Multipole Expansions

Recommended as prerequisites

- **Vector Calculus**
- **Coordinate systems**
- **Separation of PDEs - Laplace's equation**

Concepts of primary interest:

- **Consistent expansions and small parameters**
- **Multipole moments**
  - **Monopole moment**
  - **Dipole moment**
  - **Quadrupole moment**

Sample calculations:

- **Decomposition example**
- **Quadrupole expansion with details**
- **To Add: Energy of a distribution in an external potential**
- **Using Symmetry to Avoid Calculations**

Tools of the trade:

- **Taylor's series expansions in three dimensions**
- **Relation to the Legendre Expansion in Griffiths**
- **Using Symmetry to Avoid Calculations**

A multipole expansion provides a set of parameters that characterize the potential due to a charge distribution of finite size at large distances from that distribution. The goal is to represent the potential by a series expansion of the form:

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$$V(\vec{r}) \quad = \quad V_0(\vec{r}) \quad + \quad V_1(\vec{r}) \quad + \quad V_2(\vec{r}) + \ldots \qquad \text{[MP.1]}$$

total = zeroth order + first order + second order + …

where each term $V_n(\vec{r})$ is proportional to $(Coulomb's\,Constant)\frac{charge}{distance}\left(\frac{size\ of\ the\ distribution}{distance\{to\ field\ point\}}\right)^n$.

The small expansion parameter is $\left(\frac{size\ of\ the\ distribution}{distance\{to\ field\ point\}}\right)$ or $\left(\frac{typical\,linear\,dimension}{distance\{to\ field\ point\}}\right)$ and n is the order of the term in the small parameter.

The Coulomb integral provides an exact expression for the electrostatic potential at points in space.

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')\ d^3r'}{|\ \vec{r}-\vec{r}'\ |} \qquad \text{[MP.2]}$$

The symbol d^3r' represents a volume element for the source (primed) coordinates, $\vec{r}'$ is the position vector for the source charge and $\vec{r}$ is the position vector for the field point at which the potential is to be computed. The equation directs that a "sum" over the source charges be computed. Repeating this sum for each field point $\vec{r}$ would be a daunting task. Instead, the denominator is to be expanded in a power series in the primed sources coordinates, the coordinates over which one must integrate, so that a few integrals over the source coordinates can be computed once and then reused to approximate to the potential at many field points $\vec{r}$. $\Rightarrow$ We want an approximation that can be evaluated again and again for new field points without any requirement to compute any additional integrals over the source coordinates.

$$\text{Def}^{\text{n}}:\quad D(\vec{r};\vec{r}') = \frac{1}{|\ \vec{r}-\vec{r}'\ |} = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = D(x,y,z;x',y',z')$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int D(\vec{r};\vec{r}')\rho(\vec{r}')\ d^3r' = \frac{1}{4\pi\varepsilon_0}\int [D_0(\vec{r};\vec{r}')+D_1(\vec{r};\vec{r}')+D_2(\vec{r};\vec{r}')+\ \ldots]\rho(\vec{r}')\ d^3r'$$

Given that the field point coordinates x, y, z are fixed during the integrations, a Taylor's series expansion in the source point coordinates x', y', z' about (0,0,0) provides an approximation of the source coordinate dependence of $D(\vec{r};\vec{r}')$. Adopt the standard subscript-component conventions: [$x' = x_1'$, $y' = x_2'$ and $z' = x_3'$]

$$D(\vec{r};\vec{r}') = D(\vec{r};\vec{0}) + \sum_{i=1}^{3} \frac{\partial D}{\partial x_i'}\bigg|_{\vec{r}'=0} \left(x_i' - 0\right) + \frac{1}{2!}\sum_{i,j=1}^{3} \frac{\partial^2 D}{\partial x_i'\,\partial x_j'}\bigg|_{\vec{r}'=0} \left(x_i' - 0\right)\left(x_j' - 0\right) + \ .....$$

$$D(\vec{r};\vec{r}') = D_0(\vec{r};\vec{r}') \quad + \quad D_1(\vec{r};\vec{r}') \quad + \quad D_2(\vec{r};\vec{r}') \quad + \ ....$$

Note that the derivatives are evaluated at $\vec{r}' = \vec{0}$. This expansion is expected to be useful only if the successive terms get smaller rapidly. We will find that if no source charges are farther than $r'_{\max}$ from the origin then the expansion may be useful for $r > r'_{\max}$. **That is**: at field points *outside* the source distribution.

To begin, $D(\vec{r};\vec{r}') = \dfrac{1}{\sqrt{\left(x-x'\right)^2 + \left(y-y'\right)^2 + \left(z-z'\right)^2}}$ and $D_o(\vec{r};\vec{r}') = D(\vec{r};0) = \dfrac{1}{r}$.

Using the zero order D, the monopole (2^0 – pole) contribution follows:

$$V_0(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int D_0(\vec{r},\vec{r}')\rho(\vec{r}')\ d^3r' = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')\ d^3r'}{r} = \frac{Q_{total}}{4\pi\varepsilon_o r} \qquad \text{[MP.3]}$$

Point Question: The integration is over source coordinates. After the integration, the result $V_0(\vec{r})$ only depends on which coordinates?

It is not surprising that the simplest approximation for large distances is to consider the distribution to be a point charge Q_{total} located at the origin. The distribution looks like a point charge of charge equal to the net or total charge of the distribution when viewed from very large distances.

Computing $\dfrac{\partial D}{\partial x_1'} = \dfrac{\partial D}{\partial x'}$ reveals the form of the next term in the expansion.

$$\frac{\partial}{\partial x'}\left[\frac{1}{\left|\vec{r}-\vec{r}'\right|}\right] = \frac{-\left(\tfrac{1}{2}\right)2\left(x-x'\right)\left(\frac{\partial\left(x-x'\right)}{\partial x'}\right)}{\left[\left(x-x'\right)^2 + \left(y-y'\right)^2 + \left(z-z'\right)^2\right]^{3/2}}$$

$$= \frac{\left(x-x'\right)}{\left[\left(x-x'\right)^2 + \left(y-y'\right)^2 + \left(z-z'\right)^2\right]^{3/2}} \rightarrow \frac{x}{\left[x^2+y^2+z^2\right]^{3/2}} \ \text{ for } (x',y',z') = (0,0,0).$$

Exercise: Given $\frac{\partial D}{\partial x_1'} = \frac{\partial D}{\partial x'}$, what are $\frac{\partial D}{\partial x_2'}$ and $\frac{\partial D}{\partial x_3'}$?

As always, the derivatives are to be taken using a form of $D(\vec{r};\vec{r}')$ that explicitly displays its dependences on all six coordinates. The results are evaluated for $x' = y' = z' = 0$ only after the derivatives have been taken. *Take care not to evaluate a function using specific values of its arguments prior to computing derivatives.*

The dipole (2^1 - pole) contribution:

$$D_1(\vec{r},\vec{r}') = \sum_{i=1}^{3} \frac{\partial D}{\partial x_i'}\bigg|_{\vec{r}'=0} x_i' = r^{-3}\sum_{i=1}^{3} x_i\, x_i' = r^{-3}\left(xx' + yy' + zz'\right) = r^{-3}\left(\vec{r}\cdot\vec{r}'\right)$$

$$V_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\vec{r}\cdot\vec{r}'\rho(\vec{r}')\, d^3r'}{r^3} = \frac{\vec{r}\cdot\int \vec{r}'\rho(\vec{r}')\, d^3r'}{4\pi\varepsilon_0 r^3} = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_o r^3} \qquad \text{[MP.4]}$$

$$\text{where } \vec{p} = \int \vec{r}'\rho(\vec{r}')\, d^3r \qquad \text{[MP.5]}$$

The first correction to $V_0(\vec{r})$ is the contribution due to a point dipole $\vec{p}$ at the origin where $\vec{p}$ is the net dipole moment of the source charge distribution.

(A physical dipole could be represented by the distribution with - q at $-\frac{1}{2}\vec{\ell}$ and q at $\frac{1}{2}\vec{\ell}$ where q and ℓ are finite. This distribution has no monopole moment, but it has higher order odd moments (such as octupole) in addition to a dipole moment. A point dipole is defined as the limit of a distribution with - q at $-\frac{1}{2}\vec{\ell}$ and q at $\frac{1}{2}\vec{\ell}$ as $\ell \to 0$ and $q\vec{\ell} = \vec{p}$ is held constant ($\Rightarrow q$ is doubled when ℓ is halved). In this limit, the dipole moment remains finite as $\ell \to 0$ while the higher moments go to zero. The distribution becomes a pure or point dipole.)

Exercise: To prepare for the next step, interpret the following recalling that x_j where $j = 1 \Rightarrow x$; $= 2 \Rightarrow y$ and $= 3 \Rightarrow z$ and that field and source coordinates are independent of one another.

$$\frac{\partial x_1}{\partial x_1} = \qquad ; \frac{\partial x_2}{\partial x_1} = \qquad ; \frac{\partial x_2}{\partial x_1'} = \qquad ; \frac{\partial x_2'}{\partial x_2'} = \qquad ; \frac{\partial x_3}{\partial x_2'} = \qquad .$$

Find compact representations for: $\frac{\partial (x_i - x_i')}{\partial x_j'}$ and $\frac{\partial (x_i - x_i')}{\partial x_j}$. Write out the relations for various sets of integer values substituted for i and j if that is needed to clarify the interpretations of the symbols.

Second partials must be computed to generate the *second order* terms.. $\frac{\partial^2 D}{\partial x_i' \, \partial x_j'} =$

$$\frac{\partial^2}{\partial x_i' \, \partial x_j'}\left[\frac{1}{|\vec{r} - \vec{r}'|}\right] = \frac{3\left(x_i - x_i'\right)\left(x_j - x_j'\right)}{\left[\left(x - x'\right)^2 + \left(y - y'\right)^2 + \left(z - z'\right)^2\right]^{5/2}} + \frac{\frac{\partial}{\partial x_i'}\left(x_j - x_j'\right)}{\left[\left(x - x'\right)^2 + \left(y - y'\right)^2 + \left(z - z'\right)^2\right]^{3/2}}$$

$$= \frac{3\left(x_i - x_i'\right)\left(x_j - x_j'\right)}{\left[\left(x - x'\right)^2 + \left(y - y'\right)^2 + \left(z - z'\right)^2\right]^{5/2}} - \frac{\delta_{ij}}{\left[\left(x - x'\right)^2 + \left(y - y'\right)^2 + \left(z - z'\right)^2\right]^{3/2}}$$

$$\rightarrow \left[\frac{\partial^2}{\partial x_i' \, \partial x_j'}\left(\frac{1}{|\vec{r} - \vec{r}'|}\right)\right]_{\vec{r}' = \vec{0}} = \frac{3 x_i \, x_j - r^2 \, \delta_{ij}}{r^5} \text{ with } (x', y', z') = (0,0,0).$$

In order to reach the standard form for this term, a few summation (*dummy*) indices need to be changed. To begin we recast $\sum_{i,j}\left[3 x_i \, x_j - r^2 \, \delta_{ij}\right] x_i' \;\; x'$

$$\sum_{i,j}\left[3 x_i \, x_j - r^2 \, \delta_{ij}\right] x_i' \;\; x_j' = \sum_{i,j}\left[3 x_i \, x_j\right] x_i' \;\; x_j' - \sum_{k,\ell} \delta_{k\ell} \;\; x_k \, x_\ell \sum_{i,j} \;\; \delta_{ij} \, x_i' \;\; x_j'$$

$$= \sum_{i,j}\left[3 x_i' \;\; x_j'\right] x_i \, x_j - \sum_{i,j} \;\; \delta_{ij} \, x_i' \;\; x_j' \sum_{k,\ell} \delta_{k\ell} \;\; x_k \, x_\ell = \sum_{i,j}\left[3 x_i' \;\; x_j'\right] x_i \, x_j - \left(r'\right)^2 \sum_{k,\ell} \delta_{k\ell} \;\; x_k \, x_\ell$$

$$= \sum_{i,j}\left[3 x_i' \;\; x_j'\right] x_i \, x_j - \left(r'\right)^2 \sum_{i,j} \delta_{ij} \;\; x_i \, x_j = \sum_{i,j}\left[3 x_i' \;\; x_j' - \left(r'\right)^2 \delta_{ij}\right] \;\; x_i \, x_j$$

In the last step, the dummy labels were renamed *i* and *j*. As a result of the entire process, the forms for the primed and unprimed values are interchanged! Clearly this step is not magic; it's just not obvious.

$$D_2(\vec{r},\vec{r}') = \frac{1}{2!}\sum_{i,j=1}^{3}\frac{\partial^2 D}{\partial x_i'\,\partial x_j'}\bigg|_{\vec{r}'=0} x_i'\,x_j' = \frac{1}{2r^5}\sum_{i,j}\left[3x_i'\;x_j' - (r')^2\,\delta_{ij}\right]\,x_i\,x_j$$

After pulling the unprimed variables out of the integral,

$V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int D_2(\vec{r},\vec{r}')\rho(\vec{r}')\;d^3r'$ becomes

$$V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i,j}\frac{x_i\,x_j}{2r^5}\int\rho(\vec{r}')\left[3x_i'\;x_j' - (r')^2\,\delta_{ij}\right]d^3r' = \frac{\tfrac{1}{2}\sum_{i,j}Q_{ij}x_i\,x_j}{4\pi\,\varepsilon_0 r^5} \qquad \text{[MP.6]}$$

$$\text{where}\quad Q_{ij} = \int\rho(\vec{r}')\left[3x_i'\;x_j' - (r')^2\,\delta_{ij}\right]d^3r' \qquad \text{[MP.7]}$$

$$Q_{ij} = \sum_{\substack{Sources\\ pt.charges\ \alpha}} q_\alpha\left[3x_{\alpha i}\,x_{\alpha j} - r_\alpha^2\;\delta_{ij}\right] \quad (\textbf{\textit{for point charges}}) \qquad \text{[MP.8]}$$

Multipole summary: $q_{net} = \int_{sources} dq;\;\; \vec{p} = \int_{sources}\vec{r}\,dq;\;\; Q_{ij} = \int_{sources}\left[3x_i x_j - r^2\,\delta_{ij}\right]dq$

The Q_{ij} are the Cartesian quadrupole (2^2 - pole) moments. The 3 x 3 matrix of the Q_{ij} is traceless and symmetric and hence has 5 independent components. Continuing this horrendous expansion shows in general that the 2^ℓ - pole term has $2\ell+1$ independent quantities defined in terms of integrals over the source charge distribution. The ***reduction*** from the initial 3^ℓ terms to the (2ℓ + 1) independent parameters in each order becomes progressively more painful.

$$V(\vec{r}) = \frac{q_{total}}{4\pi\varepsilon_o r} + \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_o r^3} + \frac{1}{4\pi\varepsilon_0}\sum\frac{x_i\,x_j}{2\,r^5}\,Q_{ij} + higher\ order\ than\ two \qquad \text{[MP.9]}$$

Attention: In the expression for *V* just above, all the information about the source charges and their locations (correct to second [quadrupole] order) is held internally in the monopole, dipole and quadrupole moments. The expression for $V(\vec{r})$ is the

approximation correct through second order which is expected to be useful at the field points $\vec{r}$ if $r >> r_{\max}$. The x_i and x_j are the components of the field point position $\vec{r}$. The symbols r, $\vec{r}$, x_j and x_j have ***no relation*** to source charge coordinates. Read this statement every time you complete a multipole problem until you are absolutely sure that you know why it is suggested that you do so.

Exercise: Verify that the matrix $\mathbb{Q}$ with elements Q_{ij} is traceless and symmetric. Justify the claim that the 3 x 3 matrix has just five independent elements.

Note that: $$\sum_{i,j} x_i Q_{ij} x_j = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underset{\sim}{x}^t \underset{\approx}{Q} \underset{\sim}{x}$$

Exercise: Express $D(\vec{r};\vec{r}')$ as $\left[\sum_{m=1}^{3}(x_m - x'_m)^2\right]^{-1/2}$. Show that the first partial derivative is:

$$\frac{\partial D}{\partial x'_i} = (-\tfrac{1}{2})\left[\sum_{m=1}^{3}(x_m - x'_m)^2\right]^{-3/2} \sum_{p=1}^{3}\left[2(x_p - x'_p)\frac{\partial(x_p - x'_p)}{\partial x'_i}\right]$$

Note that the index label p is chosen to be distinct from m.

The index label i is the free or external index (not a summation index). It cannot be changed freely.

All references to a free (external) index must be identified and changed in unison. $\Rightarrow$ Leave then fixed!

Complete the evaluations of $\frac{\partial D}{\partial x'_i}$ and $\frac{\partial^2 D}{\partial x'_j \partial x'_i}$. Evaluate the final forms for $\vec{r}' = 0$.

The smallest set of parameters that completely describe an order in the expansion is an irreducible set. Later, we will see than an expansion in spherical coordinates in terms of solutions of the Laplace equation in spherical coordinates is more natural and yields reduced representations directly.

order	description	V varies as	E varies as	parameters
zero	monopole	r^{-1}	r^{-2}	1

first	dipole	r^{-2}	r^{-3}	3
second	quadrupole	r^{-3}	r^{-4}	5
ℓth	2ℓ -pole	$r^{-(\ell+1)}$	$r^{-(\ell+2)}$	$2\ell + 1$

Expressions for Computing Multipoles:

$$q_{net} = \int dq' = \begin{cases} \int \left(\rho(\vec{r}')\, dV'\right) \\ \int \left(\sigma(\vec{r}')\, dA'\right) \\ \int \left(\lambda(\vec{r}')\, d\ell'\right) \\ \sum_{sources\ \alpha} q_\alpha \end{cases} \qquad \vec{p}_{net} = \int \vec{r}'\, dq' = \begin{cases} \int \vec{r}' \left(\rho(\vec{r}')\, dV'\right) \\ \int \vec{r}' \left(\sigma(\vec{r}')\, dA'\right) \\ \int \vec{r}' \left(\lambda(\vec{r}')\, d\ell'\right) \\ \sum_{sources\ \alpha} \vec{r}_\alpha\, q_\alpha \end{cases}$$

net charge dipole moment

$$Q_{ij} = \int \left[3x_i' x_j' - (r')^2 \delta_{ij}\right] dq' = \begin{cases} \int \left[3x_i' x_j' - (r')^2 \delta_{ij}\right] \left(\rho(\vec{r}')\, dV'\right) \\ \int \left[3x_i' x_j' - (r')^2 \delta_{ij}\right] \left(\sigma(\vec{r}')\, dA'\right) \\ \int \left[3x_i' x_j' - (r')^2 \delta_{ij}\right] \left(\lambda(\vec{r}')\, d\ell'\right) \\ \sum_{sources\ \alpha} q_\alpha \left[3x_{\alpha i}\, x_{\alpha j} - (r_\alpha)^2 \delta_{ij}\right] \end{cases} \qquad \text{[MP.10]}$$

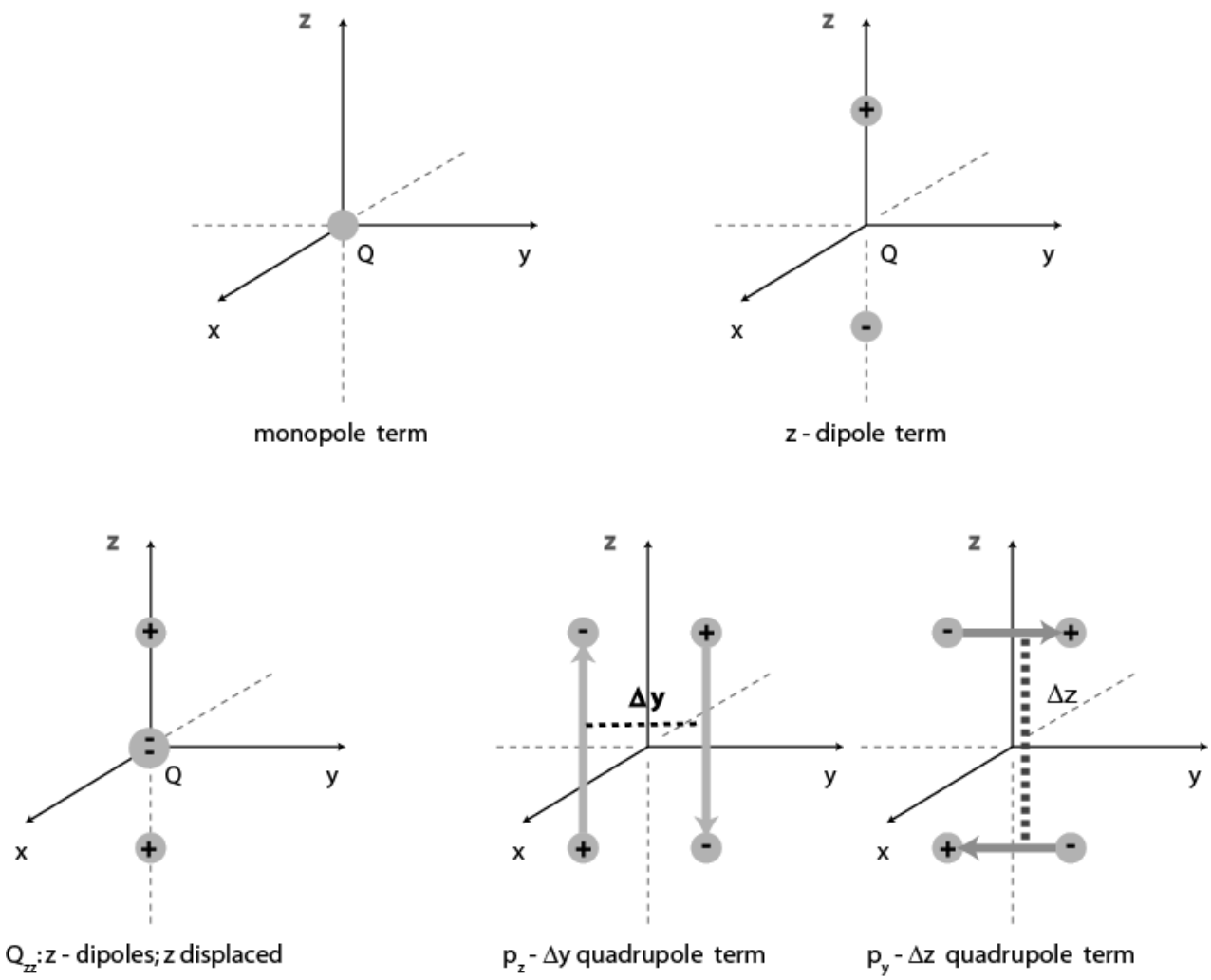
quadrupole moments

Far-Field Potential Expansion Correct through Quadrupole Terms:

$$\begin{aligned} V(\vec{r}) &= V_0(\vec{r}) + V_1(\vec{r}) + V_2(\vec{r}) + \ldots \\ &= \frac{q_{total}}{4\pi\varepsilon_o r} + \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_o r^3} + \frac{1}{4\pi\varepsilon_0} \sum \frac{x_i\, x_j}{2\, r^5}\, Q_{ij} + \ldots \end{aligned} \qquad \text{[MP.11]}$$

Note that: r, $\vec{r}$, x_i and x_j are *field point coordinate values*.

All source location data is hidden and fixed inside q_{total}, $\vec{p}$ and the Q_{ij}.

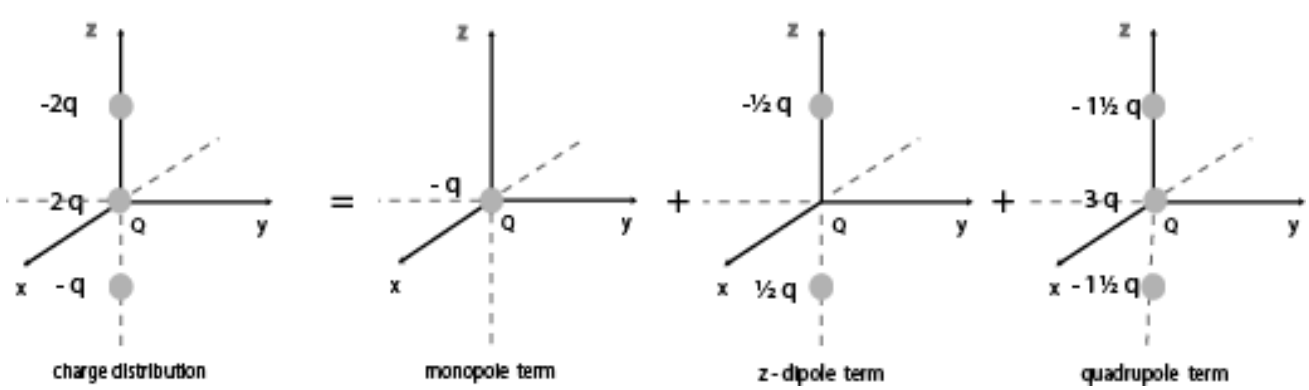


We declare that Q_{jk} is the quadrupole that results from displacing a positive p_j dipole in the positive x_k direction from an equivalent $-p_j$ dipole (equal magnitude, but oppositely directed).

Note: There is **one** value for the net charge, the monopole moment. A dipole is two equal, but opposite charges displaced from one another. The displacement can be made in each of the three independent directions, so there are **three** independent components in the electric dipole. A quadrupole is formed by displacing two equal, but opposite dipoles from one another. A basic Q_{zz} is formed by displacing two equal, but opposite z-dipoles along their direction. A pair of opposite p_z dipoles displaced in the y direction from one another yields the same charge pattern as a pair of opposite p_y dipoles displaced from one another in the z direction. It follows that Q_{zy} and Q_{yz} describe the

same charge distribution patterns. As defined, $Q_{ij} = Q_{ji}$ and the matrix of the quadrupole moments Q_{ij} is real and symmetric. Consider quadrupoles like Q_{zz} pictured above. The sum $Q_{xx} + Q_{yy} + Q_{zz}$ is a charge – 6 q at the origin plus six charge +q equidistantly from the origin along each axis. These six charges are the best *six charge approximation* to a uniformly charged shell. The field due to a point charge of charge – 6 q surrounded by a concentric spherical shell of net charge + 6 q is zero outside the 'shell'. For this reason, $Q_{xx} + Q_{yy} + Q_{zz} = 0$. The trace of the quadrupole matrix is zero. The symmetric property alone reduces the 3 x 3 quadrupole matrix to six independent components. Adding the vanishing trace constraint, the number of independent elements in the quadrupole matrix is found to be **five**. A general 2^{ℓ} multipole has $2\ell + 1$ independent elements.

S1. Decomposition Example: Consider the charge distribution - 2 q at $a\ \hat{k}$, 2 q at 0 $\hat{k}$, -q at - $a\ \hat{k}$. Find an equivalent distribution with a charge q_{total} at the origin, a dipole centered on the origin and a remainder. As a first model, restrict the charge locations to be one of the set of *actual charge locations* { $a\ \hat{k}$, 0 $\hat{k}$, - $a\ \hat{k}$ } and their inverse points with respect to the origin. A charge at $\vec{r}_1$ would means that the point $-\vec{r}_1$ would also be available for use. The net charge is $q_{\text{net}} = -q$, and the net dipole moment is: $\vec{p} =$ $(-2q)(a\ \hat{k}) + (2q)(0\ \hat{k}) + -q)(-a\ \hat{k}) = -q\,a\ \hat{k}$. It is possible to center each multipole about the origin except perhaps the highest one in the decomposition. In this example, *it just happens* that the highest multipole charge arrangement, the quadrupole, is also centered. (The exercise below *ends* with an off-center multipole.)



What is the net charge of the dipole term? What are the net charge and the net dipole moment of the quadrupole term?

A 'centered' finite dimensioned 2^{ℓ} pole has higher order terms in its expansion beginning with the $2^{\ell+2}$ - pole. A finite dimensioned 2^{ℓ} - pole that is displaced from the centered position has higher order terms in its expansion beginning with the $(2^{\ell+1})$ - pole.

Exercise: Suppose that a charge distribution is a single point charge that is not at the origin. Show that distribution is equivalent to a point charge at the origin plus a finite non-centered dipole. Show that a finite non-centered dipole is equivalent to a centered finite dipole at the origin plus a finite non-centered quadrupole.

S2. Dipole Moment and Potential:

Consider charges q at ½ $\ell\ \hat{k}$ and $-q$ at -½ $\ell\ \hat{k}$.

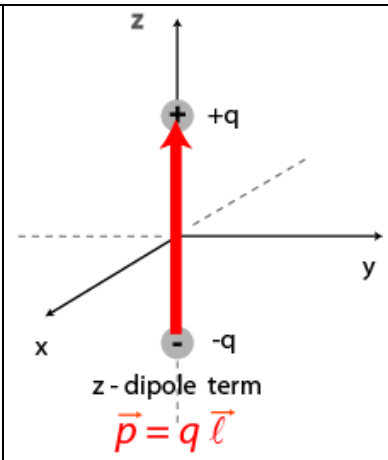
$$V(\vec{r}) = V_0(\vec{r}) + V_1(\vec{r}) + V_2(\vec{r}) + \ldots$$

$$= \frac{q_{total}}{4\pi\varepsilon_o r} + \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_o r^3} + \frac{1}{4\pi\varepsilon_0}\sum \frac{x_i\, x_j}{2\, r^5}\, Q_{ij} + \ldots$$

The net charge is zero, so out first approximation is:

$$V(\vec{r}) \approx V_0(\vec{r}) + V_1(\vec{r}) = \frac{0}{4\pi\varepsilon_o r} + \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_o r^3} = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_o r^3}$$

The dipole moment is a characteristic of the sources and contains source charge and location information: (q, ℓ,…).



$$\vec{p} = \sum_{\alpha} q_{\alpha}\, \vec{r}_{\alpha} = q\,\vec{\ell} = q\,\ell\hat{k}$$

The position vector $\vec{r}$ on the other hand is a field point position. So $\vec{r} = x\hat{i} + y\,\hat{j} + z\,\hat{k}$ where x, y, z are the field-valued coordinates of the field point at which the potential is to be evaluated.

$$V(\vec{r}) \approx \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_o r^3} = \frac{q\vec{\ell}\hat{k}\cdot(x\hat{i}+y\,\hat{j}+z\hat{k})}{4\pi\varepsilon_o r^3} = \frac{q\ell z}{4\pi\varepsilon_o[x^2+y^2+z^2]^{3/2}} = \frac{q\ell\cos\theta}{4\pi\varepsilon_o r^2}$$

(*Far-field expansions are only valid for* $|\vec{r}\,| > |\vec{r}\,'|_{\max}$.)

Exercise: Classify each symbol in $\frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_o r^3}; \frac{q\vec{\ell}\cdot(x\hat{i}+y\hat{j}+z\hat{k})}{4\pi\varepsilon_o r^3}; \frac{q\ell z}{4\pi\varepsilon_o[x^2+y^2+z^2]^{3/2}}$ as a source-related quantity, a field point related quantity or a constant (independent of the source and field point positions).

S3. Quadrupole Moment and Potential

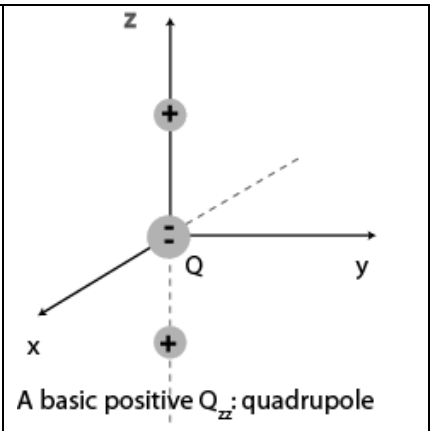
Consider charges q at $a\ \hat{k}$ and at $-a\ \hat{k}$, $-2\ q$ at $0\ \hat{k}$.

$$V(\vec{r}) = V_0(\vec{r}) + V_1(\vec{r}) + V_2(\vec{r}) + \ldots$$

$$= \frac{q_{total}}{4\pi\varepsilon_o r} + \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_o r^3} + \frac{1}{4\pi\varepsilon_0}\sum \frac{x_i\, x_j}{2\, r^5} Q_{ij} + \ldots$$

The net charge and dipole moment are zero, so the leading order contribution to the potential is the quadrupole terms:

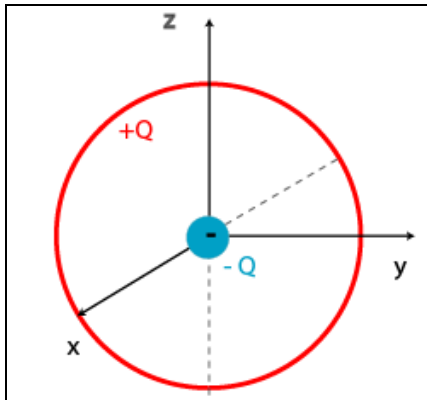
$$V(\vec{r}) \approx V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum \frac{x_i\, x_j}{2\, r^5} Q_{ij}$$



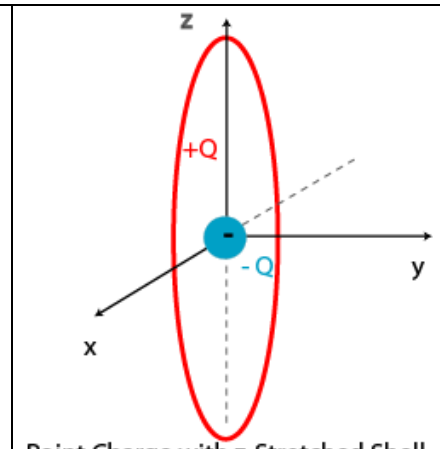
ONE MORE TIME!: Note that x_i and x_j are field point coordinates; one integrates (or sums) over the source point coordinates to compute the multipole moments. It is those resulting source moments that appear in the expression for the potential, not the source coordinates. Once a coordinate has been integrated over, it is like a summation index that has been summed over; it does not appear as a variable in the result. In the expression, $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')\, d^3r'}{|\vec{r}-\vec{r}'|}$ the integration over the source (primed) variable generates a function that depends only on the field (unprimed) variables. All of the r's, x_i's and x_j's that appear in $V(\vec{r}) = \frac{q_{total}}{4\pi\varepsilon_o r} + \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_o r^3} + \frac{1}{4\pi\varepsilon_0}\sum \frac{x_i\, x_j}{2\, r^5} Q_{ij} + \ldots$ are field point values.

A quadrupole moment can be developed for a directionally neutral (isotropic) distribution by distorting it. Consider a point charge $-Q$ surrounded by a concentric uniformly charged spherical shell with charge $+Q$. This distribution does not cause an

electric field or contribute to the potential outside of the shell. A Q_{zz} quadrupole moment can be developed by stretching the positively charged shell along the plus and minus z directions.



Point Charge with Concentric Shell causes zero field outside the shell



Point Charge with z-Stretched Shell

Stretching the positive charge along the z axes gives a positive Q_{zz}. At the same time, positive charge is pulled in along the x and y directions. The result is that Q_{xx} and Q_{yy} are negative for this distribution.

$$Q_{xx} + Q_{yy} + Q_{zz} = 0 \quad (always)$$

The charge of the shell is equal and opposite to that of the central point charge.

Calculating quadrupole moments for a distribution of point charges:

The simple pattern of point charges q at $a\ \hat{k}$ and at $-a\ \hat{k}$, $-2\ q$ at $0\ \hat{k}$ is a model with the characteristics $q_{total} = 0$, $\vec{p} = 0$ and $Q_{zz} \neq 0$. The moments, including the quadrupole moments, are to be calculated as examples.

$$q_{total} = \sum_\alpha q_\alpha \qquad \vec{p} = \sum_\alpha \vec{r}_\alpha\, q_\alpha \qquad Q_{ij} = \sum_\alpha q_\alpha \left[3x_{\alpha i}\, x_{\alpha j} - (r_\alpha)^2 \delta_{ij}\right]$$

$$q_{total} = \sum_\alpha q_\alpha = q + q + (-2q) = 0$$

$$\vec{p} = \sum_\alpha \vec{r}_\alpha\, q_\alpha = q(a\,\hat{k}) + q(-q\,\hat{k}) + (-2q)(0\,\hat{k}) = \vec{0}$$

$$Q_{33} = Q_{zz} = \sum_\alpha q_\alpha \left[3z_\alpha\, z_\alpha - (r_\alpha)^2 \delta_{33}\right] =$$
$$q[3a^2 - a^2] + q[3(-a)^2 - a^2] + (-2q)[3(0)^2 - 0^2] = 4qa^2$$

$$Q_{11} = Q_{xx} = \sum_\alpha q_\alpha \left[3x_\alpha\, x_\alpha - (r_\alpha)^2 \delta_{11}\right] =$$
$$q[3(0)^2 - a^2] + q[3(0)^2 - a^2] + (-2q)[3(0)^2 - 0^2] = -2qa^2$$

$$Q_{xz} = Q_{13} = \sum_\alpha q_\alpha \left[3x_\alpha\, z_\alpha - (r_\alpha)^2 \delta_{13}\right] = \sum_\alpha q_\alpha \left[3x_\alpha\, z_\alpha\right] =$$
$$q[3(0)a] + q[3(0)(-a)] + (-2q)[3(0)(0)] = 0$$

After more calculations: $Q_{xx} = Q_{yy} = -2qa^2$; $Q_{zz} = 4qa^2$; $Q_{xy} = Q_{yx} = Q_{xz} = Q_{zx} = Q_{yz} = Q_{zy} = 0$. Note that the quadrupole moments depend on the sources charges and locations. Each moment has the dimensions Coulomb-(meters)^2. (What are the dimensions of q_{total} and of $\vec{p}$?)

$$V(\vec{r}) = V_0(\vec{r}) + V_1(\vec{r}) + V_2(\vec{r}) \approx \frac{0}{4\pi\varepsilon_o r} + \frac{0}{4\pi\varepsilon_o r^3} + \frac{1}{4\pi\varepsilon_0}\sum \frac{x_i\, x_j}{2\, r^5}\, Q_{ij} +$$

$$V(\vec{r}) \approx \left(\frac{1}{2}\right)\frac{1}{4\pi\varepsilon_0 r^5}\sum_{i,j} Q_{ij}\, x_i\, x_j = \frac{1}{8\pi\varepsilon_0 r^5}\left(Q_{xx}\, xx + Q_{xy}\, xy + Q_{xz}\, xz + Q_{yx}\, yx + Q_{yy}\, yy + \ldots\right)$$

$$V(\vec{r}) \approx \frac{1}{8\pi\varepsilon_0 r^5}\left(\left[-2qa^2\right]xx + \left[0\right]xy + \ldots + \left[-2qa^2\right]yy + \ldots + \left[4qa^2\right]zz\right)$$

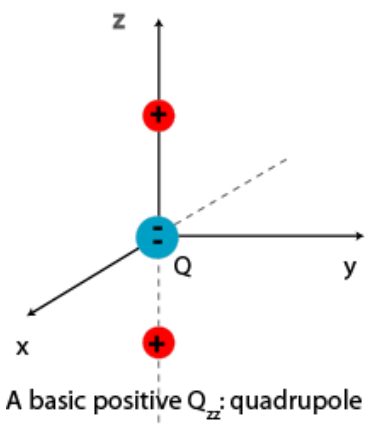
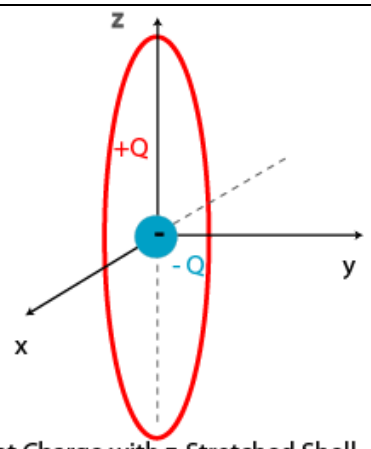
$$V(\vec{r}) \approx \frac{2qa^2[2z^2 - x^2 - y^2]}{4\pi\varepsilon_0 r^5} = \frac{2qa^2[2z^2 - x^2 - y^2]}{4\pi\varepsilon_0 [x^2 + y^2 + z^2]^{5/2}} = \frac{2qa^2[3z^2 - x^2 - y^2 - z^2]}{4\pi\varepsilon_0 [x^2 + y^2 + z^2]^{5/2}}$$

Spherical coordinates: use $z = r\cos\theta$ and $P_2(\cos\theta) = {}^3/_2 \cos^2\theta - {}^1/_2$

$$V(\vec{r}) \approx \frac{2qa^2[3r^2\cos^2\theta - r^2]}{4\pi\varepsilon_0 r^5} = \frac{4qa^2}{4\pi\varepsilon_0 r^3} P_2(\cos\theta)$$

(*Far-field expansions are only valid for* $|\vec{r}\,| > |\vec{r}\,'|_{\max}$.)

Exercise: Class each symbol in the two lines of equations above as a source-related quantity, a field point related quantity or a constant (independent of the source and field point positions).

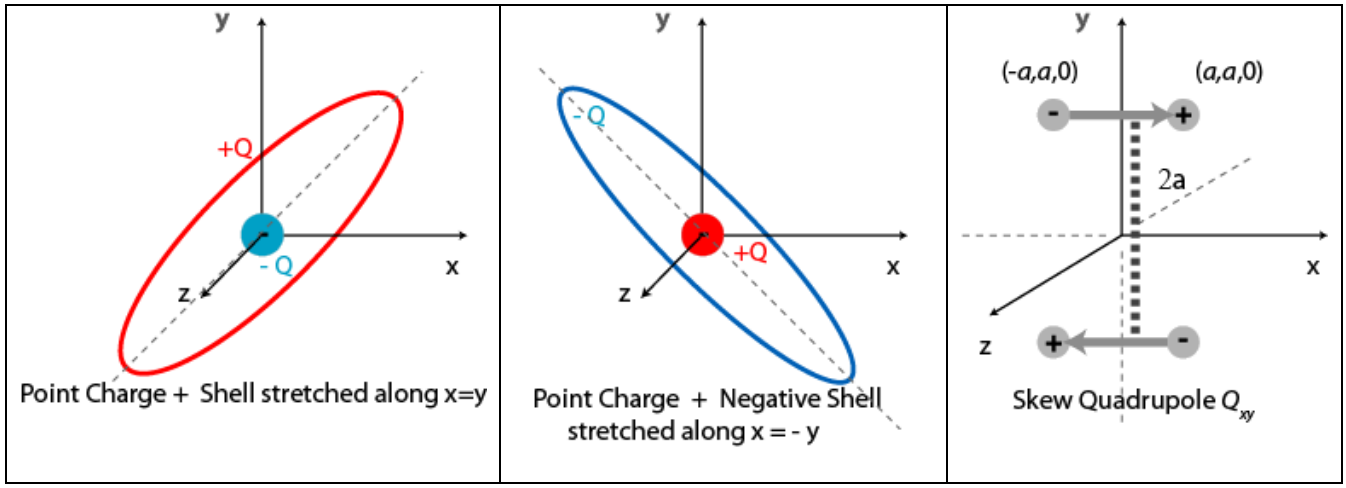
 A basic positive Q_{zz}: quadrupole	 Point Charge with z-Stretched Shell	Stretching the positive charge along the z axes gives a positive Q_{zz}. At the same time, positive charge is pulled in along the x and y directions. The result is that Q_{xx} and Q_{yy} are negative for this distribution. $Q_{xx} + Q_{yy} + Q_{zz} = 0$ (*always*)

The charge of the shell is equal and opposite to that of the central point charge.

Exercise: Note that these charge distributions might be expected to cause a positive potential in cones of space around the positive and negative z axes and a belt of negative potential around the equatorial (x-y) plane. Use the far-field potential $V(\vec{r}) \approx \frac{2qa^2[3\cos^2\theta - 1]}{4\pi\varepsilon_0 r^3} = \frac{4qa^2}{4\pi\varepsilon_0 r^3} P_2(\cos\theta)$ to find the constant θ coordinate surfaces that divide space into these three regions. Label the regions as either + or – depending on the sign of the potential in that region. (The second order or leading non-vanishing order expansion is expected to be useful for $|\vec{r}| >> |\vec{r}'|_{\max}$). Add a centered circle of radius 3 a to your drawing. Does excluding the region $r < 3a$ from consideration make sense?

Skew Moments: The quadrupoles studied to this point have had non-zero elements only on the diagonal. An off-diagonal or skew Q_{xy} quadrupole is developed from an isotropic (direction neutral) distribution when positive charge is pulled out along the line $x = y$ or negative charge is pulled out along the line $x = -y$. From another point of view, a skew moment arises when two equal but opposite dipoles are displaced from one another along a line perpendicular to their directions. A simple non-diagonal Q_{xy}

quadrupole is: q at (a,a,0), -q at (-a,a,0), q at (-a,-a,0) and -q at (-a,-a,0). This four charge distribution has the characters of pulling positive charge out along $x = y$; pulling negative charge out along $x = -y$ and of the displacement of equal and opposite dipoles along a direction perpendicular to their directions.



The total charge is zero as is the net dipole moment. The quadrupole moments are:

$$Q_{ij} = \sum_{\alpha} q_{\alpha} \left[3x_{\alpha i}\, x_{\alpha j} - (r_{\alpha})^2 \delta_{ij} \right]$$

$$Q_{33} = Q_{zz} = \sum_{\alpha} q_{\alpha} \left[3z_{\alpha}\, z_{\alpha} - (r_{\alpha})^2 \delta_{33} \right] =$$

$$q[3(0)^2 - 2a^2] + (-q)[3(0)^2 - 2a^2] + q[3(0)^2 - 2a^2] + (-q)[3(0)^2 - 2a^2] = 0$$

$$Q_{11} = Q_{xx} = \sum_{\alpha} q_{\alpha} \left[3x_{\alpha}\, x_{\alpha} - (r_{\alpha})^2 \delta_{11} \right] =$$

$$q[3(a)^2 - 2a^2] + (-q)[3(-a)^2 - 2a^2] + q[3(-a)^2 - 2a^2] + (-q)[3(a)^2 - 2a^2] = 0$$

Clearly, Q_{yy} must vanish as well. (Why?)

$$Q_{xy} = Q_{12} = \sum_{\alpha} q_{\alpha} \left[3x_{\alpha}\, y_{\alpha} - (r_{\alpha})^2 \delta_{12} \right] = \sum_{\alpha} q_{\alpha} \left[3x_{\alpha}\, y_{\alpha} \right] =$$

$$q[3(a)(a)] + (-q)[3(-a)(a)] + q[3(-a)(-a)] + (-q)[3(a)(-a)] = 12\, qa^2$$

$$Q_{xz} = Q_{13} = \sum_{\alpha} q_{\alpha} \left[3x_{\alpha}\, z_{\alpha} - (r_{\alpha})^2 \delta_{13} \right] = \sum_{\alpha} q_{\alpha} \left[3x_{\alpha}\, z_{\alpha} \right] =$$

$$q[3(a)(0)] + (-q)[3(-a)(0)] + q[3(-a)(0)] + (-q)[3(a)(0)] = 0$$

The conclusion is that $Q_{xy} = Q_{yx} = 12\ qa^2$, and all the other quadrupole moments are zero.

$$V(\vec{r}) \approx \left(\frac{1}{2}\right)\frac{1}{4\pi\varepsilon_0 r^5}\sum_{i,j} Q_{ij}\, x_i\, x_j = \frac{1}{8\pi\varepsilon_0 r^5}\left(Q_{xx}\, xx + Q_{xy}\, xy + Q_{xz}\, xz + Q_{yx}\, yx + Q_{yy}\, yy + \ldots\right)$$

$$V(\vec{r}) \approx \frac{1}{8\pi\varepsilon_0 r^5}\left([0]\, xx + [12qa^2]\, xy + [0]\, xz + [12qa^2]\, yx + [0]\, yy + \ldots\right)$$

$$V(\vec{r}) \approx \frac{12qa^2\, xy}{4\pi\varepsilon_0 r^5} = \frac{12qa^2 \sin^2\theta \sin\phi \cos\phi}{4\pi\varepsilon_0 r^3}$$

(Far-field expansions are only valid for $|\vec{r}\,| > |\vec{r}\,'|_{\max}$.)

Be able to give the Cartesian and spherical representations of each result.

Exercise: Note that this charge distribution might be expected to cause a positive potential in the far-field for certain values of ϕ and negative potential for other ranges of ϕ. Guess these ranges by studying the figure and then verify them using the result that $V(\vec{r}) \approx \frac{12qa^2 \sin^2\theta \sin\phi \cos\phi}{4\pi\varepsilon_0 r^3}$ in the far-field ($|\vec{r}\,| >> |\vec{r}\,'|_{\max}$). Confirm your results by expressing sinϕ cosϕ as a single trig function.

(Never use far-field approximations at field points inside the source distribution. They are not valid unless $|\vec{r}\,| > |\vec{r}\,'|_{\max}$.

The expansion cannot be expected to converge quickly unless $\vec{r}\,| >> |\vec{r}\,'|_{\max}$)

S4. Calculating quadrupole moments for a continuous charge distribution

Given a surface charge densityσ(θ,ϕ) on a shell of radius R:

$$\sigma(\theta,\phi) = \sigma_0 \left(\cos\theta\right)^2 \text{ located at } r = R.$$

$$q_{net} = \int dq' = \begin{cases} \int \left(\rho(\vec{r}\,')\, dV'\right) \\ \int \left(\sigma(\vec{r}\,')\, dA'\right) \\ \int \left(\lambda(\vec{r}\,')\, d\ell'\right) \\ \sum_{sources\ \alpha} q_\alpha \end{cases} \qquad \vec{p}_{net} = \int \vec{r}\,'\, dq' = \begin{cases} \int \vec{r}\,' \left(\rho(\vec{r}\,')\, dV'\right) \\ \int \vec{r}\,' \left(\sigma(\vec{r}\,')\, dA'\right) \\ \int \vec{r}\,' \left(\lambda(\vec{r}\,')\, d\ell'\right) \\ \sum_{sources\ \alpha} \vec{r}_\alpha\, q_\alpha \end{cases}$$

net charge dipole moment

$$Q_{ij} = \int \left[3x_i' x_j' - (r')^2 \delta_{ij} \right] dq' = \begin{cases} \int \left[3x_i' x_j' - (r')^2 \delta_{ij} \right] \left(\rho(\vec{r}') \, dV' \right) \\ \int \left[3x_i' x_j' - (r')^2 \delta_{ij} \right] \left(\sigma(\vec{r}') \, dA' \right) \\ \int \left[3x_i' x_j' - (r')^2 \delta_{ij} \right] \left(\lambda(\vec{r}') \, d\ell' \right) \\ \sum_{sources\ \alpha} q_\alpha \left[3x_{\alpha i} x_{\alpha j} - (r_\alpha)^2 \delta_{ij} \right] \end{cases} \qquad \text{[MP.12]}$$

quadrupole moments

a. Compute the total charge of the distribution. $\sigma(\theta,\phi) = \sigma_0 \left(\cos\theta \right)^2$ on the surface $r = R$.

$$q_{net} = \int dq \rightarrow \int \left(\sigma(\vec{r}') \, dA' \right) = \int_0^\pi \int_0^{2\pi} \sigma_0 \left(\cos\theta' \right)^2 \; R^2 \sin\theta' \, d\theta' \, d\phi'$$

$$= 2\pi\sigma_0 R^2 \int_0^\pi \left(\cos\theta' \right)^2 \; \sin\theta' \, d\theta' = \sigma_0 \tfrac{4\pi}{3} R^2$$

Used: $\sigma(\theta',\phi') = \sigma_0 \left(\cos\theta' \right)^2$ and $\mathrm{dA}' \, R^2 \sin\theta' \, d\theta' \, d\phi'$

b. Compute the electric dipole moment of the distribution.

$$\vec{p}_{net} = \int \vec{r}' \, dq = \int \vec{r}' \left(\sigma(\vec{r}') \, dA' \right)$$

$$= \int R(\sin\theta' \cos\phi' \, \hat{i} + \sin\theta' \sin\phi' \, \hat{j} + \cos\theta' \, \hat{k}) \left(\sigma(\vec{r}') \, dA' \right)$$

$$\vec{p}_{net} = \vec{0}$$

Note that $\vec{r}' = x' \hat{i} + y' \hat{j} + z' \hat{k}) = R(\sin\theta' \cos\phi' \, \hat{i} + \sin\theta' \sin\phi' \, \hat{j} + \cos\theta' \, \hat{k})$. It's not hard! Just write it out.

c. Compute Q_{zz} and Q_{xx}.

$$Q_{ij} = \int \left[3x_i' x_j' - (r')^2 \delta_{ij} \right] dq = \int \left[3x_i' x_j' - (r')^2 \delta_{ij} \right] \left(\sigma(\vec{r}') \, dA' \right)$$

$$Q_{zz} = \int \left[3R^2 (\cos\theta')^2 - R^2 \delta_{ij} \right] \left(\sigma(\vec{r}') \, dA' \right)$$

$$Q_{xx} = \int \left[3R^2 (\sin\theta' \cos\phi')^2 - R^2 \delta_{ij} \right] \left(\sigma(\vec{r}') \, dA' \right)$$

$$Q_{xy} = \int \left[3R^2 (\sin\theta')^2 \cos\phi' \sin\phi' - R^2 (0) \right] \left(\sigma(\vec{r}') \, dA' \right)$$

The remainder of the evaluation is to be assigned homework.

Used: $x' = R\sin\theta'\cos\phi'; \quad y' = R\sin\theta'\sin\phi'; \quad z' = R\cos\theta'$

Energy of a Charge Distribution in an External Electrostatic Field

A closely related calculation is that of the potential energy of the charge distribution due to an externally applied electrostatic potential.

$$\Delta U_{ext} = \int V_{ext}(\vec{r}')\ \rho(\vec{r}')\ d^3r'$$

This time, the external potential is expanded in a Taylor's series about a reference point (origin) for the charge distribution with $\vec{r}'$ representing the position relative to that origin of each charge in the distribution subject to the external potential field. External potential field means ***that the source charges for that potential are located outside the region occupied by the charge distribution of interest***. This condition ensures that the external potential satisfies the Laplace equation in the region occupied by the charge distribution. $\nabla^2 V_{ext} = 0 \Rightarrow \vec{\nabla}\cdot\vec{E}_{ext} = 0$

each of the derivatives is evaluated at $\vec{r}' = \vec{0}$

$$V_{ext}(\vec{r}') = V_{ext}(\vec{0}) + \sum_{i=1}^{3} \left.\frac{\partial V_{ext}}{\partial x_i'}\right|_{\vec{r}'=\vec{0}} \left(x_i'\right) + \frac{1}{2!}\sum_{i,j=1}^{3} \left.\frac{\partial^2 V_{ext}}{\partial x_i'\ \partial x_j'}\right|_{\vec{r}'=\vec{0}} \left(x_i'\ x_j'\right) + \ .....$$

$$\Delta U_{ext} = \int \left[V_{ext}(\vec{0}) + \sum_{i=1}^{3} \left.\frac{\partial V_{ext}}{\partial x_i'}\right|_{\vec{r}'=\vec{0}} \left(x_i'\right) + \frac{1}{2!}\sum_{i,j=1}^{3} \left.\frac{\partial^2 V_{ext}}{\partial x_i'\ \partial x_j'}\right|_{\vec{r}'=\vec{0}} \left(x_i'\ x_j'\right) + \ ..... \right] \rho(\vec{r}')\ d^3r'$$

$$\Delta U_{ext}^{(0)} = \int \left[V_{ext}(\vec{0}) \right] \rho(\vec{r}')\ d^3r' = Q_{total}\ V_{ext}(\vec{0})$$

$$\Delta U_{ext}^{(1)} = \int \left[\sum_{i=1}^{3} \left.\frac{\partial V_{ext}}{\partial x_i'}\right|_{\vec{0}} \left(x_i'\right) \right] \rho(\vec{r}')\ d^3r' = -\vec{E}_{ext}(\vec{0})\cdot\int \vec{r}'\ \rho(\vec{r}')\ d^3r' = -\vec{p}\cdot\vec{E}_{ext}(\vec{0})$$

$$\Delta U_{ext}^{(2)} = \int \left[\frac{1}{2!}\sum_{i,j=1}^{3} \left.\frac{\partial^2 V_{ext}}{\partial x_i'\ \partial x_j'}\right|_{\vec{0}} \left(x_i'\ x_j'\right) \right] \rho(\vec{r}')\ d^3r' = \frac{1}{2!}\sum_{i,j=1}^{3} \left.\frac{\partial^2 V_{ext}}{\partial x_i'\ \partial x_j'}\right|_{\vec{0}} \int x_i'\ x_j'\ \rho(\vec{r}')\ d^3r'$$

equivalently: $$\Delta U_{ext}^{(2)} = -\frac{1}{2!}\sum_{i,j=1}^{3} \left.\frac{\partial\left[\vec{E}_{ext}\right]_j}{\partial x_i'}\right|_{\vec{0}} \int x_i'\ x_j'\ \rho(\vec{r}')\ d^3r'$$

A magic trick transforms the last term to the quadrupole form. Recall that the external field is due to charges located elsewhere. The external field is caused by a charge distribution $\rho_{external}$ which is identically zero in thee region containing the charge distribution of interest (where $\rho(\vec{r}^{\,/}) \neq 0$). Gauss's Law states that: $\vec{\nabla} \cdot \vec{E}^{ext} = \varepsilon_0^{-1} \rho_{ext} = 0$. Subtracting 0 or its alias, $\frac{1}{6}\, r^2\, \vec{\nabla} \cdot \vec{E}^{ext}$, recasts the second order term as:

$$\Delta U_{ext}^{(2)} = -\frac{1}{6} \sum_{i,j=1}^{3} \frac{\partial \left[\vec{E}_{ext} \right]_j}{\partial x_i'} \bigg|_{\vec{0}} \int \left[3 x_i'\, x_j' - \left(r' \right)^2 \delta_{ij} \right] \rho(\vec{r}')\, d^3 r' = -\frac{1}{6} \sum_{i,j=1}^{3} \frac{\partial \left[\vec{E}_{ext} \right]_j}{\partial x_i'} \bigg|_{\vec{0}} Q_{ij}$$

One might be troubled by the need for a magic step and perhaps even skeptical that such a compatible step could be found. This happy circumstance is less surprising when one realizes that the expansion of V_{ext} involves $|\vec{r} - \vec{r}'|^{-1}$, the same basic function as did the far-field expansion that led to the multipole moments, and that that function is symmetric under the interchange of source and field coordinates.

In summary

$$\Delta U_{ext} = Q_{total}\, V_{ext}(\vec{0}) - \vec{p} \cdot \vec{E}_{ext}(\vec{0}) - \frac{1}{6} \sum_{i,j=1}^{3} Q_{ij} \frac{\partial \left[\vec{E}_{ext} \right]_j}{\partial x_i'} \bigg|_{\vec{0}} + \ldots \qquad \text{[MP.13]}$$

$$\Delta U_{ext} = Q_{total}\, V_{ext}(\vec{0}) + \sum_{i,j=1}^{3} p_j \frac{\partial V_{ext}}{\partial x_j'} \bigg|_{\vec{0}} + \frac{1}{6} \sum_{i,j=1}^{3} Q_{ij} \frac{\partial^2 V_{ext}}{\partial x_i'\, \partial x_j'} \bigg|_{\vec{0}} + \ldots$$

This expansion clarifies the manner in which a charge distribution interacts with an external potential. The net charge interacts with the potential; the dipole moment interacts with the gradient of the potential (the electric field); the quadrupole moments interact with the gradient of the external electric field and so on. Understand that 'the dipole moment interacts with the gradient of the potential' just means that the total energy of a dipole in a uniform potential is zero. The net interaction energy for a pair of equal, but opposite charges depends on the potential at the location of the positive charge being different from that at the location of the negative charge. There is no net

interaction energy in the absence of a gradient of the potential (That is: a non-zero electric field).

Further expansion of the potential to 3rd order leads to a requirement for a fist full of magic tricks to identify the particular combinations of the terms to yield the most meaningful 7 (= 2 ℓ + 1) independent parameters describing the charge distribution. As a rule, a pure point multipole should have zero values for all its lower multipole moments. The reduction to 2 ℓ + 1 parameters in ℓ^{th} order occurs naturally in spherical coordinates as demonstrated below.

Spherical Coordinate Expansion of the potential due to a charge distribution with finite spatial extent. The source charges are restricted to a finite region and, outside of that region, the potential satisfies the Laplace equation. As the distance from the charge distribution increases, the distribution appears more point-like, and the potential becomes more nearly spherically symmetric. This character suggests that a spherical rather than a Cartesian expansion might be more natural. Let us review the separation of Laplace's equation in spherical coordinates.

$$\nabla^2 V(\vec{r}) = \left(\frac{1}{r^2}\right)\frac{\partial}{\partial r}\left(r^2\frac{\partial V(\vec{r})}{\partial r}\right) + \left(\frac{1}{r^2\sin\theta}\right)\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V(\vec{r})}{\partial\theta}\right) + \left(\frac{1}{r^2\sin^2\theta}\right)\frac{\partial^2 V(\vec{r})}{\partial\phi^2} = 0$$

The solution is assumed to separate as $V(r,\theta,\phi) = R(r)\ \Theta(\theta)\ \Phi(\phi)$ with the results:

$$\nabla^2\left[R(r)\ \Theta(\theta)\ \Phi(\phi)\right] = \Theta(\theta)\ \Phi(\phi)\left(\frac{1}{r^2}\right)\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right) + R(r)\ \Phi(\phi)\left(\frac{1}{r^2\sin\theta}\right)\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)$$
$$+ R(r)\ \Theta(\theta)\left(\frac{1}{r^2\sin^2\theta}\right)\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = 0$$

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) = \ell\left(\ell+1\right)R(r)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)=-\left[\ell\left(\ell+1\right)-\frac{m^2}{\sin^2\theta}\right]\ \Theta(\theta)$$

$$\frac{d^2\Phi(\phi)}{d\phi^2}=-m^2\,\Phi(\phi)$$

For problems with no ϕ dependence, the solutions which are regular for $\theta = 0, \pi$ take the form:

$$V(r,\theta)=\sum_{\ell=0}^{\infty}\left[a_\ell\ r^\ell+b_\ell\ r^{-(\ell+1)}\right]P_\ell(\cos\theta)$$

The P_ℓ are the Legendre polynomials. With ϕ dependence, the solutions become (restricted to solutions regular in θ,ϕ)

$$V(r,\theta,\phi)=\sum_{\ell=0}^{\infty}\ \sum_{m=-\ell}^{\ell}\ \left[a_{\ell m}\ r^\ell+b_{\ell m}\ r^{-(\ell+1)}\right]Y_{\ell m}(\theta,\phi)$$

where the $Y_{\ell m}$ are the spherical harmonics (*that are the angular dependences of the electron wave functions for the hydrogen atom*).

$$Y_{\ell m}(\theta,\phi)=(-1)^m\sqrt{\frac{(2\,\ell+1)\ \ (\ell-m)!}{4\,\pi\quad(\ell+m)!}}\ P_\ell^m(\cos\theta)\ e^{im\phi}$$

$$P_\ell^m(x)=\left(-1\right)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell^m(x)\ ;\qquad\left(m\geq 0\right)$$

$$\int_0^{2\pi}\ \int_0^{\pi}Y^*_{\ell' m'}(\theta,\phi)Y_{\ell m}(\theta,\phi)\ \sin\theta\,d\theta\,d\phi=\delta_{\ell\ell'}\ \delta_{mm'}$$

for reference

$$Y_{\ell 0}(\theta,\phi)=\sqrt{\frac{(2\,\ell+1)}{4\,\pi}}\ P_\ell\ (\cos\theta)\quad\text{and}\ Y_{\ell m}(\theta,\phi)=\left(-1\right)^m Y_{\ell,-m}(\theta,\phi)$$

With enormous difficulty, it can be shown that for $\vec{r}(r,\theta,\phi)$ and $\vec{r}^{\,/}(r^{/},\theta^{/},\phi^{/})$ the $\frac{1}{\left|\vec{r}'-\vec{r}\right|}$ has the expansion: (*This development is to be included in the PDEs Handout.*)

$$\frac{1}{|\vec{r}^{\,/}-\vec{r}|}=\sum_{\ell=0}^{\infty}\ \sum_{m=-\ell}^{m=\ell}\frac{4\,\pi}{2\,\ell+1}\left(\frac{r_<^\ell}{r_>^{\ell+1}}\right)Y^*_{\ell m}(\theta',\phi')\ Y_{\ell m}(\theta,\phi)$$

where $r_<$ is the smaller of r and r' and $r_>$ is the greater. Applying this incredible result, a spherical multipole expansion follows as:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} \frac{4\pi}{2\ell+1} \left(\frac{1}{r^{\ell+1}} \right) q_{\ell m} \; Y_{\ell m}(\theta,\phi)$$

For points outside the finite source distribution, the multipole moments $q_{\ell m}$ are defined as:

$$q_{\ell m} = \int_{\substack{source \\ volume}} Y^*_{\ell' m'}(\theta,\phi) r'^{\ell} \; \rho(\vec{r}\,') d^3\vec{r}\,'$$

Note that as m runs from = -ℓ to +ℓ; there are 2ℓ+ 1 expansion quantities for the $r^{-(\ell+1)}$ contribution to the potential naturally!

Again, the closely related problem of computing the potential energy of the charge distribution due to an externally applied electrostatic potential leads to the spherical multipole moments.

$$\Delta U_{ext} = \int_{\substack{charge \\ distribution}} V_{ext}(\vec{r}\,') \; \rho(\vec{r}\,') d^3\vec{r}\,'$$

As the external potential is due to charges located outside the region occupied by the charge distribution $\rho(\vec{r}^{\,/})$, the external potential satisfies Laplace's equation and can be expanded in the small argument form for the spherical coordinate solutions: (*Reference the Laplace Equation in Spherical Coordinates Handout.*)

$$V_{ext}(\vec{r}\,') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} \left(v^{ext}_{\ell m} \left(r'\right)^{\ell} \right) \; Y_{\ell m}(\theta',\phi')$$

Substituting into the expression for ΔU_{ext} we find:

$$\Delta U_{ext} = \int_{\substack{charge \\ distribution}} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} \left(v^{ext}_{\ell m} \left(r'\right)^{\ell} \right) \; Y_{\ell m}(\theta',\phi') \; \rho(\vec{r}\,') d^3\vec{r}\,'$$

$$\Delta U_{ext} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} v^{ext}_{\ell m} \int_{\substack{charge \\ distribution}} \left(r'\right)^{\ell} Y_{\ell m}(\theta',\phi') \; \rho(\vec{r}\,') d^3\vec{r}\,' = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} v^{ext}_{\ell m} \, q^*_{\ell m}$$

We note the various moments of the charge distribution only "interact" with the corresponding terms in the expansion of the potential demonstrating a behavior related to orthogonality.

Cartesian Representation of the Spherical Harmonics
$$Y_{00}(\theta,\phi)=\sqrt{\frac{1}{4\pi}}$$
$$Y_{10}(\theta,\phi)=\sqrt{\frac{3}{4\pi}}\cos\theta=\sqrt{\frac{3}{4\pi}}\left(\frac{z}{r}\right)$$ $$Y_{1,\pm1}(\theta,\phi)=\mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}=\sqrt{\frac{3}{8\pi}}\left(\frac{\mp x-i\,y}{r}\right)$$
$$Y_{20}(\theta,\phi)=\sqrt{\frac{3}{16\pi}}\left[\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2}\right]=\sqrt{\frac{3}{16\pi}}\left(\tfrac{3}{2}\left[\tfrac{z}{r}\right]^2-\tfrac{1}{2}\right)$$ $$Y_{2,\pm1}(\theta,\phi)=\mp\sqrt{\frac{15}{8\pi}}\left[\sin\theta\cos\theta\, e^{\pm i\phi}\right]=\mp\sqrt{\frac{15}{8\pi}}\,\frac{z\left(x\pm i\,y\right)}{r^2}$$ $$Y_{2,\pm2}(\theta,\phi)=\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi}=\sqrt{\frac{15}{32\pi}}\left(\frac{x\pm i\,y}{r}\right)^2$$

BE CAUTIOUS: THIS HANDOUT IS UNDER DEVELOPMENT AND THE RELATIONS PRESENTED SHOULD BE CHECKED CAREFULLY BEFORE THEY ARE USED FOR IMPORTANT APPLICATIONS.

Relation between Cartesian and Spherical Moments
$$q_{00}=\sqrt{\tfrac{1}{4\pi}}\,q_{net}$$

$q_{10} = \sqrt{\frac{3}{4\pi}}\, p_z \qquad q_{11} = -\sqrt{\frac{3}{8\pi}}\left[p_x + i\, p_y\right]$
$q_{20} = \sqrt{\frac{5}{16\pi}}\, Q_{zz} \qquad q_{21} = -\sqrt{\frac{5}{24\pi}}\left[Q_{xz} - i\, Q_{yz}\right]$ $q_{22} = \sqrt{\frac{5}{96\pi}}\left[Q_{xx} - 2i\, Q_{xy} - Q_{yy}\right]$

APPLICATIONS OF MULTIPOLE EXPANSIONS:

1. ANALYSIS OF THE ENERGIES OF NUCLEAR STATES TO PROVIDE NUCLEAR STRUCTURE DATA.
2. TO CHARACTERIZE THE MAGNETIC FIELDS OF SUBMARINES AS A TOOL IN ADJUSTING ACTIVE DEGAUSSING MEASURES.
3. TO PREDICT ONE CONTRIBUTION TO THE PRECESSION OF THE PERIHELION OF MERCURY.
4. TO STUDY THE DYNAMICS OF ROTATING BODIES.
5. TO PREDICT THE ELECTROMAGNETIC RADIATION PATTERNS ASSOCIATED WITH ATOMIC AND MOLECULAR TRANSITIONS.
6. TO CHARACTERIZE THE MASS DISTRIBUTIONS, MAGNETIC FIELDS, AND GRAVITATIONAL FIELDS OF PLANETS.
7. TO PROBE THE STRUCTURES OF SOLIDS.

EXERCISE: Fill in the steps to complete the derivations of $V_1(\vec{r})$ on page 1 and of $V_2(\vec{r})$ on page 2. Verify the relations between q_{00} and q_{net} and between the first moments q_{10} and $q_{1\pm1}$ and the Cartesian dipole moments: p_x, p_y, and p_z as listed above.

NOTATION ALERT: The symbol $Y_{\ell m}$ represents our standard complex spherical harmonics. These symbols are occasionally hijacked to represent *real-valued* functions related to the spherical harmonics. We will use $Y_{\ell m}$ for the functions above, but other may use $Y_\ell^m(\theta, \phi)$ to emphasize that the *standard compelx-valued* spherical harmonics are to be substituted.

Physical and Point Multipoles:

A physical monopole is a finite charge distribution with a non-zero net charge q_{net}. The point moment limit is achieved by keeping the monopole moment constant while spatial (the distance of each charge from the origin) extent the distance of each charge from the origin) is reduced to zero. This limit leads to a point charge q_{net} at the origin. A physical dipole can be formed by separating two equal by opposite point monopoles q and -q. The dipole moment of the distribution is $\vec{p} = q\vec{\ell}$ where $\vec{\ell}$ is the displacement from the negative charge to the positive charge. If opposite charges are symmetrically displaced from the origin, the distribution can also have non-zero odd multipole moments of higher order. If the charges are not symmetrically displaced from the origin, the distribution can also have non-zero values for all multipole moments of higher order. A pure dipole is developed from a physical dipole by taking the limit that ℓ approaches zero while $\vec{p}$ is keep constant. That is each charge is doubled as its distance from the origin is halved. The 2^n - pole moment is proportional to the linear extent of the charge distribution taken to the nth power. Hence the higher order multipoles vanish in the limit that the spatial extent of the distribution is halved while all charges are doubled repeatedly.

The mulitpoles of a charge distribution are related to its symmetry. A symmetric charge distribution only has even order moments. A distribution that is anti-symmetric under inversion only has odd multipole moments. A finite multipole has its *base* multipole moment plus higher multipole moments. A dipole with its plus and minus charges symmetrically located (*an odd symmetry distribution*) has a base dipole moment plus odd order higher moments. A point charge at the origin is a pure monopole. A quadrupole distribution(one with zero net charge and zero net dipole moment) has a base quadrupole (2^2 – pole) moment and corrections that begin with hexadecupole (2^4 – pole) if the charge distribution is symmetric. If the distribution has mixed symmetry, there can be octupole (2^3 – pole) corrections as well.

If the charges are restricted to lie on a spherical shell, the position of the charges is fully symmetric and does not *complicate* that multipole structure. If the surface charge distribution on the shell is proportional to a single spherical harmonic $Y_{\ell m}$, then only the corresponding ℓ^{th} order moment is non-zero. More complicated distributions can be analyzed by invoking superposition. If the spherical shell with a fixed surface charge distribution on the shell is proportional to a single spherical harmonic $Y_{\ell m}$ is given a small radial distortion following a single spherical harmonic $Y_{\ell' m'}$ then the potential is expected to include small additional moments with orders in the range $|\ell - \ell'|$ to $|\ell + \ell'|$ following the rules for the addition of angular momenta.

Exercise: How would one form a physical quadrupole from two equal, but opposite dipoles?

Translating a charge distribution does not change ***its lowest non-zero multipole moment***, but it can change the values of all the higher order multipoles of the distribution.

Tools of the Trade:

3D Taylor's series: The task at hand is to develop an extension of the Taylor's series expansion appropriate for function of three variables, the Cartesian coordinates in three dimensions. The value of the function is known at $(x_0 , y_0,, z_0)$ and the value at (x_0+u, y_0+v, z_0+w) is to be predicted using derivatives computed in a small neighborhood of the point $(x_0,, y_0,, z_0)$. The distance between the points is

$s = \sqrt{u^2 + v^2 + w^2}$. The plan is to map the problem to a set of primed coordinate axes rotated such that the *x'* axis lies parallel to the line from (x_0,,y_0,,z_0) to (x_0+u,y_0+v,z_0+w). For points along this line, the function is a function of the single variable *x'*. The function is $f\,[x(x'),y(x'),z(x')]$ where $x(x'_o\ )=x_o$; $y(x'_o\ )=y_o$; $z(x'_o\ )=z_o$ and $x(x'_o\ +s\,)=x_o+u$; $y(x'_o+s\,)=y_o+v$; $z(x'_o+s\,)=z_o+w$. The rotation is a linear mapping.

$$\frac{dx}{dx'} = \frac{u}{s};\ \frac{dy}{dx'} = \frac{v}{s}\ \ \frac{dz}{dx'} = \frac{w}{s} \quad \textit{constant values}$$

Recalling the 1D Taylor's series

$$f(x') = f(x'_0) + \left.\frac{df}{dx'}\right|_{x'_0}(x' - x'_0) + \frac{1}{2!}\left.\frac{d^2 f}{d\left(x'\right)^2}\right|_{x'_0}(x' - x'_0)^2 + \ldots + \frac{1}{n!}\left.\frac{d^n f}{d\left(x'\right)^n}\right|_{x'_0}(x' - x'_0)^n + \ldots$$

and applying the chain rule:

$$\frac{d}{dx'} = \frac{\partial}{\partial x}\left(\frac{dx}{dx'}\right) + \frac{\partial}{\partial y}\left(\frac{dy}{dx'}\right) + \frac{\partial}{\partial z}\left(\frac{dz}{dx'}\right) = \left(\frac{u}{s}\right)\frac{\partial}{\partial x} + \left(\frac{v}{s}\right)\frac{\partial}{\partial y} + \left(\frac{w}{s}\right)\frac{\partial}{\partial z}$$

Noting that $x' - x'_0 = s$, the chain rule is substitutes into the Taylor's series.

$$\begin{aligned} f(x') = f(x_o, y_o, z_o) &+ \left.\left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right] f(x,y,z)\right|_{(x_o,y_o,z_o)} \\ &+ \frac{1}{2!}\left.\left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]^2 f(x,y,z)\right|_{(x_o,y_o,z_o)} + \ldots \\ &+ \frac{1}{n!}\left.\left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]^n f(x,y,z)\right|_{(x_o,y_o,z_o)} + \ldots \end{aligned}$$

Adopting the subscript notation $x_1 = x$, $x_2 = y$ *and* $x_3 = z$, the expansion becomes
Adopting the subscript notation $x_1 = x$, $x_2 = y$ *and* $x_3 = z$, the expansion becomes

$$\begin{aligned} f(x,y,z) = f(x_o, y_o, z_o) &+ \sum_{i=1}^{3}\left.\frac{\partial f}{\partial x_i}\right|_{(x_o,y_o,z_o)}\Delta x_i + \frac{1}{2!}\sum_{i,j=1}^{3}\left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{(x_o,y_o,z_o)}\Delta x_i\,\Delta x_j \\ &+ \frac{1}{3!}\sum_{i,j,k=1}^{3}\left.\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}\right|_{(x_o,y_o,z_o)}\Delta x_i\,\Delta x_j \Delta x_k + \ldots \end{aligned}$$

where $\Delta x_1 = \Delta x = u$, $\Delta x_2 = \Delta y = v$ *and* $\Delta x_3 = \Delta z = w$. The formula above is the three-dimensional Taylor's series expansion.

An alternative notation follows by rotating the axis to so that the new axis, the *x'*-axis lies along the line joining $\vec{r}_0$ and the point of interest $\vec{r}$. Let $\vec{r} - \vec{r}_0 = s\,\hat{e}$. The 1D Taylor's series

$$f(x') = f(x_0') + \left.\frac{df}{dx'}\right|_{x_0'} (x' - x_0') + \frac{1}{2!}\left.\frac{d^2 f}{d\left(x'\right)^2}\right|_{x_0'} (x' - x_0')^2 + \ldots + \frac{1}{n!}\left.\frac{d^n f}{d\left(x'\right)^n}\right|_{x_0'} (x' - x_0')^n + \ldots$$

can be expressed using the directional derivative, $\hat{e}\cdot\vec{\nabla} = e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}$, as represented in the original coordinate system. (*Reference the Vector Calculus handout or RHB p.163-6.)*

$$f(\vec{r}) = f(\vec{r}_0) + \left[\left(\hat{e}\cdot\vec{\nabla}\right) f(\vec{r})\right]\Big|_{\vec{r}_0} s + \tfrac{1}{2!}\left[\left(\hat{e}\cdot\vec{\nabla}\right)^2 f(\vec{r})\right]\Big|_{\vec{r}_0} s^2 + \tfrac{1}{3!}\left[\left(\hat{e}\cdot\vec{\nabla}\right)^3 f(\vec{r})\right]\Big|_{\vec{r}_0} s^3 + \ldots$$

$$f(\vec{r}) = f(\vec{r}_0) + \left[\left(\left[\vec{r} - \vec{r}_0\right]\cdot\vec{\nabla}\right) f(\vec{r})\right]\Big|_{\vec{r}_0} + \tfrac{1}{2!}\left[\left(\left[\vec{r} - \vec{r}_0\right]\cdot\vec{\nabla}\right)^2 f(\vec{r})\right]\Big|_{\vec{r}_0} + \tfrac{1}{3!}\left[\left(\left[\vec{r} - \vec{r}_0\right]\cdot\vec{\nabla}\right)^3 f(\vec{r})\right]\Big|_{\vec{r}_0} + \ldots$$

where $\Delta x_1 = \Delta x = u$, $\Delta x_2 = \Delta y = v$ *and* $\Delta x_3 = \Delta z = w$. The formula above is the three dimensional Taylor's series expansion and is the basis for the Cartesian multipole expansion presented in this handout.

The Relation to the Method of Griffiths: The relation:

$\frac{1}{\left|\vec{r} - \vec{r}'\right|} = \sum_{\ell=0}^{\infty} \frac{\left(r'\right)^{\ell}}{r^{(\ell+1)}} P_\ell(\hat{r}\cdot\hat{r}')$ *for* $r' < r$ is the basis for the multipole expansion presented in the E&M text written by David Griffiths. ***This identity is developed in the Laplace Equation handout.*** The expression for the potential becomes:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')\, d^3r'}{\left|\,\vec{r} - \vec{r}'\,\right|} = \frac{1}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\int \rho(\vec{r}')\,\frac{\left(r'\right)^{\ell}}{r^{(\ell+1)}}\, P_\ell(\hat{r}\cdot\hat{r}')\, d^3r' \quad \textit{for } r' < r$$

For example, the lowest order term is:

$$V_0(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}^{\,/})\frac{(r')^0}{r^{(0+1)}}\ P_0(\hat{r}\cdot\hat{r}')\ d^3r' = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{r}\ (1)\ d^3r' = \frac{\int \rho(\vec{r}')\ d^3r'}{4\pi\varepsilon_0 r}$$

The first order term is:

$$V_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')\ r'}{r^2}\ P_1(\hat{r}\cdot\hat{r}')\ d^3r' = \frac{1}{4\pi\varepsilon_0 r^3}\int \rho(\vec{r}')\ r\ \ r'\ \left(\hat{r}\cdot\hat{r}'\right)\ d^3r'$$

As $P_1(x) = x$, $P_1(\hat{r}\cdot\hat{r}') = \hat{r}\cdot\hat{r}'$.

$$V_1(\vec{r})^{/} = \frac{1}{4\pi\varepsilon_0 r^3}\int \ \rho(\vec{r}')\ \vec{r}\cdot\vec{r}'\ d^3r' = \frac{\vec{r}\cdot\int \rho(\vec{r}')\,\vec{r}'\,d^3r'}{4\pi\varepsilon_0 r^3} = \frac{\vec{r}\cdot\vec{p}}{4\pi\varepsilon_0 r^3}$$

Continuing the expansion generates higher order terms. The following relations prove to be useful.

$$\hat{r}\cdot\hat{r}' = \frac{\vec{r}\cdot\vec{r}'}{r\ \ r'} = \frac{\sum_{j=1}^{3} x_j\ x'_j}{r\ \ r'} \Rightarrow P_2(\hat{r}\cdot\hat{r}') = \tfrac{3}{2}\left(\hat{r}\cdot\hat{r}'\right)^2 - \tfrac{1}{2} = \tfrac{3}{2}\left(\frac{\sum_{j,m=1}^{3} x_j\ x_m\ \ x'_j\ x'_m}{\left(r\ \ r^{/}\right)^2}\right) - \tfrac{1}{2}$$

With a few more twists, the form $\tfrac{1}{2}\sum_{i,j}\left[3x'_i\ \ x'_j\ -\left(r'\right)^2\delta_{ij}\ \right]\ x_i\,x_j$ appears as required to match the definition of the Cartesian quadrupole moments.

Exercise: Show that $\sum_{j,m=1}^{3} 3x_j\,x_m\ \ x'_j\,x'_m - r'^2 r^2$ can be written as: $\sum_{j,m=1}^{3}\left[3x'_j\,x'_m - r'^2\delta_{jm}\right]x_j\,x_m$.

Recall that $\vec{A}\cdot\vec{B} = \sum_{j,m=1}^{3} A_j B_m \delta_{jm}$.

Using Symmetry to Avoid Calculations: Cartesian Multipoles

Consider a problem of the type:

Given a surface charge density on a shell of radius R:

$$\sigma(\theta,\phi) = \left[\sigma_0\ \right]\left[\cos^2\theta + \sin\phi\right]$$

a. Compute the total charge of the distribution.
b. Compute the electric dipole moment of the distribution.
c. Compute all the Cartesian quadrupole moments of the distribution.
d. Give the expansion of the potential due to this distribution including terms through second order.

Computing the quadrupole moments can be a chore. One can reduce the work load by noting that the $Q_{ij} = \int \left[3x_i' x_j' - (r')^2 \delta_{ij} \right] dq$ are the Cartesian multipole moments and that the charge density can be separated into two Cartesian-form components with the overall result following from superposition.

$$\sigma(\theta,\phi) = \left[\sigma_0 \right] \left[\cos^2\theta + \sin\phi \right] = \sigma_1 + \sigma_2 = \sigma_0 \frac{z^2}{r^2} + \sigma_0 \frac{y}{\sqrt{x^2+y^2}}$$

Symmetry arguments can be applied more directly in these forms. Consider Q_{zz}. The operator has the form $3\,z^2 - r^2$ and hence is even in x, y and z. The density σ_1 is even in all three coordinates as well so integration over the even domain can give a non-zero result for Q_{zz}. The density σ_2 is odd with respect to x so integration over the even domain with the even Q_{zz} operator yields zero additional contribution to Q_{zz}. Exploiting the fact that a zero contribution will result in the case that any coordinate integration is of an odd function over an even domain, it follows that only the diagonal elements of Q_{zz} are non-zero and that only σ_1 contributes to them. Extending this reasoning to the dipole moment indicates that only σ_2 contributes and only p_y is non-zero. Finally, the operator for the monopole moment (1) is even so only σ_1 yields a net contribution. (See problem SL-6.)

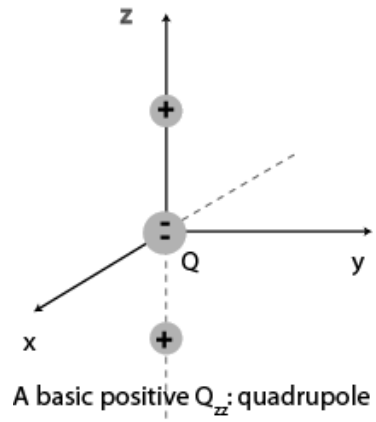
Consider a basic quadrupole:

$$Q_{ij} = \sum_{\substack{sources \\ \alpha}} q_\alpha D_{ij}$$

Q_{zz} quadrupole ($Q_{xx} = Q_{yy} = -½\, Q_{zz}$).

The charge distribution is even under the symmetry operations: $x \to -x$; $y \to -y$ and $z \to -z$. The operators $D_{xx} = 2\,x^2 - y^2 - z^2$,$D_{yy} = 2\,x^2 - y^2 - z^2$ and $D_{zz} = 2\,z^2 - y^2 - x^2$ are even under these symmetry operations. Thus, $Q_{zz} = \sum_{sources\ \alpha} q_\alpha \left[2\,z_\alpha^2 - x_\alpha^2 - y_\alpha^2 \right]$ can be non-zero as can Q_{xx} and Q_{yy}. On the other hand $D_{xz} = -\,3xz$ is odd under the symmetry

operations: $x \to -x$ and $z \to -z$ while the charge distribution is even. Thus Q_{xz} must be zero. For example, the operation $z \to -z$ changes the sign of the expression for Q_{xz} while moving each charge q_α from its position $(x_\alpha, y_\alpha, z_\alpha)$ to $(x_\alpha, y_\alpha, -z_\alpha)$ does not alter the physical charge distribution. It is the same distribution, but the expression for Q_{xz} has changed sign $\Rightarrow$ Q_{xz} must be zero.

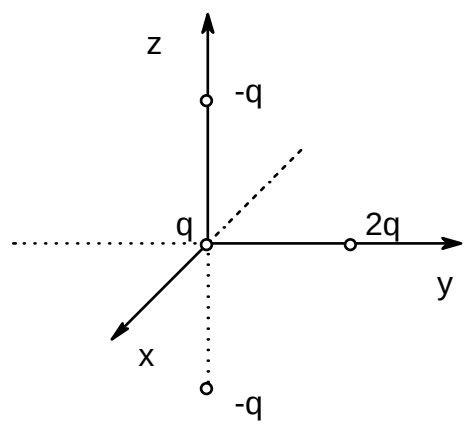


$$D_{ij}=3\, x_i x_j - r^2\, \delta_{ij}$$

Problems:

1. Four charges are located as shown:

   - q @ (0,0,a)
   - q @ (0,0,-a)
   q @ (0,0,0) and
   2q @ (0,a,0)



$$\sum_\alpha q_\alpha \left[3x_{\alpha i}\, x_{\alpha j} - (r_\alpha)^2 \delta_{ij} \right]$$

a.) Compute the monopole moment q_{total} for the charge distribution.

b.) Compute the dipole moment of the charge distribution.

c.) Give the large distance approximation for the potential due to the charges correct to order $1/r^2$.

d. Compute Q_{ZZ} for the distribution.

e. Show the details as you calculate the contribution of the $2q$ charge to Q_{zz}.

f. Argue that $Q_{xy} = Q_{xz} = Q_{yz} = 0$. See: **Using Symmetry to Avoid Calculations in Tools … .**

g.) Calculate Q_{xx} and Q_{yy} using $\sum_{\alpha} q_{\alpha}\left[3x_{\alpha}\, x_{\alpha} - (r_{\alpha})^2 \delta_{ij}\right]$. Show that $Q_{xx} + Q_{yy} + Q_{zz} = 0$.

h.) Compute the quadrupole contribution to the far-field approximation of the potential. See equation[MP.11].

2. Use $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$. Express $\cos(3\theta)$ as a sum of the $P_{\ell}(\cos\theta)$; that is: of the Legendre polynomials of $\cos\theta$ for $\ell < 4$.

   Desired form: $\cos(3\theta) = a_o P_o(\cos\theta) + a_1 P_1(\cos\theta) + a_2 P_2(\cos\theta) + a_3 P_3(\cos\theta)$.

3. Given a charge distribution:

$$\rho(\vec{r}) = \begin{cases} \left[\rho_0\left(1-\left(r/a\right)\right)\right]\left[\sin^2\theta - \cos\theta\right] & for \quad r < a \\ 0 & for \quad r < a \end{cases}$$

a. Compute the total charge of the distribution.

b. Compute the electric dipole moment of the distribution.

c. Compute all the Cartesian quadrupole moments of the distribution.

d. Give the expansion of the potential due to this distribution including terms through second order.

4. Given a surface charge density on a shell of radius R:

$$\sigma(\theta,\phi) = \sigma_0 \cos\theta \ \text{ located at } r = R.$$

a. Compute the total charge of the distribution.

b. Compute the electric dipole moment of the distribution.

c. Compute Q_{zz} and Q_{xy}. All the Cartesian quadrupole moments of the distribution vanish. It is a pure dipole distribution.

d. Give the expansion of the potential due to this distribution including terms through second order.

Note: The charge density $\sigma(\theta,\phi) = \sigma_o \cos\theta$ on a shell of radius R is identical to that of a uniformly polarized sphere of radius R. In your E&M text, the volume polarization charge density is ρ_{pol} is the negative of the divergence of the

polarization density $\rho_{pol} = -\vec{\nabla} \cdot \vec{P}$ and $\sigma_{pol} = \vec{P} \cdot \hat{n}$ evaluated on the surface of the polarized object. Another representation is two spheres with uniform charge densities $+\rho_o$ and $-\rho_o$ with the center of the positive sphere at $d\,\hat{k}$ relative to that of the negative sphere. The convention is that polarization direction defines the *z* direction: $\hat{k}$. The combination has a net surface charge density of $\rho_o\, d\,\hat{k} \cdot \hat{n}$. (What is the divergence of a uniform (constant) vector field?)

5. Given a surface charge density on a shell of radius R:

$$\sigma(\theta,\phi) = \sigma_0 \left(\cos\theta\right)^2 \text{ located at } r = R.$$

a. Compute the total charge of the distribution.

b. Compute the electric dipole moment of the distribution.

c. Compute Q_{zz} and Q_{xx}. It happens that $Q_{xx} = Q_{yy}$. Why must this be true for this distribution? All the other Cartesian quadrupole moments of the distribution vanish.

d. Give the expansion of the potential due to this distribution including terms through second order.

6. Given a surface charge density on a shell of radius R:

$$\sigma(\theta,\phi) = \left[\sigma_0\right]\left[\cos^2\theta + \sin\phi\right]$$

a. Compute the total charge of the distribution.

b. Compute the electric dipole moment of the distribution.

c. Compute all the Cartesian quadrupole moments of the distribution.

d. Give the expansion of the potential due to this distribution including terms through second order.

7. Given a collection of point charges on the *z*-axis, *q* at $2a\hat{k}$, - *q* at $a\hat{k}$, - *q* at $-a\hat{k}$ and *q* at $-2a\hat{k}$.

a. Compute the total charge of the distribution.

b. Compute the electric dipole moment of the distribution.

c. Compute all the Cartesian quadrupole moments of the distribution.

d. Give the expansion of the potential due to this distribution including terms through second order.

8. Given a collection of point charges in the *x*-*y* plane: q at $a\,\hat{i}+a\,\hat{j}$, - q at $-a\,\hat{i}+a\,\hat{j}$, - q at $a\,\hat{i}-a\,\hat{j}$ and q at $-a\,\hat{i}-a\,\hat{j}$.

a. Compute the total charge of the distribution.

b. Compute the electric dipole moment of the distribution.

c. Compute all the Cartesian quadrupole moments of the distribution.

d. Give the expansion of the potential due to this distribution including terms through second (quadrupole) order.

{+ q at (a, a, 0), + q at (- a, - a, 0), - q at (- a, a, 0), - q at (a, - a, 0)}.

10. Use the Griffiths method to generate the Cartesian quadrupole terms in the expansion of the potential. See the Tools of the Trade section.

11. ******* ***This Identity has not been VERIFIED ! !!!! ****** The following identity is proposed.

$$P_\ell(\hat{r}'\cdot\hat{r}) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{+\ell} Y^*_{\ell m}(\theta,\phi)Y_{\ell m}(\theta',\phi')$$

where $\hat{r}' = \sin\theta'\left[\cos\phi'\,\hat{i}+\sin\phi'\,\hat{j}\right]+\cos\theta'\,\hat{k}$ and $\hat{r} = \sin\theta\left[\cos\phi\,\hat{i}+\sin\phi\,\hat{j}\right]+\cos\theta\,\hat{k}$. Establish the equality for $\ell =$ 0 and for $\ell =$ 1 by brute force comparison. It may be useful to note that $\cos(\phi-\phi') = \cos(\phi')\cos(\phi)+\sin(\phi')\sin(\phi)$. Some of the $Y_{\ell m}$'s are tabulated in this handout.

13. Given a collection of point charges on the *z*-axis: 2 q at $0\hat{k}$, and - 2 q at $-a\hat{k}$.

a. Compute the total charge of the distribution.

b. Compute the electric dipole moment of the distribution.

c. Compute all the Q_{zz} and Q_{xy} Cartesian quadrupole moments of the distribution.

d. Give the expansion of the potential due to this distribution including terms through second order.

Use: - ½ $Q_{zz} = Q_{xx} = Q_{yy}$ and $Q_{ij} = 0$ for $i \neq j$. (See the solution to problem 5.)

e. Represent the charge distribution as the sum of a monopole at the origin and a dipole (slightly off center). Compare the moments with those calculated in parts a and b. Represent the charge distribution as the sum of a monopole at the origin, a dipole centered on the origin, and an off-center quadrupole.

f. The quadrupole moments of that off-center quadrupole identified in the previous part should match those computed in part c. Compute the Q_{zz} and Q_{xy} quadrupole moments for the quadrupole part of your distribution. (*Note that finite multipoles have non-vanishing higher order multipole moments.*) Compute the Q_{zz} and Q_{xy} quadrupole moments of the off-center dipole part of your distribution. *If 2^{ℓ}-pole is symmetrically located, its first higher moment is a $2^{\ell+2}$-pole. If it is not symmetrically distributed relative to the origin, it can have a $2^{\ell+1}$-pole and higher moments. Further note that, as your monopole was placed at the origin, the monopole is pure and infinitesimal.*

14. Show that the four charges:

$$3q \text{ @ } (0,0,a),\ q \text{ @ } (0,0,-a),\ -2q \text{ @ } (0,-a,0) \text{ and } -2q \text{ @ } (0, a,0)$$

are a physical octupole plus a dipole $2qa\hat{k}$. Only two of the diagonal elements of Q_{ij} are nonzero. All the off diagonal elements vanish. Verify these statements.

15. Consider a solid sphere of radius R and charge density $\rho_0\,({}^{r}/_{R})\cos\theta$. Compute the electric dipole moment of the distribution and find the leading non-zero term in the expression for the electric field for points on the *polar axis with* $z >> R$ and on the x axis in the equatorial plane with $x >> R$.

16. Consider a cylindrical shell of radius R, height h and charge density σ_0 (z/h) that is concentric with the z axis and the x-y plane as its midplane (the cylinder lies between $z = -\frac{1}{2}h$ and $+\frac{1}{2}h$). Compute the electric dipole moment of the distribution and find the leading non-zero term in the expression for the electric field for point on the *z axis with* $z >> R$ and on the x axis in the mid-plane with $x >> R$.

17. Consider starting with equal, but opposite pairs of dipoles with separations ℓ oriented along each of the Cartesian directions. Separate the pairs by displacements of magnitude ℓ in each of the three Cartesian directions. Make sketches to identify and count the distinct charge distributions that are formed. The final reduction from six to five independent moments follows because all the moments through quadrupole* vanish for the distribution of charges: $-6q$ at the origin plus six charges q at $\left\{\pm\frac{\ell}{2}\hat{i};\pm\frac{\ell}{2}\hat{j};\pm\frac{\ell}{2}\hat{k}\right\}$. Verify that all the moments through quadrupole vanish for this distribution. Discuss the results in the context of the number of independent quadrupole moments.

* The distribution: q at $\left\{\pm\frac{\ell}{2}\hat{i};\pm\frac{\ell}{2}\hat{j};\pm\frac{\ell}{2}\hat{k}\right\}$ is the best approximation to a uniform spherical shell of radius $\frac{1}{2}\ell$ and net charge $+6q$ that can be made using six point charges. What is the electric field due to a uniform spherical shell of radius $\frac{1}{2}\ell$ and net charge $+6q$ concentric with a point charge $-6q$ in the region outside the shell?

18. It was claimed that translating a charge distribution does not change its lowest non-zero multipole moment, but can change the values of all the higher order multipoles of the distribution. Consider a physical dipole: q at $+\frac{\ell}{2}\hat{i}$ and $-q$ at $-\frac{\ell}{2}\hat{i}$. Compute the monopole, dipole and quadrupole moments of the distribution. Next, consider the translated physical dipole: q at $\frac{\ell}{2}\left(\hat{i}+\hat{j}\right)$ and $-q$ at $\frac{\ell}{2}\left(-\hat{i}+\hat{j}\right)$. Compute the monopole, dipole and quadrupole moments of the distribution. Prepare a drawing displaying the initial and the translated charge distribution. Deduce the charge distribution that must

be added to the initial distribution to yield the translated distribution. Compute the monopole, dipole and quadrupole moments of that added distribution. In general, translating a multipole moment can be accomplished by adding a distribution with the next higher multipole pus even higher order multipoles.

19. Definition: $D(x,y,z;x',y',z') = D(\vec{r};\vec{r}') = \dfrac{1}{|\ \vec{r}-\vec{r}'\ |} = \dfrac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$,

The field coordinates *x*,*y*,*z* are fixed, a Taylor's series expansion in *x*', *y*', *z*' about (0,0,0) provides an approximation of the source coordinate dependence of $D(\vec{r};\vec{r}')$. [$x' = x_1'$, $y' = x_2'$ and $z' = x_3'$]

$$D(\vec{r};\vec{r}') = D(\vec{r};\vec{0}) + \sum_{i=1}^{3} \left.\frac{\partial D}{\partial x_i'}\right|_{\vec{r}'=0} (x_i'-0) + \frac{1}{2!}\sum_{i,j=1}^{3} \left.\frac{\partial^2 D}{\partial x_i' \partial x_j'}\right|_{\vec{r}'=0} (x_i'-0)(x_j'-0) + \ .....$$

Compute $\left.\dfrac{\partial D}{\partial x_i'}\right|_{\vec{r}'=0}$ and $\left.\dfrac{\partial^2 D}{\partial x_i'\ \partial x_j'}\right|_{\vec{r}'=0}$. Give the expansion for *D(x,y,x,x',y',z')* about $\vec{r}' = \vec{0}$ correct to second order.

20.) Given point charges on the *z*-axis: 2 *q* at $0\hat{k}$, - *q* at $+a\hat{k}$ and - *q* at $-a\hat{k}$.

a.) Find the net charge and the net dipole moment of the distribution.

b.) Compute all the quadrupole moments of the distribution.

c.) Give the form of the potential for points (*x*, *y*, *z*) that are far from the origin compared to *a*.

d.) Find the electric field on the *z* axis for |*z*| >> *a*.

e.) Find the electric field on the *x* axis for |*x*| >> *a*. The field on the positive *x* - axis is weak and in the *x* - direction. Justify this result.

f.) The net electric field flux away from the distribution is zero as the net charge is zero (Gauss's Law). The field on the positive *x* - axis is weak and in the *x* - direction. Using this fact, the ideas of flux and of symmetry, what is the expected direction of the field on the *z* - axis.

21.) As an extension to the previous problem, use the Coulomb's Law method to compute the electric field at points on the x axis and at points on the z axis. Use the binomial theorem to extract the leading order contribution to the electric field for large x and z. Note that the multipole expansion provides a method to calculate the leading contribution to the field at all distant points ($|r| >> a$).

22.) A finite multipole has its *base* multipole moment plus higher multipole moments. A charge distribution with inversion symmetry has even moments only; one that is odd under inversion has odd moments only. A dipole with its plus and minus charges symmetrically located (*an odd symmetry distribution*) has a base dipole moment plus odd order higher moments. A point charge at the origin is a pure monopole.
Consider the charge distribution 2 q at $0\hat{k}$, - 2 q at $+a\hat{k}$ and - q at $-a\hat{k}$. Represent the distribution as the sum of a monopole at the origin, a dipole with its plus and minus charges symmetrically located and a quadrupole distribution.
NOTE: The *symmetrically located* dipole has a dipole moment and a leading correction that is octupole. The quadrupole has a base quadrupole moment and corrections that begin with hexadecupole if the charge distribution is symmetric. If the distribution has mixed symmetry, there can be octupole corrections as well.

24. Consider the array of charges:

$$\{+q \text{ at } (a, a, a), +q \text{ at } (-a, -a, a), -q \text{ at } (-a, a, a), -q \text{ at } (a, -a, a)\} \text{ plus}$$

$$\{-q \text{ at } (a, a, -a), -q \text{ at } (-a, -a, -a), +q \text{ at } (-a, a, -a), +q \text{ at } (a, -a, -a)\}$$

Give all the moments of the distribution through quadrupole. Argue for your answers if possible rather than computing them brute force.

25. Show that the dipole moment of a charge distribution is invariant when the origin for its calculation is shifted if the net charge of the distribution is zero. Shifting the

origin is equivalent to changing the location of a charge q_α from $\vec{r}_\alpha$ to $\vec{r}_\alpha + \vec{d}$ where $\vec{d}$ is a constant vector.

26. Show that the quadrupole moments of a charge distribution are invariant when the origin for its calculation is shifted if the net charge and dipole moment of the distribution are zero. Shifting the origin is equivalent to changing the location of a charge q_α from $\vec{r}_\alpha$ to $\vec{r}_\alpha + \vec{d}$ where $\vec{d}$ is a constant vector. In general, the 2^n - pole moments of a charge distribution are invariant under a shift of origin if all the lower order multipole moments of the distribution vanish.

Hint: Consider the new coordinate values to be primed.

$$Q_{ij} = \sum_{sources\ \alpha} q_\alpha \left[3x_{\alpha i}\, x_{\alpha j} - (r_\alpha)^2 \delta_{ij} \right] \rightarrow Q'_{ij} = \sum_{sources\ \alpha} q_\alpha \left[3x'_{\alpha i}\, x'_{\alpha j} - (r'_\alpha)^2 \delta_{ij} \right]$$

$x'_{\alpha j} = x_{\alpha j} + d_j$ and $r'^2 = (\vec{r} + \vec{d}) \cdot (\vec{r} + \vec{d})$ where $\vec{d}$ and its components are constant.

Lesson learned: Always begin by writing down the definitions.

27. **Critical Observation:** Consider the expression

$$\vec{F}(\vec{r}, \vec{r}') = \vec{F}(x, y, z, x', y', z') = \frac{k\, dq'}{\left| \vec{r} - \vec{r}' \right|^3} \left(\vec{r} - \vec{r}' \right)$$

The function $\vec{F}(\vec{r}, \vec{r}')$ is a *two-point* function. It depends of the coordinates of the field point $\vec{r}$ and those of the source point $\vec{r}'$. The charge dq' is at the source point and so carries the *source* subscript. When one evaluates the electric field, an integral over sources of the form below is executed.

$$\int_{sources} \vec{F}(\vec{r}, \vec{r}') = \int_{sources} \frac{k\ \rho(\vec{r}')\, d^3 r'}{\left| \vec{r} - \vec{r}' \right|^3} \left(\vec{r} - \vec{r}' \right) = \vec{\text{Result}}(??)$$

Upon which variables does the *Result* depend after the $d^3 r'$ integration?

28. The expansion of $\left| \vec{r} - \vec{r}' \right|^{-1}$ has been expressed in terms of Legendre polynomials. Note that $\left| \vec{r} - \vec{r}' \right| = \sqrt{r^2 + r'^2 - 2\, r\, r' (\hat{r} \cdot \hat{r}')} = r \sqrt{1 + a^2 - 2\, a\, (\hat{r} \cdot \hat{r}')}$ where $a = {}^{r'}/_{r}$ for $r > r'$. Note

that $\hat{r}\cdot\hat{r}'$ is the cosine of the angle between the directions of the vectors. Consider the functions $\frac{1}{\sqrt{1+a^2-2a\cos\theta}}$ where a < ½. Use the binomial theorem to expand this function by identifying x as $a^2 - 2\,a\cos\theta$. Expand the expression to second order in x. (Note that this process generates all contributions up to second order in a plus some terms of order a^3 and a^4.) Collect the terms by powers of a. Identify the coefficients of a^0, a^1 and a^2. Note that the terms of order a^3 and a^4 should be discarded because not all terms of those orders have been calculated. It should be clear that an expansion to order x^m is required to get all the terms of order a^m. Some of the terms through order a^{2m} will be generated, but not all of them! Expansions should always be reported correct to a given order. That is: the expansion should include all terms up to and including that order and none higher. Compare your results to the assertion that:

$$\frac{1}{\sqrt{1+a^2-2a\cos\theta}} = \sum_{m=0}^{\infty} a^m\, P_m(\cos\theta) \quad \text{for } a < 1.$$

Note that: $P_2(x) = {}^3/_2\, x^2 - ½ \Rightarrow P_2(\cos\theta) = {}^3/_2 \cos^2\theta - ½$; $P_0(x) = 1$; $P_1(x) = x$

Use the assertion to show that $\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{m=0}^{\infty} \frac{(r')^m}{r^{m+1}} P_m(\cos\theta)$ for $r > r'$.

**** Add problems on the energy of a distribution in an external potential field.

29. A potential for a uniform electric field has the form $V(\vec{r}) = V_0 - \vec{E}_0\cdot\vec{r}$. Use the potential energy formula $U = \sum_{\text{charges } \alpha} q_\alpha V(\vec{r}_\alpha)$ to compute the potential energy of a dipole consisting of a charge +q at $\vec{r}_d + ½\,\vec{\ell}$ and a charge –q at $\vec{r}_d - ½\,\vec{\ell}$. Express your result in terms of $\vec{p} = q\vec{\ell}$ and $\vec{E}_0$. Next consider a quadrupole consisting of q at $a\ \hat{k}$, -2q at 0 $\hat{k}$ and +q at $a\ \hat{k}$. Compute its potential energy.

30. Consider a quadrupole (q at $a\ \hat{k}$, -2q at 0 $\hat{k}$ and +q at $a\ \hat{k}$) and a dipole (q at $(d + ½\,\ell)\hat{k}$ and –q at $(d - ½\,\ell)\hat{k}$. Use the *far-field* approximations (d >> a or ℓ). a.)

Compute the energy of the dipole in the field of the quadrupole. b.) Compute the energy of the quadrupole in the field due to the dipole.

c. Compute the force on the dipole due to the field of the quadrupole ($\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E} \rightarrow q\ell\frac{\partial}{\partial z}\vec{E}$ for this case.). Calculate the work required to move the dipole from $\infty\hat{k}$ to $d\ \hat{k}$ with its orientation fixed. d. Start with the dipole at infinity on the *z* axis, but oriented in the *x* direction. How much work would be required to move it into $d\ \hat{k}$ along the *z* axis with its orientation fixed? How much work would be required to rotate the dipole to lie along the *z* axis keeping its center fixed ar $d\ \hat{k}$?

31.) Use the base formula that $U_{collection} = \sum_{pairs} \frac{q_i q_j}{4\pi\varepsilon_o r_{ij}}$ to repeat the energy calculations for parts a and b in problem 30. Use the fact that *a* and ℓ are small to cast differences in terms of derivatives.

32.) Consider the charge distribution: a single point charge *q* located at $d\ \hat{k}$.

a.) Compute all the moments through quadrupole order and give the far-field expansion of the potential through quadrupole terms.

b.) In the Laplace equation section of this handout series, the potential of a point charge at $d\ \hat{k}$ is shown to be:

$V(x,y,z) = \frac{q}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z-d)^2}} = V(r,\theta) = \sum_{\ell=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\right)\frac{d^\ell}{r^{(\ell+1)}}\ P_\ell(\cos\theta)$. Write out the first three terms in the expansion of the left and compare them with the far-field expansion of part a. Use the standard relation $z = r\cos\theta$.

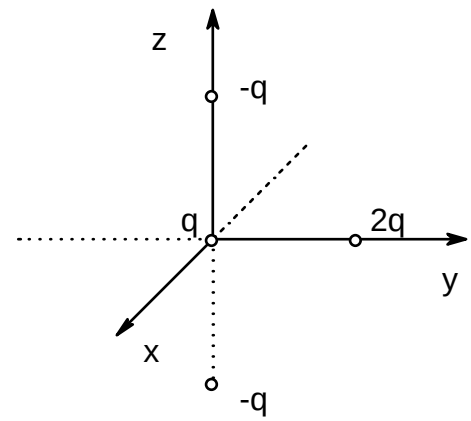
$$P_0(z) = 1;\quad P_1(z) = z;\quad P_2(z) = {}^3/_2\, z^2 - {}^1/_2$$

c.) What order multipoles would by non-zero in the complete expansion of the potential of a single point charge located at *d* .

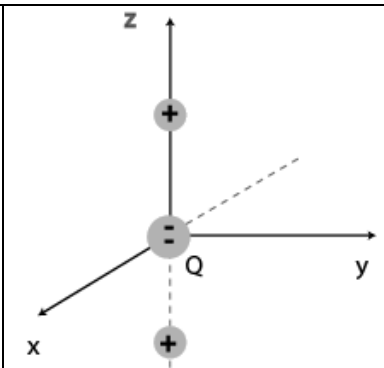
33. Four charges are located as shown:

- q @ (0,0,a)
- q @ (0,0,-a)
  q @ (0,0,0) and
  2q @ (0,a,0)

$$\sum_{\alpha} q_{\alpha}\left[3x_{\alpha i}\, x_{\alpha j} - (r_{\alpha})^2 \delta_{ij}\right]$$



Show that this distribution can be decomposed into *i*.) the net charge at the origin, *ii*.) the net dipole moment configured* to be centered about the origin and *iii*.) two basic quadrupoles. The quadrupoles are centered although the highest order multipoles in a decomposition are not necessarily centered. The only positions to be used as charge locations are the actual positions of charges, the $\vec{r}_{\alpha}$, and there reflected positions, the $-\vec{r}_{\alpha}$.

Recall that a basic Q_{zz} quadrupole contributes $+4qa^2$ to Q_{zz} and $-2qa^2$ to Q_{xx} and Q_{yy}. Its skew moments vanish. Use this to motivate that the quadrupole moments for the distribution above are $Q_{yy} = -Q_{zz} = 6qa^2$; $Q_{xx} = 0$; all skew moments vanish. The other answers: $q_{total} = q$; $\vec{p} = 2qa\,\hat{j}$	 A basic positive Q_{zz}: quadrupole

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3. David J. Griffiths, ***Introduction to Electrodynamics***, 3rd Edition, Prentice Hall (1999).