Tools of the Trade

Cyclic permutation of the rotation matrices:

There are many examples in which the labels *x, y, z* are permuted to generate the corresponding form for another index once one is known. The matrix form representing a rotation about the *z*-axis is known.

$$\mathbb{R}_{\mathbf{z}}(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To generate the form for a rotation about the *x*-axis, the indices are permuted or cycled using the rule $x \to y \to z \to x \to \ldots$. This change shifts each element down one space and to the right one space. The elements *roll-over* to the top and left-side as they exit the bottom and right side.

$$\mathbb{R}_{\mathbf{z}}(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \mathbb{R}_{\mathbf{x}}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \to \mathbb{R}_{\mathbf{y}}(\phi) = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}$$

Projection Operators:

A projection mapping of 3D vectors into the *x-y* –plane served as an example in this section. The projection can be realized as an operator $P_{\to x-y} = \hat{i}\,(\hat{i}\cdot\,) + \hat{j}\,(\hat{j}\cdot\,) = \sum_{m=1}^{2} \hat{e}_m\,(\hat{e}_m\cdot\,)$. A common operation is to project one vector onto another using the inner product. The following operator projects vector onto the direction of $\vec{B}$. $P_{dir\,\vec{B}} = (\vec{B}\cdot\vec{B})^{-1}\ \vec{B}(\vec{B}\cdot\ ) = \left[\langle B|B\rangle\right]^{-1}\ |B\rangle\langle B|$. The hallmark of a projection operator *P* is that *PP* = *P*. That is: repeating the projection any number of times yields the same result as applying the operation once.

Warm-Up Problems:

WUP1.) Draw a set of Cartesian axes to represent the unprimed coordinate system. The primed system is rotated by θ about the original *z* axis. That is: the primed and unprimed systems share a *z* direction, and the *x′* and *y′* directions make angles of +θ with their unprimed counterparts. Add the primed axis to your drawing and identify the angles θ. Prepare a second drawing that displays and details the *x-y* plane only. Use this drawing to identify the angles that separated each unprimed-primed direction pair. Use your drawing to replace each element of $\mathbb{R}_z(\theta)$ by the appropriate function of θ: ($\mathbb{R}_z(\theta) \Rightarrow$ rotation by θ about the *z* axis)

$$\mathbb{R}_z(\theta) = \begin{bmatrix} (\hat{i}'\cdot\hat{i}) & (\hat{i}'\cdot\hat{j}) & (\hat{i}'\cdot\hat{k}) \\ (\hat{j}'\cdot\hat{i}) & (\hat{j}'\cdot\hat{j}) & (\hat{j}'\cdot\hat{k}) \\ (\hat{k}'\cdot\hat{i}) & (\hat{k}'\cdot\hat{j}) & (\hat{k}'\cdot\hat{k}) \end{bmatrix}$$

WUP2.) Use the process described in WUP1, but adapted to generate the form of $\mathbb{R}_x(\phi)$.

WUP3.) Given that: $\mathbb{R}_{\mathbf{y}}(\phi) = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}$, show that $\mathbb{R}_{\mathbf{y}}(-\phi) = [\mathbb{R}_{\mathbf{y}}(\phi)]^t$.

WUP4.) Given that: $\mathbb{R}_{\mathbf{y}}(\phi) = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}$, show that $\mathbb{R}_{\mathbf{y}}(-\phi) = [\mathbb{R}_{\mathbf{y}}(\phi)]^{-1}$. Compute the matrix product $\mathbb{R}_{\mathbf{y}}(\phi)\,\mathbb{R}_{\mathbf{y}}(-\phi)$ explicitly.

WUP5.) Given that: $\mathbb{R}_{\mathbf{y}}(\phi) = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}$, compute the matrix product $\mathbb{R}_{\mathbf{y}}(\alpha)\,\mathbb{R}_{\mathbf{y}}(\beta)$ explicitly.

Show that $\mathbb{R}_{\mathbf{y}}(\alpha)\,\mathbb{R}_{\mathbf{y}}(\beta) = \mathbb{R}_{\mathbf{y}}(\alpha + \beta)$. Rotations about a *fixed* axis add.

WUP6. The transformation matrix from an initial frame to one rotated about the origin has been identified as $\lambda = \begin{bmatrix} (\hat{i}'\cdot\hat{i}) & (\hat{i}'\cdot\hat{j}) & (\hat{i}'\cdot\hat{k}) \\ (\hat{j}'\cdot\hat{i}) & (\hat{j}'\cdot\hat{j}) & (\hat{j}'\cdot\hat{k}) \\ (\hat{k}'\cdot\hat{i}) & (\hat{k}'\cdot\hat{j}) & (\hat{k}'\cdot\hat{k}) \end{bmatrix}$.

a.) Compute $\mathbb{R}_{\mathbf{y}}(\pi)\,\mathbb{R}_{\mathbf{x}}(\frac{1}{2}\pi)$ and $\mathbb{R}_{\mathbf{x}}(\frac{1}{2}\pi)\,\mathbb{R}_{\mathbf{y}}(\pi)$. Comment on the outcomes.

b.) Consider a rotation of $\frac{1}{2}\pi$ about the *x* axis followed by a rotation π about the *y* axis. Note that: λ for the overall rotation is ($\mathbb{R}_{\mathbf{y}}(\pi)\,\mathbb{R}_{\mathbf{x}}(\frac{1}{2}\pi)$. Sketch of the original coordinate triad $\{\hat{i}, \hat{j}, \hat{k}\}$ followed by the triad as it appears after the first and second rotation. Compute the inner products of the final coordinate directions with the original ones to fill in the elements in the matrix λ. Compare the result with your matrix product result.

b.) Consider a rotation π about the *y* axis followed by a rotation of $\frac{1}{2}\pi$ about the *x* axis. Note that: λ for the overall rotation is $\mathbb{R}_{\mathbf{x}}(\frac{1}{2}\pi)\,\mathbb{R}_{\mathbf{y}}(\pi)$. Make a sketch of the original coordinate triad $\{\hat{i}, \hat{j}, \hat{k}\}$

followed by the triad as it appears after the first and second rotation. Compute the inner products of the final coordinate directions with the original ones to fill in the elements in the matrix λ. Compare your result with $\mathbb{R}_{\mathbf{x}}(\frac{1}{2}\pi)\,\mathbb{R}_{\mathbf{y}}(\pi)$.

Problems

1.) Consider the case of a primed frame rotated by θ about the z-axis relative to the unprimed frame and a double-primed frame rotated by ϕ about the z-axis relative to the primed frame.

$$\begin{bmatrix} a_1^{/} \\ a_2^{/} \\ a_3^{/} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \begin{bmatrix} a_1^{//} \\ a_2^{//} \\ a_3^{//} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1^{/} \\ a_2^{/} \\ a_3^{/} \end{bmatrix}$$

Apply the transformations in sequence to show that:

$$\begin{bmatrix} a_1^{//} \\ a_2^{//} \\ a_3^{//} \end{bmatrix} = \begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) & 0 \\ -\sin(\theta+\phi) & \cos(\theta+\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

'In sequence' means that later operations (rotation matrices) are multiplied on the left. Use well-known trig identities to cast your result in the final form. Why is the result obvious?

In this case, the two rotations can be applied in either order.

$$\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The order of rotations *about a fixed rotation axis* is unimportant. The order matters if the rotation axis directions are different because the first rotation reorients the axis for the second.

2.) Show that a similarity transformation of a real symmetric matrix with a real orthogonal matrix and its inverse results in a real symmetric matrix. Similarity: $\mathbb{M}^{/} = \mathbb{O}\,\mathbb{M}\,\mathbb{O}^{-1}$; Orthogonal: $\mathbb{O}^{t} = \mathbb{O}^{-1}$. Show that a (real) congruent transformation of a real symmetric matrix yields a real symmetric matrix. Congruent: $\mathbb{M}^{/} = \mathbb{C}\,\mathbb{M}\,\mathbb{C}^{t}$. Note that a congruent transformation with a real orthogonal matrix

is a similarity transformation. It is expected that real symmetric matrices represent classical physical properties. It is therefore important that the real symmetric property is preserved.

3.) Show that a similarity transformation of a Hermitian matrix using a unitary transformation matrix and its inverse results in a Hermitian matrix. Hermitian: $\mathbb{H}^{\dagger} = [(\mathbb{H})^{*}]^{t} = \mathbb{H}$; Unitary: $\mathbb{U}^{-1} = \mathbb{U}^{\dagger}$.

4.) Given the equation for an ellipse in standard form with major and minor axes parallel to the x and y axes, transform to the view from a frame rotated by θ about the z-axis. Show that:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) x^{/2} + \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\sin(2\theta)\, x^{/} y^{/} + \left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right) y^{/2} = 1$$

that gives: $A = \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)$; $B = \left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)$; $C = \left(\frac{1}{2}\right)\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\sin(2\theta)$ in the expression for an ellipse $A x^{/2} + 2C\, x^{/} y^{/} + B\, y^{/2} = 1$

Is the angle θ used here the same as the one used in the discussion of the rotated ellipse in this linear transformations section?

5.) Considering $\tilde{\tilde{\alpha}}$ as a matrix, what is the relation between the determinants $|\tilde{\tilde{\alpha}}|$ and $|\tilde{\tilde{\alpha}}^{/}|$ where $\tilde{\tilde{\alpha}}$ is the polarizability matrix and $\tilde{\tilde{\alpha}}^{/} = \mathbb{S}\ \tilde{\tilde{\alpha}}\ \mathbb{S}^{-1}$, the polarizabilty component matrix as measured in another coordinate system? The determinant of a matrix is a scalar as is the magnitude of a vector. How do we expect scalars to transform?

6.) Consider a linear mapping of a vector space $\mathbb{V}$ into a space $\mathbb{W}$. Show that the image of $\mathbb{V}$ in $\mathbb{W}$ under the mapping is a vector space (the full space $\mathbb{W}$ or a subspace of $\mathbb{W}$). Assume that $\mathbb{V}$ and $\mathbb{W}$ are vector spaces over the same field that is also the field for the linear map. Note that most of the properties of addition and multiplication are not issues. Why? What are the issues that must be addressed?

7.) Consider a linear mapping of a vector space $\mathbb{V}$ into a space $\mathbb{W}$. Show that the kernel of the mapping is a subspace of $\mathbb{V}$. Assume that V and $\mathbb{W}$ are vector spaces over the same field that is also the field for the linear map. Note that most of the properties of addition and multiplication are not issues. Why? What are the issues that must be addressed?

8.) Consider a linear mapping of a vector space $\mathbb{V}$ into a space $\mathbb{W}$. Show that the set of images of a basis set $\mathbb{V}$ in $\mathbb{W}$ under the mapping is a spanning set for the image of $\mathbb{V}$ in $\mathbb{W}$. Assume that $\mathbb{V}$ and $\mathbb{W}$ are vector spaces over the same field which is also the field for the linear map.

9.) Consider a linear mapping of a vector space $\mathbb{V}$ into a space $\mathbb{W}$. Show that the dimension of $\mathbb{V}$ minus the dimension of the kernel of the mapping is equal to the dimension of the image of $\mathbb{V}$ in $\mathbb{W}$ under the mapping. Assume that $\mathbb{V}$ and $\mathbb{W}$ are vector spaces over the same field that is also the field for the linear map.

10.) In the moment of inertia example, the claim is made that $\mathbb{I}$ could be computed by finding the transformed coordinates of the particles and exercising the formula:

$$I'_{ij} = \sum_{masses\ \alpha} m_\alpha \left[r'^2_\alpha \ \delta_{ij} - x'_{\alpha i} x'_{\alpha j} \right]$$

Please do so. Use the transformation equations: **(remember: $\theta = -\sin^{-1}(^3/_5)$)**

$$x' = x\cos\theta + y\sin\theta, \qquad y' = x\left(-\sin\theta\right) + y\cos\theta, \qquad z' = z$$

to compute the coordinates of the masses in the rotated frame. Compute the moment of inertia tensor as observed in the rotated frame using the transformed coordinates. Masses are neither vectors nor (rank two) tensors. What is the transformation type for masses and inner products of vectors? How do the values of the masses of each object compare as determined by the original observer and by the observer in the rotated frame?

11.) Referring to the Eulerian angles, compute the rotation corresponding to $\phi = {}^{\pi}/_{2}$; $\theta = {}^{\pi}/_{4}$; $\psi = 0$ and to $\phi = 0$; $\theta = {}^{\pi}/_{4}$; $\psi = {}^{\pi}/_{2}$. How are the two rotation sequences related? Are the resulting overall rotations equivalent? What statement can you make that summarizes your observations?

$$\lambda(\phi,\theta,\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

12.) It has been demonstrated that the coordinate dyadic transforms as a rank two tensor.

$$\underset{\approx}{d}' = \lambda\ \underset{\approx}{d}\ \lambda^{-1} \text{ where } [\underset{\approx}{d}]_{ij} \text{ is } x_i\, x_j$$

$$\text{Recall } x_i' = \sum_j \lambda_{ij}\ x_j \text{ so:}$$

$$d_{ij}' = x_i' x_j' = \sum_{m,\ell} \lambda_{im}\ x_m \lambda_{j\ell} x_\ell = \sum_{m,\ell} \lambda_{im}\ x_m x_\ell \lambda_{j\ell}$$

$$= \sum_{m,\ell} \lambda_{im}\ x_m x_\ell \lambda^t_{\ell j} = \sum_{m,\ell} \lambda_{im}\ x_m x_\ell \lambda^{-1}_{\ell j} = \sum_{m,\ell} \lambda_{im}\ d_{m\ell}\ \lambda^{-1}_{\ell j}$$

a.) Show that the Kronecker delta represents the elements of a rank two tensor in the sense that the elements transform according to the similarity transformation rule.

b.) Conclude that $\mathbb{I}$ a matrix with elements of the form: $I_{ij} = \sum_{masses\ \alpha} m_\alpha \left[r_\alpha^2\ \delta_{ij} - x_{\alpha i} x_{\alpha j} \right]$ transforms as a rank two tensor given that m_α and r_α^2 are scalars.

c.) Conclude that $\mathbb{Q}$ a matrix with elements of the form: $Q_{ij} = \sum_{\text{charges}\ \alpha} q_\alpha \left[3\, x_{\alpha i} x_{\alpha j} - r_\alpha^2\ \delta_{ij} \right]$ transforms as a rank two tensor given that q_α and r_α^2 are scalars.

13.) An operator $\mathbb{A}$ has the matrix representation $\underset{\approx}{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ with eigenvalues $\{1, 3, 3\}$.

a.) Pick mutually orthogonal eigenvectors for these eigenvalues and use them to develop an orthogonal transformation matrix λ that diagonalizes $\underset{\approx}{A}$. Verify that your eigenvector sequence is in right-hand order. If not, multiply the third vector by negative one to make it so..

b.) Apply the resulting similarity transformation to demonstrate that $\underset{\approx}{A}$ is transformed to a diagonal form with the eigenvalues as the diagonal elements. $\underset{\approx}{A}' = \lambda\ \underset{\approx}{A}\ \lambda^{-1}$; $\underset{\approx}{A}'$ diagonal.

c.) Verify that each eigenvector is transformed to a canonical form for a coordinate direction by the transformation; that is: as viewed in the rotated frame in which $\mathbb{A}$ appears in a diagonal form.

d.) Choose an alternative ordering of the eigenvectors that is also right-handed. Form the new λ and diagonalize $\mathbb{A}$.

e.) Give the equations for the transformation rules for a scalar, a vector and for a matrix operator under the action of the rotation described by λ.

f.) Note that λ can be identified as a rotation about one of the Cartesian axes by some angle if you chose the original *y* axis as one of the eigen-directions. If you made some other choice, try to find a sequence of rotations of the $\{\mathbb{R}_x, \mathbb{R}_y, \mathbb{R}_z\}$ family that rotate the original coordinate directions into the eigen-directions. Make a sketch to show that this rotation or rotation sequence would take the original axes into the eigenvector directions. Note that the angle depends on the order that you chose for the eigenvectors.

14.) Show that a rotation of $^{\pi}/_2$ about the *x*-axis can be effected by rotating by $^{\pi}/_2$ about the y-axis, followed by $^{\pi}/_2$ about the *z*-axis and followed finally by $-^{\pi}/_2$ about the *y*-axis. That is, show that: $\mathbb{R}_y(-^{\pi}/_2)\, \mathbb{R}_z(^{\pi}/_2)\, \mathbb{R}_y(^{\pi}/_2) = \mathbb{R}_x(^{\pi}/_2)$. Draw a coordinate triad and apply each action to verify the interpretation of the matrix multiplication. Continue to verify that: $\mathbb{R}_y(-^{\pi}/_2)\, \mathbb{R}_z(\theta)\, \mathbb{R}_y(^{\pi}/_2) = \mathbb{R}_x(\theta)$ by multiplying the matrices.

Note that the rotation matrices themselves transform using a similarity transformation. In a coordinate system in which the *z*-axis had been rotated to the original x-direction, a rotation by θ about the *x*-axis would have the form of a rotation about the *z*-axis by θ. $\mathbb{R}_z(\theta) = [\mathbb{R}_x(\theta)]' = \lambda\, \mathbb{R}_x(\theta)\lambda^{-1}$ where λ is the transformation that rotates the original *z*-axis to the original *x*-direction ($\lambda = \mathbb{R}_y(^{\pi}/_2)$). It follows that:

$$\lambda^{-1}\, \mathbb{R}_z(\theta)\lambda = \lambda^{-1}\, [\mathbb{R}_x(\theta)]'\, \lambda = \mathbb{R}_y(-^{\pi}/_2)\, \mathbb{R}_z(\theta)\, \mathbb{R}_y(^{\pi}/_2) = \mathbb{R}_x(\theta)$$

Flipping the sense of the transformation, a rotation of θ about the direction $\hat{n}$ by rotating $\hat{n}$ into the *z* direction, applying $\mathbb{R}_z(\theta)$, and then rotating the *z* direction back to the $\hat{n}$ direction.

$$\mathbb{R}_n(\theta)\lambda = (\lambda_{n\to z})^{-1}\, \mathbb{R}_z(\theta)\, \lambda_{n\to z}$$

15.) Find a matrix λ that represents a rotation taking the *z* direction into the direction $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Prepare a careful sketch and argue that a rotation of $-^{\pi}/_4$ about the *z* axis followed by a rotation by α about the *x* axis where $\sin(\alpha) = -\sqrt{\frac{2}{3}}$ and $\cos(\alpha) = \frac{1}{\sqrt{3}}$ does the job. $\lambda = \mathbb{R}_x(\alpha)\, \mathbb{R}_z(-^{\pi}/_4)$. Verify the result by showing that the vector $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ in the original frame has components (0,0,1) after the λ rotation. Compute the form of a rotation by θ about the direction $\hat{n} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ as $\mathbb{R}_n(\theta)$

$= \lambda^{-1}\, \mathbb{R}_z(\theta)\, \lambda$. Verify the result by showing that $\mathbb{R}_n(^{2\pi}/_3)$ acts to permute the components of a vector: ($x \to y \to z \to x$). Make a sketch showing that a $^{2\pi}/_3$ rotation about $\hat{n}$ just interchanges the positions of the axes.

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16.) Consider the function space with basis set {cos(x), cos($2x$) , cos($3x$) , sin(x), sin($2x$), sin($3x$)}, and inner product $\left(\frac{1}{\pi}\right)\int_{-\pi}^{\pi} f_i(x)\, f_j(x)\, dx$ consisting of functions of the form

$$f(x) = a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(3x) + a_4 \sin(x) + a_5 \sin(2x) + a_6 \sin(3x)$$

with function vectors:

$$\left| f(x) \right\rangle = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \qquad \left\langle f(x) \right| = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{bmatrix}$$

Show that the matrices for the linear operations $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ have representations as (the ^ is read as 'the operator matrix for'):

$$\frac{\hat{d}}{dx} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \frac{\hat{d}^2}{dx^2} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9 \end{bmatrix}$$

Mote that the elements of these matrices could be computed using $\left(\frac{1}{\pi}\right)\int_{-\pi}^{\pi} f_i(x)\hat{O}\, f_j(x)\, dx$ as the basis set is orthonormal.

a.) Evaluate the result of each operator on $f(x) = \cos(2x) - \sin(3x)$ which is

$$|\, f(x)\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

b.) Compute the matrix for $\frac{\hat{d}}{dx}$ followed by $\frac{\hat{d}}{dx}$.

17.) Consider the space with basis set $\{ x^0, x^1, \ldots, x^n, \ldots \}$ of functions of the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with function vectors similar to those for the previous problem, but with index numbers running from *zero* to *infinity*. Note that the basis set is not orthonormal and that no proper inner product has been defined. However, the basis set is independent so that unique operator matrices exist.

a.) Show that the matrices for the linear operations $\frac{d}{dx}$ and multiplication by x have representations as (the ^ is read as 'the operator matrix for'):

$$\frac{\hat{d}}{dx} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 2 & 0 & \ldots \\ 0 & 0 & 0 & 3 & \ldots \\ 0 & 0 & 0 & 0 & 4 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \quad \text{and} \quad \hat{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 \\ \ldots & \ldots & \ldots & 1 & \ldots \end{bmatrix}$$

That is: the operator matrix for the derivative with respect to x has zero elements with the exception of the counting numbers on the first super-diagonal. The matrix for multiplication by x has zero elements with the exception of the ones on the first sub-diagonal. *Recall that rows and columns are numbered from ZERO.*

b.) Show that the matrix for $\frac{d}{dx}\, x - x\, \frac{d}{dx}$ is the matrix all zero elements except for ones on the diagonal. The difference of operators acting in reversed order is called a commutator. The result above stated that $\frac{d}{dx}$ and x do not commute.

c.) Show that $\frac{d}{dx}\left(x\ f(x)\right) - x\frac{d}{dx}f(x) = f(x)\ \forall\ f(x)$. Equivalently, $\frac{d}{dx}\ x\ -\ x\frac{d}{dx}$ is the identity operator for all differentiable functions.

18. The trace of a matrix is defined to be the sum of its diagonal elements. Tr[$\mathbb{M}$] $= \sum_{ij} \delta_{ij}\, m_{ij}$

Show that the trace of a matrix is invariant under a similarity transformation.

19. A standard problem in quantum mechanics* examines the energies of two states that are initially degenerate (have the same energy) when a small additional interaction couples the two states. It begins with the matrix:

$$\begin{bmatrix} E_0 & \Delta \\ \Delta & E_0 \end{bmatrix}$$

a.) Find the eigenvalues and eigenvectors for this matrix. Represent the eigenvectors as column vectors.

b.) Find the transformation matrix that diagonalizes the matrix. Show that it works. It happens that the matrix for the transformation is orthogonal, a special case of unitary.

* One embodiment is the energies of the 2S and $2P_z$ state of a hydrogen atom placed in a uniform external electric field $\mathcal{E}\ \hat{k}$. See pages 252-255 of the text by Schiff.

20. An interesting variant of the previous problem studies the matrix:

$$\begin{bmatrix} E_0 + \varepsilon & \Delta \\ \Delta & E_0 - \varepsilon \end{bmatrix}$$

Examine the eigenvectors and eigenvalues for this problem. Represent the eigenvectors as two element column vectors. Compare and contrast the results for the limits $\Delta << \varepsilon$ and $\varepsilon << \Delta$. This matrix is related to a problem in which two nearly degenerate levels are coupled by a perturbation. Eigenvalues: $E_0 \pm \sqrt{\Delta^2 + \varepsilon^2}$. If $\varepsilon = 0$, the original states are degenerate and a small perturbation *significantly* mixes the two states leading to eigenvalue shifts that are linear in Δ. If $\Delta << \varepsilon$, then the eigenvalue difference $2\ \varepsilon$provides a stiffness against mixing and the mixing fraction goes to zero as Δ goes to zero. Further, the eigenvalues shift is quadratic in ε in this limit. In fact the shifts have the form

$\Delta^2/2\varepsilon$ that displays the original eigenvalue difference as a stiffness against change.

21. Search for the word *theorem* and the words *preserve inner* in this handout. Compare and contrast the roles of orthogonal and real symmetric matrices in real spaces and unitary and Hermitian matrices in complex spaces. Recall the situation for real vector spaces. Compare the process to construct the transformation that diagonalizes the real and Hermitian operators in the two cases.

REAL SPACES RESULTS: It is common that real symmetric matrices represent concepts (material properties) in classical physics. The eigenvectors of these matrices are orthogonal if they are associated with distinct eigenvalues. The eigenvectors that correspond to a degenerate eigenvalues may be chosen to be orthogonal, and processes such as Gram-Schmidt can produce a suitable orthogonal set from an initial independent set.
The Cartesian form of the inner product is our usual starting point. Orthogonal transformations preserve inner products so the standard definition as the sum of the products of the corresponding components remains valid. => The eigenvectors can be unity normalized

$$\begin{bmatrix} a_i & b_i & c_i \end{bmatrix} \begin{bmatrix} a_j \\ b_j \\ c_j \end{bmatrix} = a_i\, a_j + b_i\, b_j + c_i\, c_j = \tilde{v}_i^t\, \tilde{v}_j = \delta_{ij}$$

This form implies that the column vector is the KET for a vector while the row vector is the BRA. A symmetric matrix is its own transpose and can be diagonalized by an orthogonal transformation, a transformation represented by a matrix that has its own transpose as its inverse. The form of the transformation matrix is the collection of the eigenvectors as the rows of the matrix. The orthonormal form of the eigenvectors ensures that the matrix is orthogonal.

The transformation matrix that diagonalizes a real symmetric matrix has the eigenvectors of the matrix as the rows of the transformation matrix.

COMPLEX VECTOR SPACES: For complex spaces the individual components of a vector can be complex. A column vector of the components can still be chosen to represent the KET for a vector. However, in order to ensure that property I4 is satisfied by the inner product, the conversion from KET to BRA requires an adjustment in addition to transposing the KET. **The complex conjugate of the transpose is taken.** The combined process of taking the transpose plus complex conjugate

adjustment is designated as taking the Hermitian conjugate. This process is the ***dagger*** (Hermitian conjugate) operation for the matrix representation of a complex vector space. The operators for (real) physical observables are represented by Hermitian matrices in matrix representations of a complex space. **A unitary** transformation matrix is one that has its Hermitian conjugate as its inverse. **Unitary transformations preserve the values of inner products.** A Hermitian matrix can be diagonalized by a *unitary* transformation. **T**he eigenvalues of a Hermitian matrix are real. **T**he eigenvectors of a Hermitian matrix that correspond to distinct eigenvalues are orthogonal.The complex conjugates of the eigenvectors of a Hermitian operator appear ar the rows of the transformation that diagonalizes that operator's matrix. That is each row of the transformation is the complex conjugate of a column vector that represents an eigenvector. **The resulting transformation matrix is unitary.**

22. **Operators and Change of Basis Functions** ****** problem needs to be checked

Consider the space of functions of the form

$$\begin{aligned} f(x) &= a_1\sqrt{2}\cos(x) + a_2\sqrt{2}\sin(x) + a_3\sqrt{2}\cos(2x) + a_4\sqrt{2}\sin(2x) \\ &= a_1 B_1(x) + a_2 B_2(x) + a_3 B_3(x) + a_4 B_4(x) \end{aligned}$$

with function vectors:

$$\left| f(x) \right\rangle = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \qquad \left\langle f(x) \right| = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}$$

a.) Show that the matrices for the linear operations $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ have representations as (the ^ is read as 'the operator matrix for'):

$$\frac{\hat{d}}{dx} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \quad \text{and} \quad \frac{\hat{d}^2}{dx^2} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

Notice that the operator matrices separate into 2 by 2 diagonal blocks corresponding to each frequency subspace.

b.) What is the value of the inner product for the various functions pairs?

$$\left\langle B_i \middle| B_j \right\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} B_i^*(x)\, B_j(x)\, dx$$

c.) Evaluate the result of each operator on $f(x) = \sqrt{2}\cos(x) - \sqrt{2}\sin(2x)$ which is

$$\left| f(x) \right\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

d.) Compute the matrix for $\frac{\hat{d}}{dx}$ followed by $\frac{\hat{d}}{dx}$.

Next consider the change of basis functions to the form

$$\begin{aligned} f(x) &= b_1\left(\cos(x) + \sin(x)\right) + b_2\left(\cos(x) - \sin(x)\right) \\ &\quad + b_3\left(\cos(2x) + \sin(2x)\right) + b_4\left(\cos(2x) - \sin(2x)\right) \\ &= b_1\, B_1^{/}(x) + b_2\, B_2^{/}(x) + b_3\, B_3^{/}(x) + b_4\, B_4^{/}(x) \end{aligned}$$

with function vectors:

$$\left| f(x) \right\rangle^{/} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \left\langle f(x) \right|^{/} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

e.) What are the values of the inner products of the various basis function pairs?

$$\left\langle B_i^{/} \middle| B_j^{/} \right\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} B_i^{/*}(x)\, B_j^{/}(x)\, dx$$

f.) Compute the transformation matrix $\mathbb{U}$ that maps the original basis set to this new basis set. Each element U_{ij} is the $B^{/}_I(x)$ behavior in $B_j(x)$. For example,

$$U_{11} = \left\langle B_i^{/} \middle| B_j \right\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} B_i^{/*}(x)\, B_j(x)\, dx = \frac{1}{\sqrt{2}}.$$

g.) Apply the transformation to $f(x) = \sqrt{2}\cos(x) - \sqrt{2}\sin(2x)$ which is

$$\left| f(x) \right\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad \text{to find } \left| f(x) \right\rangle^{/} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

h.) Develop the matrices that represent the operators $\frac{\hat{d}}{dx}$ and $\frac{\hat{d}^2}{dx^2}$ for the new basis set by computing the derivatives and building the matrices.

i.) Apply a similarity transform using $\mathbb{U}$ to transform the operator matrices from the original basis representations to the representations in the new basis.

23.) Verify that the eigenvectors are list in right-hand order in the moment of inertia example above. Interchange the eigenvector rows to form any other order. Check that the order is right-handed. If not multiply row three by negative one. Verify that the new eigenvector set is right-handed. Apply the new rotation to diagonalize the original moment of inertia. Note that the result is diagonal, but different. What has changed?

24.) Consider the *n x n* matrices $\mathbb{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\mathbb{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Compare $\mathbb{A}\,\mathbb{B}$ and $\mathbb{B}\,\mathbb{A}$.

Matrices with the property that $\mathbb{A}\,\mathbb{B} = -\,\mathbb{B}\,\mathbb{A}$ are said to anti-commute.

25.) The trace of a matrix is defined to be the sum of its diagonal elements.

$$\mathrm{Tr}[\,\mathbb{M}\,] = \sum_{i\,j} \delta_{i\,j}\, m_{i\,j} \quad \text{(Assume that all matrices are } n \times n.\text{)}$$

(a.) Show that $\mathrm{Tr}[\,\mathbb{A}\,\mathbb{B}\,] = \mathrm{Tr}[\,\mathbb{B}\,\mathbb{A}\,]$.

(b.) Show that $\mathrm{Tr}[\,\mathbb{A}\,\mathbb{B}\,\mathbb{C}] = \mathrm{Tr}[\,\mathbb{C}\,\mathbb{A}\,\mathbb{B}]$.

(c.) Show that $\mathrm{Tr}[\,\mathbb{A}\,\mathbb{B}\,\mathbb{C}\,]$ need not equal $\mathrm{Tr}[\,\mathbb{B}\,\mathbb{A}\,\mathbb{C}\,]$. See the problem above about matrices that do not commute. Choose $\mathbb{C}$ to yield a product with a trace.

26.) Show that the product of two diagonal matrices is a diagonal matrix. Give the form of the diagonal elements of the product matrix. What is the determinant of a diagonal matrix? (the diagonal matrices are square $\Rightarrow$ *n x n*.)

27.) Consider a transformation of coordinates that preserves the length of a vector in two dimensions. We have $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} V_x' \\ V_x' \end{bmatrix}$ and need to preserve the Euclidean length:

$$\left(V_x^{/}\right)^2+\left(V_y^{/}\right)^2=\left(a_{11}V_x+a_{12}V_y\right)^2+\left(a_{21}V_x+a_{22}V_y\right)^2=\left(V_x\right)^2+\left(V_y\right)^2$$

Three relationships follow from matching the coefficients of V_x^2, V_y^2 and V_xV_y on both sides leaving one degree of freedom. Argue that the relations could be satisfied by choosing one pair of the a_{ij} equal to sinα and cosα and another pair equal to sinβ and cosβ. The coefficient of V_xV_y leads to a relation leading to two possible values for β given α. One must choose the solution for which the identity transformation [$a_{11} = a_{22} = 1$; $a_{12} = a_{21} = 0$] is a possibility. Use these guides to assign the values of the a_{ij}. Conclude that the form is:

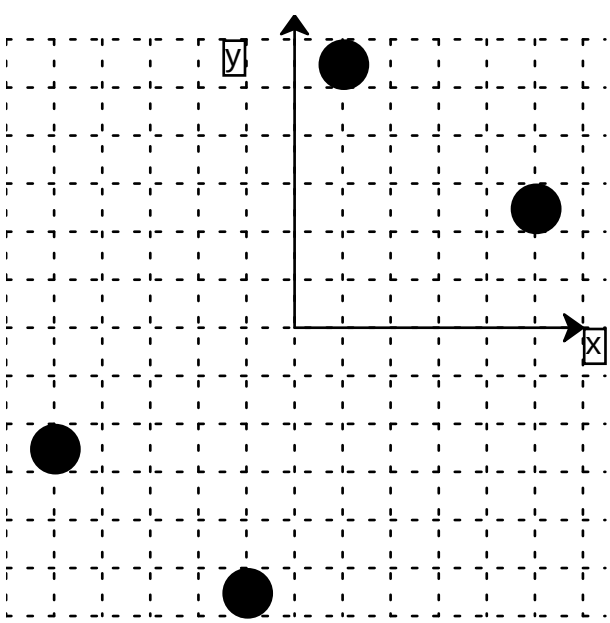
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \pm \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

With the overall plus sign, the transformation is a rotation with the degree of freedom is identified as the angle of rotation. The overall minus adds an inversion.

28.) Moments of inertia for a collection of point masses: (See the Introduction to Eigenvalues handout.)

$$I_{ij} = \sum_{masses\ \alpha} m_\alpha \left[r_\alpha^2\ \delta_{ij} - x_{\alpha i} x_{\alpha j} \right]$$

Using the mapping: $x_1 \leftrightarrow x; x_2 \leftrightarrow y; x_3 \leftrightarrow z$, for example x_{72} is the y coordinate of particle 7.



m_1: 1 kg @ (10m, 5m, 0m)

m_2: 1 kg @ (2m, 11m, 0m)

m_3: 1 kg @ (-10m, -5m, 0m)

m_4: 1 kg @ (-2m, -11m, 0m)

a.) COMPUTE the moment of inertia tensor for the four-mass distribution shown. The units

should be *kg-m*^{2}.

$$\underset{\approx}{I} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \; kg\text{-}m^2$$

b.) $\mathbb{I}\vec{\omega} = \vec{L}$, the angular momentum of the rigid assembly of masses rotating at $\vec{\omega}$. The assembly can only rotate smoothly (torque-free) if $\vec{L}$ is parallel to $\vec{\omega}$. Consider $\vec{\omega} = \left[1\,\hat{i} + 1\,\hat{j} + 1\hat{k}\right](rad/s)$. Compute the angular momentum for this angular velocity. Compute the angle between the directions of $\vec{\omega}$ and $\vec{L}$。

c.) The inertial tensor can be normalized by a rotation about the *z* axis by ϕ where $\cos\phi = {}^4/_5$ and $\sin\phi = {}^3/_5$. Find the values of the masses and their coordinates as measured by an observer in the frame rotated about the *z* axis by ϕ. Compute the elements of the inertia tensor as measured by that observer.

29.) **Moments of Inertia for Bodies with Rotational Symmetry**: The moment of inertial is real symmetric so it may be represented as: $\mathbb{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$. Give the array of values observed in a frame rotated by ${}^{\pi}/_2$ about the *z* axis relative to the original frame. Next suppose that the *z* axis is a four-fold rotational symmetry axis of the body. That means that the new form and the original form are identical. Conclude that: $\mathbb{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{xx} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$. If the body has two distinct *n*-fold rotational symmetry axes with $n \geq 3$, then the inertia tensor is proportionate the 3 x 3 identity.

30.) **Rotations:** Given the rotation matrices

$$\mathbb{R}_{\mathbf{x}}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \mathbb{R}_{\mathbf{y}}(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \text{ and } \mathbb{R}_{\mathbf{z}}(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

a.) Give the matrix product that represents a rotation of ${}^{\pi}/_2$ about the *x*-axis followed by a rotation

of $^{\pi}/_{2}$ about the then *z*-axis followed by $^{\pi}/_{2}$ about the then *y*-axis. ***You need not multiply the matrices.***

b.) Sketch an axis triad representing the *x, y,* and *z* directions before any rotations and then after each rotation described above. What additional rotation would return the triad to its original orientation?

c.) A vector has a component representation $\vec{V} = 2m\,\hat{i} + 11\,m\,\hat{j} + 7m\hat{k}$ in an original frame. Give the component representation of this vector in a frame rotated by $\phi = \cos^{-1}(0.6) = +\,53.13^{0}$ about the *z* axis relative to the original frame.

31. **Orthogonal Transformation:**

a.) The matrix $\mathbb{A} = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 4 \end{bmatrix}$ has eigenvectors: $\left\{ \frac{\hat{i}+\hat{k}}{\sqrt{2}};\ \frac{\hat{i}-\hat{k}}{\sqrt{2}};\ \hat{j} \right\}$. Find the eigenvalue for the second of these directions.

b.) How are the eigenvectors related to λ, the orthogonal transformation matrix that diagonalizes A. Give the explicit form of λ.

c.) Apply the resulting similarity transformation to demonstrate that A is transformed to a diagonal form with the eigenvalues as the diagonal elements. $A^{/} = \lambda\, A\, \lambda^{-1}$; $A^{/}$ diagonal.

32.) Consider two arbitrary vectors ($\Rightarrow$ their components transform according to the vector rule) and show that their inner product is preserved by orthogonal transformations.

33.) Show that if the eigenvectors are normalized and listed in RHR order, that the determinant of the associated transformations from the original axis set to the eigen-set is orthogonal with determinant one. That is: the transformation is a rotation.

34.) Show that a rotation of $^{\pi}/_{2}$ about the *y*-axis can be effected by rotating by - $^{\pi}/_{2}$ about the *x*-axis, followed by $^{\pi}/_{2}$ about the z-axis and followed finally by + $^{\pi}/_{2}$ about the *x*-axis. That is, show that: $\mathbb{R}_x(^{\pi}/_{2})\, \mathbb{R}_z(^{\pi}/_{2})\, \mathbb{R}_x(^{-}\,^{\pi}/_{2}) = \mathbb{R}_y(^{\pi}/_{2})$. Draw a coordinate triad and apply each action in turn to generate new triads to verify the interpretation of the matrix multiplication. Continue to verify that: $\mathbb{R}_x(^{\pi}/_{2})$

$\mathbb{R}_z(\theta)\ \mathbb{R}_x(\ ^{-\pi}/_2) = \mathbb{R}_y(\theta)$. Note that the rotation matrices themselves transform using a similarity transformation. In a coordinate system in which the *z*-axis had been rotated to the original *y*-direction, a rotation by θ about the *y*-axis would have the form of a rotation about the *z*-axis by θ.

35.) a.) Show that: $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$. Conclude that $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \geq 0$ for vectors in right-hand-rule order. b.) A proper orthogonal transformation (a rotation with no inversion) has a determinant of positive one. What does this tell you about the order in which the eigenvectors are inserted as rows to form the matrix λ representing the transformation?

36.) Apply a rotation of θ about the *z* axis to the matrix $\mathbb{M} = \begin{bmatrix} 5 & \sqrt{3} & 0 \\ \sqrt{3} & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ to reach a diagonal form $\mathbb{M}^{/} = \lambda\, \mathbb{M}\, \lambda^{-1}$. Use the method and results form the **"Why would we ever use a rotation? - Consider an ellipse:"** section. Identify the angle θ required and give the explicit form of λ. Next use the matrix eigenvalue approach to find the eigenvalues and eigenvectors. Create a 3 x 3 matrix $\mathbb{S}$ by arranging the eigenvectors in a RHR order and using them in sequence as the rows of $\mathbb{S}$. Does $\mathbb{S}$ correspond to a rotation? If so, by what angle? How is that angle related to θ? Compare the eigenvalues with the diagonal values in $\mathbb{M}^{/} = \lambda\, \mathbb{M}\, \lambda^{-1}$. Compute $\mathbb{M}^{//} = \mathbb{S}\, \mathbb{M}\, \mathbb{S}^{-1}$. Create a 3 x 3 matrix $\mathbb{T}$ by arranging the eigenvectors in a ***distinct*** RHR order and using them in sequence as the rows of $\mathbb{T}$. Does $\mathbb{T}$ correspond to a rotation? If so, by what angle? How is that angle related to θ? Compare the diagonal elements in Compute $\mathbb{M}^{///} = \mathbb{T}\, \mathbb{M}\, \mathbb{T}^{-1}$ with those in $\mathbb{M}^{//} = \mathbb{S}\, \mathbb{M}\, \mathbb{S}^{-1}$. Make and report an astute observation. Study the original matrix $\mathbb{M}$. What suggest that a rotation about the *z* axis might be adequate to reach a diagonal form rather than a more general rotation?

37.) Compute $e^{\mathbb{M}}$ where $\mathbb{M}$ is the matrix below. Use: $\mathbb{M}^0 \equiv \mathbb{1}$; $\mathbb{M}^1 \equiv \mathbb{M}$; $\mathbb{M}^2 \equiv \mathbb{M}\,\mathbb{M}$; . . . and a function of a matrix is evaluated as the power series expansion of the function with the matrix replacing its argument. (*See problem 39 for a more important result*.)

$$\mathbb{M} = \mathbb{M}^1 = \begin{bmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbb{M}^2 = ? \qquad \mathbb{M}^3 = ? \qquad \mathbb{M}^4 = ?$$

Begin by computing $\mathbb{M}^0$, $\mathbb{M}^1$, $\mathbb{M}^2$, $\mathbb{M}^3$ and $\mathbb{M}^4$. Extrapolate to $\mathbb{M}^{1+4n}$, $\mathbb{M}^{2+4n}$, $\mathbb{M}^{3+4n}$ and $\mathbb{M}^{4n}$.

You may use: $\mathbb{M}^n = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^n & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix}$

Write out the power series expansions for e^x, $\cos(x)$ and $\sin(x)$. Write out the first 6 terms in the expansion for e^x explicitly with $\mathbb{M}$ substituted for x. Collect the terms in a single matrix with elements that are the sum of the corresponding elements of the various terms in the expansion of $e^{\mathbb{M}}$. Sum the expressions in that appear as each element in the summed matrices.

$$\text{Answer: } e^{\mathbb{M}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & e \end{bmatrix}$$

38.) Prove that: $\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} & 0 \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and hence that

$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^n & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix}$. Note: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$

39.) A well-know representation of e^x is $\lim_{n\to\infty}\left[1+{}^{x}\!/_{n}\right]^n$. Using the binomial theorem, one can expand the expression as:

$$\left[1+{}^{x}\!/_{n}\right]^n = 1 + n({}^{x}\!/_{n}) + \frac{n(n-1)}{2!}({}^{x}\!/_{n})^2 + \frac{n(n-1)(n-2)}{3!}({}^{x}\!/_{n})^3 + \ldots$$

$$\left[1+{}^{x}\!/_{n}\right]^n = 1 + x + \frac{(1-n^{-1})}{2!}(x)^2 + \frac{(1-n^{-1})(1-2n^{-1})}{3!}(x)^3 + \ldots$$

In the large n limit, this becomes:

$$\left[1+{}^{x}\!/_{n}\right]^{n} = 1 + x + \tfrac{1}{2!}(x)^2 + \tfrac{1}{3!}(x)^3 + \ldots = \sum_{m=0}^{\infty} \frac{x^m}{m!} = e^x$$

The matrix for an infinitesimal rotation $d\theta$ about the z axis is:

$$\lim_{d\theta\, small} \begin{bmatrix} \cos(d\theta) & \sin(d\theta) & 0 \\ -\sin(d\theta) & \cos(d\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & d\theta & 0 \\ -d\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix for a rotation by a finite angle θ about the z axis follows by making a sequence of n rotations by the small angle ${}^{\theta}\!/_{n}$.

$$\mathbb{R}_z(\theta) = \lim_{n\to\infty} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & {}^{\theta}\!/_{n} & 0 \\ -{}^{\theta}\!/_{n} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^{n}$$

Following the reasoning above,

$$\mathbb{R}_z(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sum_{m=1}^{\infty} \frac{1}{m!} \begin{bmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{m}$$

Begin by computing the various powers of the z rotation generator.

$$\mathbb{L} = \mathbb{L}^1 = \begin{bmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbb{L}^2 = ? \qquad \mathbb{L}^3 = ? \qquad \mathbb{L}^4 = ?$$

Begin by computing: $\mathbb{L}^1$, $\mathbb{L}^2$, $\mathbb{L}^3$ and $\mathbb{L}^4$. Extrapolate to $\mathbb{L}^{1+4n}$, $\mathbb{L}^{2+4n}$, $\mathbb{L}^{3+4n}$ and $\mathbb{L}^{4n}$.

$$\text{You may use: } \mathbb{L}^n = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^n & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix}$$

Write out the power series expansions for cos(x) and sin(x). Write out the first several terms in the expansion for sin(x) explicitly with $\mathbb{L}$ substituted for x. Collect the terms in a single matrix with elements that are the sum of the corresponding elements of the various terms. Sum the expressions in that appear as each element in the summed matrices. Congratulations, you have shown that a rotation by θ can be accomplished by making n rotations by ${}^{\theta}\!/_{n}$.

$$\lim_{n\to\infty}\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\begin{bmatrix}0&\theta/n&0\\-\theta/n&0&0\\0&0&0\end{bmatrix}\right)^n=\begin{bmatrix}\cos(\theta)&\sin(\theta)&0\\-\sin(\theta)&\cos(\theta)&0\\0&0&1\end{bmatrix}$$

40.) Solve the eigenvalue problem associated with the moment of inertia example.

$$\begin{bmatrix}(73-\mu)&-36&0\\-36&(52-\mu)&0\\0&0&(125-\mu)\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\qquad \mathbb{I}=\begin{bmatrix}73&-36&0\\-36&52&0\\0&0&125\end{bmatrix}$$

Find the eigenvalues and eigenvectors. Use them to generate the form of λ, the transformation that diagonalizes the original matrix $\mathbb{I}$. Compute $\lambda\,\mathbb{I}\,\lambda^{-1} = \lambda\,\mathbb{I}\,\lambda^{t}$ to display the diagonal form. Verify that the eigenvalues appear on the diagonal in the row in which the corresponding eigenvector appeared in λ.

41.) **Special Quantum Mechanics Sample Problem**. A standard problem in quantum mechanics examines the energies of two states that are initially degenerate (have the same energy) when a small additional interaction couples the two states. It begins with the Hamiltonian matrix:

$$\hat{H}=\begin{bmatrix}E_0&i\Delta\\-i\Delta&E_0\end{bmatrix}\text{ where } E_o \text{ and } \Delta \text{ are real values}$$

Note: Δ is a small positive energy; the symbol does not indicate an increment.

The quantum states would be represented in terms of a set of basis wavefunctions – say the hydrogenic $|2S\rangle$ and $|2P_o\rangle$. A column vector represents a mixed state.

$$|\psi\rangle = a\,|2S\rangle + b\,|2P_o\rangle \;\rightarrow \underset{\sim}{\psi}=\begin{bmatrix}a\\b\end{bmatrix}\quad\text{where } |a|^2=|b|^2=1 \text{ so that } |2S\rangle\rightarrow\begin{bmatrix}1\\0\end{bmatrix}\text{ and } |2P_o\rangle\rightarrow\begin{bmatrix}0\\1\end{bmatrix}$$

The inner product $\langle\phi|\psi\rangle = \begin{bmatrix}a_\phi^* & b_\phi^*\end{bmatrix}\begin{bmatrix}a_\psi\\b_\psi\end{bmatrix}=\underset{\sim}{\phi}^\dagger\,\underset{\sim}{\psi}$. The eigenvalue relation and eigenvalue equation become: $\begin{bmatrix}E_0&i\Delta\\-i\Delta&E_0\end{bmatrix}\begin{bmatrix}a_j\\b_j\end{bmatrix}=\varepsilon_j\begin{bmatrix}a_j\\b_j\end{bmatrix}$ and $\begin{vmatrix}E_0-\varepsilon&i\Delta\\-i\Delta&E_0-\varepsilon\end{vmatrix}=0$. The normalized eigenvectors are formed as: $\underset{\sim}{e}_i=\dfrac{1}{\sqrt{|a_i|^2+|b_i|^2}}\begin{bmatrix}a_i\\b_i\end{bmatrix}$.

The problem is the same as our classical physics eigenvalue problems except that real symmetric is replaced by Hermitian (a matrix equal to the complex conjugate of its transpose), and orthogonal is

replaced by unitary (the inverse is the complex conjugate of the transpose), and *a* and *b* can be complex as can the elements of all matrices. The rows of the unitary matrix that diagonalizes the hamiltonian are the complex conjugates of the coefficients for the eigenvectors.

a.) Show that the Hamiltonian matrix is Hermitian. That is: $\hat{H} = \hat{H}^{\dagger}$.

b.) Find the eigenvalues for the Hamiltonian matrix. Are they real?

c.) Find the normalized eigenvectors and note which has which eigenvalue.

d.) Form the transformation matrix. $\mathbb{U} = \begin{bmatrix} a_1^* & b_1^* \\ a_2^* & b_2^* \end{bmatrix}$. It can be said that each row is the Hermitian conjugate of an eigenvector ($\Rightarrow$ one of the $\underset{\sim}{e}_i^{\dagger}$).

e.) Show that $\mathbb{U}$ is unitary. Compute $\mathbb{U}\,\mathbb{U}^{\dagger}$.

f.) Compute $\mathbb{U}\ \underset{\sim}{e}_1$ and $\mathbb{U}\underset{\sim}{e}_2$. Comment on the forms of the results.

g.) Compute $\underset{\approx}{H}' = \mathbb{U}\ \underset{\approx}{H}\ \mathbb{U}^{\dagger}$. Comment on the form of the result.

$$\hat{H} = \begin{bmatrix} E_0 & i\Delta \\ -i\Delta & E_0 \end{bmatrix},\ \hat{H}\,\underset{\sim}{\psi} = i\hbar\frac{\partial}{\partial t}\underset{\sim}{\psi} \ \Rightarrow\ \begin{bmatrix} E_0 & i\Delta \\ -i\Delta & E_0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} E_0\,a + i\Delta b \\ -i\Delta a + E_0 b \end{bmatrix} = i\hbar\begin{bmatrix} \partial a/\partial t \\ \partial b/\partial t \end{bmatrix}$$

h.) Write out the equations for $\partial a/\partial t$ and $\partial b/\partial t$. In the absence of mixing ($\Delta = 0$), show that: $a(t) = a(0)\ e^{-i(E_0/\hbar)t}$ and $b(t) = b(0)\ e^{-i(E_0/\hbar)t}$. This means that $|a|^2$ and $|b|^2$, the probabilities for the system to be found in the 2S and $2P_o$ states, do not change in time if Δ is zero (the hamiltonian appears in a diagonal form).

i.) In the case of mixing ($\Delta \neq 0$) with the system starting in the 2S state at $t = 0$:

$$\underset{\sim}{\psi}(t=0) = \left|2S\right\rangle_{t=0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Show that b(t) is approximately: $\partial b/\partial t = -\left[\frac{\Delta a}{\hbar}\right]_{a=1}$ and $b(t) \approx \left(-\Delta/\hbar\right)t$ for very small times *t*. Hence the probability for the system to be found in the $2P_o$ state, $|b|^2$ grows as a quadratic in time for very small times.

j.) What units must the elements in the Hamiltonian matrix have? What are the units of the combination $\left(-\Delta/\hbar\right)t$ which represents $b(t)$ for small times t?

The perturbation, the small additional piece added to the hamiltonian, has off-diagonal elements. The action of these elements is to mix the states that correspond to the index values for those elements.

42.) **Almost a Vector**: Consider angular momentum defined as the cross product $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{r}$ and $\vec{p}$ are both vectors with proper transformation properties. Show that $L_i = |\lambda|\, \lambda_{ij} L_j$.

That is: angular momentum transforms like a vector under orthogonal transformations with det =1, and it changes signs under orthogonal transformations with det = -1. Orthogonal transformations with det = 1 are called proper and are the rotations. Orthogonal transformations with det = -1 are called improper and are the rotations plus a single (or odd number of) inversions.

As background, one should review the definition of determinant. Implicit summation notation may be confusing in this case so you should adopt an explicit summation convention. Using this convention, use $L_i = \varepsilon_{ijk}\, x_j\, p_k$ and establish that $\sum_{i,j,k=1}^{3} \varepsilon_{ijk}\, \lambda_{\ell i} \lambda_{mj} \lambda_{nk} = \varepsilon_{\ell mn}\, |\lambda|$.

A quantity that transforms according to the rule $v_i = |\lambda|\, \lambda_{ij}\, v_j$ is called a pseudo-vector or an axial vector. This potential for a sign change is thought provoking, but not a real issue with regard to the physical outcome of a situation. Cross products are always used in pairs! In the case of angular momentum, the *LAW* is $\frac{\partial \vec{L}}{\partial t} = \frac{\partial}{\partial t}(\vec{r} \times \vec{p}) = \vec{r} \times \vec{F}$. In the case of magnetism, a cross product is used to define the field as: $\vec{B}(\vec{r}_p) = \frac{q_s\, \vec{v}_s \times \vec{r}_{sp}}{4\pi\varepsilon_0\, r_{sp}^3}$. The field is just our bookkeeping device. We don't have physics until we find the force on another charge. $\vec{F}_{q_p} = q_p\, \vec{v}_p \times \underset{\text{due to others}}{\vec{B}(\vec{r}_p)}$. Cross products appear in pairs so the final physics is not changed as there are always two factors of $|\lambda|$ between the cause and the effect. One could even adopt the *left-hand* rule as long as one used it consistently.

43.) Show that the inner product of two vectors is a scalar. Show that the inner product of two vectors $\vec{A} \cdot \vec{B} = \sum_{m=1}^{3} A_m\, B_m$ transforms as a scalar under an orthogonal transformation.

44.) Given a displacement: $\Delta\vec{r}$, someone proposed a metric $\Delta x + \Delta y + \Delta z$. Consider the vector $\Delta\vec{r} = 1\,\hat{i} + 1\,\hat{j} + 1\hat{k}$. Use it to show that $\tau = \Delta r_x + \Delta r_y + \Delta r_z$ is not a scalar. Do this by computing τ' for the special case the λ is $\mathbb{R}_z(\theta)$. Show that τ fails to transform as a scalar. An alternative proposal for a metric is $\sigma = |\Delta x| + |\Delta y| + |\Delta z|$. Why is this *alternative* call the *taxi cab metric*? Could it be a proper scalar?

45.) The components of vectors transform according to the rule $\tilde{a}' = \lambda\ \tilde{a}$. For a sequence of transformations such as λ_1 followed by λ_2, the overall transformation follows the rule $\tilde{a}'' = \lambda\ \tilde{a}$ where $\lambda = \lambda_2\ \lambda_1$. The component matrix for the polarizabilty follows a more complex rule. $\tilde{\tilde{\alpha}}' = \lambda\ \tilde{\tilde{\alpha}}\ \lambda^{-1}$. Apply the transformations in sequence: $\tilde{\tilde{\alpha}}' = \lambda_1\ \tilde{\tilde{\alpha}}\ \lambda_1^{-1}$ followed by $\tilde{\tilde{\alpha}}'' = \lambda_2\ \tilde{\tilde{\alpha}}'\ \lambda_2^{-1}$. Does the rule $\lambda\ \tilde{\tilde{\alpha}}\ \lambda^{-1}$ with $\lambda = \lambda_2\ \lambda_1$ agree with your result? Explain.

46.) Consider the physical matrix $\mathbb{M} = \begin{bmatrix} 7 & \sqrt{3} & 0 \\ \sqrt{3} & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix}$. What property does $\mathbb{M}$ posses that we expect of a matrix the represents a physical property? Use the rotation results from the moment of inertia example to identify the rotation about the *z* axis that would diagonalize this matrix. Compute $\mathbb{R}_z(\phi)\ \mathbb{M}\ [\mathbb{R}_z(\phi)]^{-1}$. Find the eigenvalues and eigenvectors for $\mathbb{M}$ and discuss the results. Form a transformation matrix λ by placing the eigenvectors in RHR order and then using them in turn as the rows of λ. Compare the result with $\mathbb{R}_z(\phi)$.

47.) Tensors exist or they do not. They do not appear in one coordinate frame and disappear in another. Read the complete problem statement and then think.
The rule for transforming the representation of a vector is:

$$\tilde{F}' = \lambda\ \tilde{F} \qquad\qquad \tilde{F}_i' = \sum_j \lambda_{i\,j}\ \tilde{F}_j$$

All physical vectors transform using the identical rule. As vectors transform using one power of λ they are designated tensors of rank one.

Tensors of rank two such as the (moment of) inertia tensor or the electric quadrupole tensor are physical entities or fields that have as many components as the dimension of the universe raised to the second power with these components following the transformation rule are tensors.

$$\tilde{\tilde{\alpha}}' = \lambda\ \tilde{\tilde{\alpha}}\ \ \lambda^{-1} = \lambda\ \tilde{\tilde{\alpha}}\ \ \lambda^{t}\ ;\ \ \tilde{\tilde{\alpha}}'_{ij} = \sum_{m,\ell} \lambda_{im}\, \tilde{\tilde{\alpha}}_{m\ell}\, \lambda^{t}_{\ell j} = \sum_{m,\ell} \lambda_{im}\, \tilde{\tilde{\alpha}}_{m\ell}\, \lambda_{j\ell} = \sum_{m,\ell} \lambda_{im}\, \lambda_{j\ell}\, \tilde{\tilde{\alpha}}_{m\ell}$$

Tensors of rank *r*: Physical entities or fields that have as many components as the dimension of the universe to the *r* power **and** have components that obey the transformation rule:

$$\tilde{\tilde{\tilde{\alpha}}}'_{i_1 i_2 \ldots i_r} = \sum_{j_1 j_2 \ldots j_r} \lambda_{i_1 j_1} \lambda_{i_1 j_2} \ldots \lambda_{i_r j_r}\, \tilde{\tilde{\tilde{\alpha}}}_{j_1 j_2 \ldots j_r}$$

Show that if all the components of a tensor are zero in one coordinate system, that they are all zero in any coordinate system. Assume that λ is the linear transformation for the change of coordinates. **Conclude**: If a physical tensor has at least one non-zero component in a coordinate system, then it must have at least one non-zero component in every coordinate system. We have focused on rotations, but any linear transformation should be allowed.

48.) The matrix representations for rotations about the Cartesian axes are:

$$\mathbb{R}_{\mathbf{x}}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}\ ;\ \mathbb{R}_{\mathbf{y}}(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \text{ and } \mathbb{R}_{\mathbf{z}}(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a.) Give the matrix that represents a rotation of $^{\pi}/_{2}$ about the *y*-axis followed by a rotation of $^{\pi}/_{2}$ about the then *x*-axis. Complete the matrix multiplication.

b.) A vector $\vec{V}$ has a component representation $\underset{\sim}{V} = 75\,m\ \hat{i} + 50\,m\ \hat{j} + 25\,m\,\hat{k}$ in an original frame. Give the component representation $\underset{\sim}{V}'$ of this vector in a frame rotated by $\phi = \cos^{-1}(0.8) \approx$ $+\ 36.87^{0}$ about the ***y* axis** relative to the original frame. Compute the square root of the sum of the squares of the components in two coordinate system representations: $\underset{\sim}{V}$ and $\underset{\sim}{V}'$.

c.) A tensor $\mathbb{A}$ has the representation $\Rightarrow \underset{\approx}{A} = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ in the original frame. Using the

transformation matrix $\lambda = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$, write down the explicit matrix multiplication that

would give $\underset{\sim}{A}'$, the form of the matrix as expressed with respect to the basis vectors of the rotated frame.

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5B.) Prove that if a real symmetric matrix is transformed using a similarity transformation that the result is also a symmetric matrix.

5C.) Prove that the trace of a matrix is invariant under a similarity transformation using an orthogonal transformation.

X2. (*2%*) Prove that $|\lambda| = \pm 1$ for an orthogonal relation. Use $\lambda^t = \lambda^{-1}$ and the properties of determinants. (Note that a rotation matrix has determinant +1. Orthogonal transformations are the rotations $\oplus$ inversions.)

X3.) Consider two arbitrary vectors ($\Rightarrow$ their components transform according to the vector rule) and show that their inner product is preserved by orthogonal transformations.

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Hermitian operator: An operator is Hermitian if it is its own Hermitian conjugate. The example above $^{d}/_{dx}$ did not meet this standard, but all operators that represent physical observables do so let us press onward. What can be said about the matrix that represents a Hermitian operator? The matrix elements have the form $\langle e_k | \hat{H} | e_m \rangle = \langle e_k | \hat{H}\, e_m \rangle = \langle \hat{H}^{\dagger} e_k \,|\, e_m \rangle$. The last equality follows from the definition of the Hermitian conjugate operator. **IMPORTANT:** There is another process that moves things from the *KET* to the *BRA*. The third property of the inner product interchanges the *BRA* and *KET* at the cost of one complex conjugation.

I3. $\langle I | M \rangle = \langle M | I \rangle^{*}$ asterisk means complex conjugate *'Hermitian'*

$$\langle e_k | \hat{H} | e_m \rangle = \langle e_k | \hat{H}\, e_m \rangle = \langle \hat{H}^{\dagger} e_k \,|\, e_m \rangle = \langle e_m \,|\, \hat{H}^{\dagger} e_k \rangle^{*}$$

If the operator is Hermitian, then $\hat{H} = \hat{H}^{\dagger}$.

$$\langle e_k | \hat{H} | e_m \rangle = \langle e_k | \hat{H}\, e_m \rangle = \langle \hat{H}^{\dagger} e_k \,|\, e_m \rangle = \langle e_m \,|\, \hat{H}^{\dagger} e_k \rangle^{*} = \langle e_m | \hat{H}^{\dagger} | e_k \rangle^{*} = \langle e_m | \hat{H} | e_k \rangle^{*}$$

$$H_{km} = (H_{mk})^{*} \quad \text{or} \quad \mathbb{H} = (\mathbb{H}^{t})^{*} = \mathbb{H}^{t}$$

The matrix representing the operator is equal to the complex conjugate of its transpose. That is: it's Hermitian. Hermitian is used to describe a bundle of features because all those features are related.

Exercise: Consider the space of all functions of *x* defined on the interval (*a, b*) that are continuously differentiable to all orders. Further all functions in the space have the property that *f(a) = f(b)*. The operator is $\hat{d} = -i\left({}^{d}/_{dx}\right)$ and the inner product is the function space standard. The factor $i = \sqrt{-1}$.

$$\langle g | h \rangle = \frac{1}{N} \int_a^b \left(g(x)\right)^{*} h(x)\, dx$$

Find the differential operator representing: $\hat{d}^{\dagger}$. Is $\hat{d}$ Hermitian? Explain.

Note: The condition *f(a) = f(b)* is crucial. It can be arbitrarily imposed, but it occurs more naturally for functions that are periodic with period *b - a* or that vanish at *a* and *b*.

Exercise: Consider the space of all functions of *x*, a single real variable, defined on the interval (*a, b*) that are continuously differentiable to all orders. Further all functions in the space have the property that *f(a) = f(b)*. The operator is $\hat{\ell} = x$ and the inner product is the function space standard.

$$\langle g | h \rangle = \frac{1}{N} \int_a^b \left(g(x) \right)^* h(x)\, dx$$

Find the operator representing: $\hat{\ell}^\dagger$. Is $\hat{\ell}$ Hermitian? Explain.

Operations that equal the identity when applied a second time are call *idempotent*.

Unary minus:	- (- A) = A
Multiplicative inverse:	1/(1/A) = A
Complex conjugate:	$(A^*)^* = A$
Matrix transpose:	$(\mathbb{M}^t)^t = \mathbb{M}$
Matrix Hermitian Conjugate:	$(\mathbb{M}^t)^t = \mathbb{M}$

Exercise: Use the definition of Hermitian conjugate and property I3 of the inner product to show that $\left(\hat{o}^\dagger\right)^\dagger = \hat{o}$. Use

$$\left\langle \hat{o}^\dagger V_m \middle| V_j \right\rangle = \left\langle V_m \middle| \hat{o}\, V_j \right\rangle \quad \forall \left| V_j \right\rangle, \left| V_m \right\rangle \in \mathbb{V}.$$

It is understood that operators act to the right so $\left\langle V_m \middle| \hat{o} \middle| V_j \right\rangle \Rightarrow \left\langle V_m \middle| \hat{o}\, V_j \right\rangle$. Adding the Hermitian conjugate operator:

$\left\langle V_m \middle| \hat{o} \middle| V_j \right\rangle \Rightarrow \left\langle V_m \middle| \hat{o}\, V_j \right\rangle = \left\langle \hat{o}^\dagger V_m \middle| V_j \right\rangle$. Beware of a careless interpretation that arises from too much familiarity with Hermitian operators for which

$$\left\langle V_m \middle| \hat{h} \middle| V_j \right\rangle \Rightarrow \left\langle V_m \middle| \hat{h}\, V_j \right\rangle = \left\langle \hat{h} V_m \middle| V_j \right\rangle \text{ as } \hat{h}^\dagger = \hat{h}.$$

In general, the action direction and ordering of operators matters. To the right is right.