace transform of t^2.

11.) Compute the Laplace transform of sin(ω t).

12.) Prove that $\mathcal{L}[\int_a^t f(t')\ dt'\ ] = s^{-1}\,\mathcal{L}[\ f(t)] + s^{-1}\ \int_a^0 f(t')\ dt'$. Use the defining integral for the Laplace transform and integration by parts.

13.) Iterate the derivative property of the Laplace transform to show that:

$$\mathcal{L}[f^{[2]}(t)\ ] = s^2\,\mathcal{L}[\ f(t)] - s\,f(\ 0^+) - f^{[1]}(\ 0^+)$$

14.) A partial fraction problem arose during one of the Laplace transform applications.

$$\frac{1}{(s+r)(s-2i)(s+2i)} = \frac{A}{(s+r)} + \frac{B}{(s-2i)} + \frac{C}{(s+2i)}$$

Find the values of the complex constants A, B and C. The equation is equivalent to:

$$A(\ s - 2i)\ (\ s + 2i) + B\ (\ s + r)\ (\ s + 2i) + C\ (\ s + r)\ (\ s + 2i) = 1$$

The coefficient of s^2 should vanish as should the coefficient of s. The constant term should be 1. *Partial Answer*: $C = \dfrac{-r-2i}{2i\left(8+2r^2\right)}$

15.) Solve the following DE using Laplace transform methods. Interpret the answer.

$$L\frac{di}{dt} + Ri = E(t) \text{ with } i(t) = i_0 \text{ and } E(t) = \begin{cases} V_0 & for\ 0 \le t < \pi \\ 0 & for\quad \pi \le t \end{cases}$$

That is $E(t) = V_0\ [u(t) - u(t - \pi)]$.

a.) Compute $\mathcal{L}[E(t)]$. You should do the using the table and the theorems and by direct computation.

b.) Transform the equation and find $\mathcal{L}[i(t)]$. Group the terms to represent the response to the change at $t = 0$, the response to the change at $t = \pi$ and the homogeneous solution piece.

Answer: $i(t) = \left(V_0/R\right)\left[1 - e^{-R/L\,t}\right] - \left(V_0/R\right)u(t-\pi)\left[1 - e^{-R/L\,(t-\pi)}\right] + i_0\,e^{-R/L\,t}$

c.) Interpret the terms in the expression for $i(t)$. Explain the role of the unit step function in the second term.

16.) Compute the Laplace transform of $f(t) = \begin{cases} 1 & for\ 0 \le t < 1 \\ e^t & for\ \ 1 \le t \end{cases}$. The definition of *ft*) can be rewritten as $f(t) = [u(t) - u(t-1)] + e\,[u(t-1)\,e^{(t-1)}]$.

a.) Using the table and shift property #2, show that $\mathcal{L}[f(t)] = s^{-1} - s^{-1}\,e^{-s} + e\,\{\,e^{-s}/(s-1)\}$. Explain the use of each resource and property.

b.) Show that the result also follows from direct calculation -- $\int_0^\infty f(t)\,e^{-st}\,dt$.

$$\int_0^\infty f(t)\,e^{-st}\,dt = \int_0^1 e^{-st}\,dt + \int_1^\infty e^t\,e^{-st}\,dt = \frac{1-e^{-s}}{s} + \left.\frac{e^{(1-s)t}}{1-s}\right|_1^\infty = \frac{1-e^{-s}}{s} + \frac{e^{-(s-1)}}{s-1}$$

17.) The auto-correlation of a function $f(x)$ is defined as:

$$A(x) = \int_{-\infty}^{\infty} f(x')^*\ f(x'+x)\,dx'$$

Note that an autocorrelation is similar to the inner product of a function with itself. It differs in that the function at x is compared to the function at $x + x'$ rather than for the same argument value. The inner product gauges the degree to which the two functions wiggle in the same pattern. The auto-correlation gauges the degree to which a function's local wiggle pattern persists as the argument changes. Show that the Fourier transform of a functions autocorrelation is the product of that function's Fourier transform with its complex conjugate.

$$\{\tilde{A}(x)\} = \{\int_{-\infty}^{\infty} f(x')^*\ \tilde{f}(x'+x)\,dx'\} = \tilde{f}(k)\ \tilde{f}(k)^* = |\,\tilde{f}(k)\,|^2$$

18.) The cross-correlation of two function $f(x)$ and $g(x)$ is defined as:

$$C(x) = \int_{-\infty}^{\infty} f(x')^*\ g(x'+x)\,dx'$$

Express the Fourier transform of the cross-correlation of the functions in terms of the Fourier transforms of the individual functions. A fairly direct solution follows if you replace $f(x)$ and $g(x)$ by their equivalent Fourier integrals. Next interchange the orders of integration and use the delta function property. $\{\tilde{C}(x)\} =$ **?**

19.) Compute the Fourier Transform of $f(t) = \frac{1}{\sqrt{\tau\sqrt{\pi}}} e^{-(t-t_0)^2/2\tau^2} e^{-i\omega_o t}$. * Problem changed since spring 2006.* Verify that the product of the temporal width of the function τ and the spectral width of the transform Δω is of order 1. Compare with the result that the Gaussian $\left(\frac{1}{a^{1/2}\pi^{1/4}}\right) e^{-t^2/2a^2}$ transforms to $\left(\frac{a^{1/2}}{\pi^{1/4}}\right) e^{-(\omega a)^2/2}$. Discuss the result in terms of the translation or the linear phase property of the Fourier transform. The temporal function $\frac{1}{\sqrt{\tau\sqrt{\pi}}} e^{-t^2/2\tau^2}$ has been translated from $t = 0$ to $t = t_o$ (which leads to a linear phase times the transform of $f(t) = \frac{1}{\sqrt{\tau\sqrt{\pi}}} e^{-t^2/2\tau^2}$) and then the temporal function is multiplied by a linear phase which translates the transform from a result centered on $\omega = 0$ to one centered on $\omega = \omega_o$.

20.) Compute the Fourier Transform of $f(t) = e^{-i\Omega t} e^{-2(t-t_0)^2/\tau^2} e^{-i\omega_o t}$. Use the quantum conventions to compute the uncertainty product $\Delta\omega\, \Delta t$ for the function. Search for: "Following conventions adopted in Quantum Mechanics".

Use the properties to start with $e^{-(t/\tau)^2}$. multiply by a linear phase and then translate by t_o to reproduce the result.

21.) A standard trigonometric Fourier series for a function $f(x)$ with period L has the form: $f(x) = c_0(1) + \sum_{m=1}^{\infty} a_m \cos[mk_0 x] + \sum_{m=1}^{\infty} b_m \sin[mk_0 x]$ where $k_o = {}^{2\pi}/_{L}$.

a.) Show that this can be cast in the form:

$$f(x) = c_0\, e^{i(0)} + \sum_{m=1}^{\infty} \tfrac{1}{2}(a_m - ib_m) e^{imk_0 x} + \tfrac{1}{2}(a_m + ib_m) e^{-imk_0 x} = \sum_{m=-\infty}^{\infty} \alpha_m\, e^{imk_0 x}$$

This result justifies the form of the complex Fourier series, $\sum_{m=-\infty}^{\infty} \alpha_m\, e^{imk_0 x}$.

b.) Show that $(L^{-1})\int_{-L/2}^{L/2}(e^{imk_0x})^* e^{ink_0x}\,dx = \delta_{mn}$.

c.) Pre-multiply $f(x) = \sum_{m=-\infty}^{\infty} \alpha_m e^{imk_0x}$ by $(e^{ipk_0x})^*$ and use the orthogonality relation developed in part b to project out the coefficient α_p.

22. Sample Calculation SC4 used the linear phase property and then used the translation property to compute the Fourier transform of $g(t) = \left(\frac{1}{a^{1/2}\pi^{1/4}}\right) e^{-\left((t-t_0)^2/2a^2\right)} e^{-i\omega_0 t}$. Repeat the problem using the translation property first and the linear phase property second.

23. Sample Calculation SC4 used the linear phase and linear phase properties to compute the Fourier transform of $g(x) = \left(\frac{1}{a^{1/2}\pi^{1/4}}\right) e^{-\left((x-x_0)^2/2a^2\right)} e^{ik_0x}$.

24. Compute the Fourier transform of $g(x) = \left(\frac{1}{a^{1/2}\pi^{1/4}}\right) e^{-\left((x-x_0)^2/2a^2\right)} e^{ik_0x}$ directly using the defining equation for and using the completing the square in the exponent method.

25. Consider the Laplace transform $[(s+a)\,(s+b)]^{-1}$.

a.) Use partial fractions to reach a form that can be inverted using tables. Invert it.

b.) Use the convolution theorem $f \otimes g(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau \Rightarrow \mathcal{L}[\ f \otimes g(t)\ ] = \mathcal{L}[f(t)]$ $\mathcal{L}[g(t)]$

to invert the Laplace transform.

c.) The inverse of a Laplace transform can be computed directly using the Bromwich integral

$f(t) = (2\pi i)^{-1}\int_C \tilde{f}(z)e^{tz}dz$ where $\tilde{f}(z)$ is $\mathcal{L}[g(t)]_{s=z}$. The contour consists of a straight path up parallel to the imaginary axis which is to the right of all the poles of $\tilde{f}(z)$ and which is closed by a large circular arc closing on the left to enclose all those poles. Do so.

25. Consider the Laplace transform $[(s^2+1)(s-1)]^{-1}$.

a.) Use partial fractions to reach a form that can be inverted using tables. Invert it.

b.) Use the convolution theorem $f \otimes g(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau \Rightarrow \mathcal{L}[\ f \otimes g(t)\ ] = \mathcal{L}[f(t)]$

$\mathcal{L}[g(t)]$

to invert the Laplace transform.

c.) The inverse of a Laplace transform can be computed directly using complex integration methods and the Bromwich integral $f(t) = (2\pi i)^{-1}\int_C \tilde{f}(z)e^{tz}dz$ where $\tilde{f}(z)$ is $\mathcal{L}[f(t)]_{s=z}$. The contour consists of a straight path up parallel to the imaginary axis which is to the right of all the poles of $\tilde{f}(z)$ and which is closed by a large circular arc closing on the left to enclose all those poles. Do so. Note: The arc closing to the left does not contribute so long as $t > 0$. For $t > 0$, the contour must be closed on the right leading to a result of 0. The $t > 0$ case is all that is of direct interest.

26.) The inverse of a Laplace transform can be computed directly using complex integration methods and the Bromwich integral $f(t) = (2\pi i)^{-1}\int_C \tilde{f}(z)e^{tz}dz$ where $\tilde{f}(z)$ is $\mathcal{L}[f(t)]_{s=z}$. The contour consists of a straight path up ***parallel to the imaginary axis and to the right of all the poles*** of $\tilde{f}(z)$ and which is closed by a large circular arc closing

on the left thus enclosing all the poles of $\tilde{f}(z)$. Compute the inverse Laplace transforms of: a.) $(s-k)^{-1}$ b.) $(s-k)^{-2}$ c.) $(s-a)^{-1}(s-b)^{-1}$ d.) $s\,(s^2+k^2)^{-1}$ e.) $k\,(s^2+k^2)^{-1}$

Answers: e^{kt}, $t\,e^{kt}$, $(a-b)^{-1}\,[e^{-bt}-e^{-at}]$, $\cos(kt)$, $\sin(kt)$

The arc closing to the left does not contribute so long as $t > 0$. For $t > 0$, the contour must be closed on the right leading to a result of 0. The $t > 0$ case is all that is of direct interest.

27.) The expression $f(x) = \sum_{m=-\infty}^{\infty} \alpha_m \exp\left[i\,m\left(2\pi/L\right)x\right]$ can be represented as an integral by noting that Δm between terms is just 1. That is:

$$f(x) = \sum_{m=-\infty}^{\infty} \alpha_m \exp\left[i\,m\left(2\pi/L\right)x\right]\Delta m \approx \int_{-\infty}^{\infty} A(m)\exp\left[i\,m\left(2\pi/L\right)x\right]dm$$

Note that $A(m)$ is a continuous function with the property that $A(m) = \alpha_m$. In order that the integral form be proven accurate, it must be shown that the integrand changes by an infinitesimal amount for a change of $dm = 1$.

a.) Show that $\exp[im(^{2\pi}/_L)x] \to 1$ as $L \to \infty$.

b.) Apply the change of variable $k = (^{2\pi}/_L)\,m$ in the integral to show that the standard expression for the Fourier transform is recovered.

Note that $\tilde{f}(k) = L\,A(\frac{L}{2\pi}m)\,A(m)$.

References:

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3. K. F. Riley, M. P. Hobson and S. J. Bence, ***Mathematical Methods for Physics and Engineering***, 2nd Ed., Cambridge, Cambridge UK (2002).

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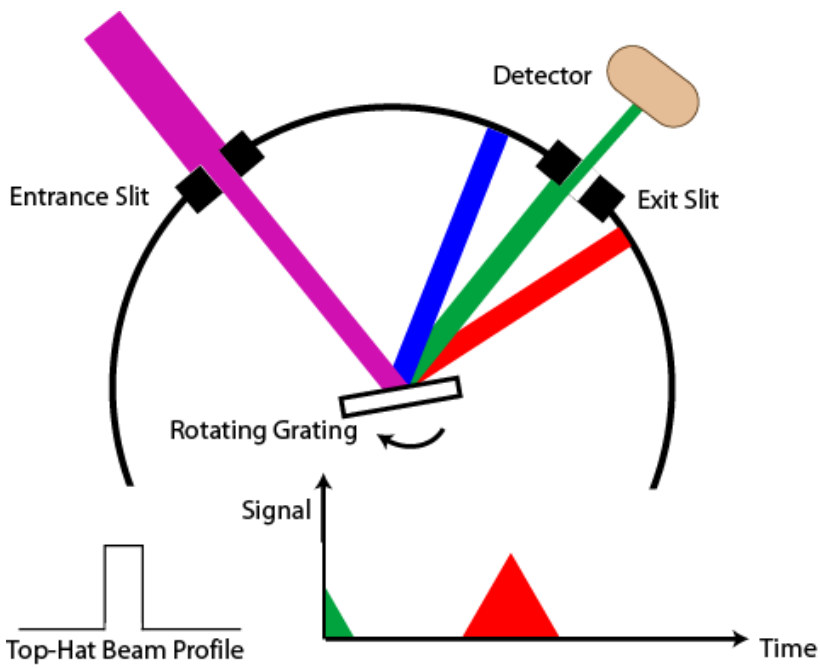
Spectrometers and Convolution



Model In Mathematica:

The actual spectral line pattern

Diffraction limited transfer

Slit to Slit convolution

Diffraction * slit- to –slit

Full * spectrum

AUTOFOCUS optimize high k part of FT

Hartley transform

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In mathematics, the **Hartley transform** is an integral transform closely related to the Fourier transform, but which transforms real-valued functions to real-valued functions. It was proposed as an alternative to the Fourier transform by R. V. L. Hartley in 1942, and is one of many known Fourier-related transforms. Compared to the Fourier transform, the Hartley transform has the advantages of transforming real functions to real functions (as opposed to requiring complex numbers) and of being its own inverse.

The discrete version of the transform, the Discrete Hartley transform, was introduced by R. N. Bracewell in 1983.

The two-dimensional Hartley transform can be computed by an analog optical process similar to an optical Fourier transform, with the proposed advantage that only its amplitude and sign need to be determined rather than its complex phase (Villasenor, 1994). However, optical Hartley transforms do not seem to have seen widespread use.

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Definition

The Hartley transform of a function $f(t)$ is defined by:

$$H(\omega) = \{\mathcal{H}f\}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\mathrm{cas}(\omega t)\mathrm{d}t,$$

where ω can in applications be an angular frequency and

$$\mathrm{cas}(t) = \cos(t) + \sin(t) = \sqrt{2}\sin(t + \pi/4) = \sqrt{2}\cos(t - \pi/4)$$

is the cosine-and-sine or *Hartley* kernel. In engineering terms, this transform takes a signal (function) from the time-domain to the Hartley spectral domain (frequency domain).

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Inverse transform

The Hartley transform has the convenient property of being its own inverse (an involution):

$$f = \{\mathcal{H}\{\mathcal{H}f\}\}$$

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Conventions

The above is in accord with Hartley's original definition, but (as with the Fourier transform) various minor details are matters of convention and can be changed without altering the essential properties:

- Instead of using the same transform for forward and inverse, one can remove the $1/\sqrt{2\pi}$ from the forward transform and use 1 / 2π for the inverse—or, indeed, any pair of normalizations whose product is 1 / 2π. (Such asymmetrical normalizations are sometimes found in both purely mathematical and engineering contexts.)
- One can also use 2π*vt* instead of ω*t* (i.e., frequency instead of angular frequency), in which case the $1/\sqrt{2\pi}$ coefficient is omitted entirely.
- One can use cos−sin instead of cos+sin as the kernel.

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Relation to Fourier transform

This transform differs from the classic Fourier transform $F(\omega) = \{\mathcal{F} f(t)\}(\omega)$ in the choice of the kernel. In the Fourier transform, we have the exponential kernel: $\exp(-i\omega t) = \cos(\omega t) - i\sin(\omega t)$, where *i* is the imaginary unit.

The two transforms are closely related, however, and the Fourier transform (assuming it uses the same $1/\sqrt{2\pi}$ normalization convention) can be computed from the Hartley transform via:

$$F(\omega) = \frac{H(\omega) + H(-\omega)}{2} - i\frac{H(\omega) - H(-\omega)}{2}$$

That is, the real and imaginary parts of the Fourier transform are simply given by the even and odd parts of the Hartley transform, respectively.

Conversely, for real-valued functions *f*(*t*), the Hartley transform is given from the Fourier transform's real and imaginary parts:

$$\{\mathcal{H} f\} = \Re\{\mathcal{F} f\} - \Im\{\mathcal{F} f\} = \Re\{\mathcal{F} f \cdot (1+i)\}$$

where $\Re$ and $\Im$ denote the real and imaginary parts of the complex Fourier transform.

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Properties

One can see immediately from the definition that the Hartley transform is a real linear operator, and is symmetric (and Hermitian). From the symmetric and self-inverse properties, it follows that the transform is a unitary operator (indeed, orthogonal).

There is also an analogue of the convolution theorem for the Hartley transform. If two functions *x*(*t*) and *y*(*t*) have Hartley transforms *X*(ω) and *Y*(ω), respectively, then their convolution *z*(*t*) = *x* * *y* has the Hartley transform:

$$Z(\omega) = \{\mathcal{H}(x*y)\} = \sqrt{2\pi}\left(X(\omega)\left[Y(\omega) + Y(-\omega)\right] + X(-\omega)\left[Y(\omega) - Y(-\omega)\right]\right)/2$$

Similar to the Fourier transform, the Hartley transform of an even/odd function is even/odd, respectively.

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cas

The properties of the *cas* function follow directly from trigonometry, and its definition as a phase-shifted trigonometric function $\mathrm{cas}(t) = \sqrt{2}\sin(t + \pi/4)$. For example, it has an angle-addition identity of:

$$2\mathrm{cas}(a+b) = \mathrm{cas}(a)\mathrm{cas}(b) + \mathrm{cas}(-a)\mathrm{cas}(b) + \mathrm{cas}(a)\mathrm{cas}(-b) - \mathrm{cas}(-a)\mathrm{cas}(-b)$$

Additionally:

$$\mathrm{cas}(a+b) = \cos(a)\mathrm{cas}(b) + \sin(a)\mathrm{cas}(-b) = \cos(b)\mathrm{cas}(a) + \sin(b)\mathrm{cas}(-a)$$

and its derivative is given by:

$$\mathrm{cas}'(a) = \frac{\mathrm{d}}{\mathrm{d}a}\mathrm{cas}(a) = \cos(a) - \sin(a) = \mathrm{cas}(-a)$$

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References