$\boxed{\text{HW\#2}}$

(a) VS #15

a. $\{\hat{\imath}+\hat{k},\ \hat{\imath}+\hat{\jmath},\ \hat{k}\} \Rightarrow \hat{B}_1 = \boxed{\dfrac{\hat{\imath}+\hat{k}}{\sqrt{2}}}$

$$\vec{B}_2' = \vec{B}_2 - (\vec{B}_2\cdot\hat{B}_1)\hat{B}_1$$

$$= (\hat{\imath}+\hat{\jmath}) - (\hat{\imath}+\hat{\jmath})\cdot\left(\frac{\hat{\imath}+\hat{k}}{\sqrt{2}}\right)\left(\frac{\hat{\imath}+\hat{k}}{\sqrt{2}}\right)$$

$$= \hat{\imath}+\hat{\jmath} - \tfrac{1}{2}(1)(\hat{\imath}+\hat{k}) = \tfrac{1}{2}\hat{\imath}+\hat{\jmath}-\tfrac{1}{2}\hat{k} \rightarrow \hat{\imath}+2\hat{\jmath}-\hat{k}$$

$$\therefore\ \hat{B}_2' = \boxed{\frac{\hat{\imath}+2\hat{\jmath}-\hat{k}}{\sqrt{6}}}$$

$$\vec{B}_3' = \vec{B}_3 - (\vec{B}_3\cdot\hat{B}_1)\hat{B}_1 - (\vec{B}_3\cdot\hat{B}_2')\hat{B}_2'$$

$$= \hat{k} - \hat{k}\cdot\left(\frac{\hat{\imath}+\hat{k}}{\sqrt{2}}\right)\left(\frac{\hat{\imath}+\hat{k}}{\sqrt{2}}\right) - \hat{k}\cdot\left(\frac{\hat{\imath}+2\hat{\jmath}-\hat{k}}{\sqrt{6}}\right)\left(\frac{\hat{\imath}+2\hat{\jmath}-\hat{k}}{\sqrt{6}}\right)$$

$$= \hat{k} - \tfrac{1}{2}(1)(\hat{\imath}+\hat{k}) - \tfrac{1}{6}(-1)(\hat{\imath}+2\hat{\jmath}-\hat{k})$$

$$= -\tfrac{1}{3}\hat{\imath}+\tfrac{1}{3}\hat{\jmath}+\tfrac{1}{3}\hat{k} \rightarrow -\hat{\imath}+\hat{\jmath}+\hat{k}$$

$$\therefore\ \hat{B}_3' = \boxed{\frac{-\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}}$$

cks// $\hat{B}_1\cdot\hat{B}_1 = \hat{B}_2'\cdot\hat{B}_2' = \hat{B}_3'\cdot\hat{B}_3' = 1$

$\hat{B}_1\cdot\hat{B}_2' = \hat{B}_1\cdot\hat{B}_3' = \hat{B}_2'\cdot\hat{B}_3' = 0$

b. $\{\hat{\imath}+\hat{k},\ 2(\hat{\imath}+\hat{\jmath}),\ 3\hat{k}\}$

Will get the $\boxed{\text{Same answer}}$ because the scaling of the vectors is irrelevant.

c. $\{\hat{\imath}+\hat{\jmath},\ \hat{\imath}+\hat{k},\ \hat{k}\}$

For the first two vectors we swap $\hat{\jmath}\leftrightarrow\hat{k}$:

$$\hat{B}_1 = \boxed{\frac{\hat{\imath}+\hat{\jmath}}{\sqrt{2}}} \quad\text{and}\quad \hat{B}_2' = \boxed{\frac{\hat{\imath}-\hat{\jmath}+2\hat{k}}{\sqrt{6}}}$$

but we leave the third vector alone:

$$\hat{B}_3' = \boxed{\frac{-\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}}$$

d. $\{\hat{\imath}+\hat{k},\ \hat{\imath}+\hat{\jmath},\ \hat{\imath}\}$

The first two vectors are unchanged:

$$\hat{B}_1 = \boxed{\frac{\hat{\imath}+\hat{k}}{\sqrt{2}}} \quad \text{and} \quad \hat{B}_2' = \boxed{\frac{\hat{\imath}+2\hat{\jmath}-\hat{k}}{\sqrt{6}}}$$

and we recalculate the third vector:

$$\vec{B}_3' = \hat{\imath} - \hat{\imath}\cdot\left(\frac{\hat{\imath}+\hat{k}}{\sqrt{2}}\right)\left(\frac{\hat{\imath}+\hat{k}}{\sqrt{2}}\right) - \hat{\imath}\cdot\left(\frac{\hat{\imath}+2\hat{\jmath}-\hat{k}}{\sqrt{6}}\right)\left(\frac{\hat{\imath}+2\hat{\jmath}-\hat{k}}{\sqrt{6}}\right)$$

$$= \hat{\imath} - \tfrac{1}{2}(1)(\hat{\imath}+\hat{k}) - \tfrac{1}{6}(1)(\hat{\imath}+2\hat{\jmath}-\hat{k})$$

$$= \tfrac{1}{3}\hat{\imath} - \tfrac{1}{3}\hat{\jmath} - \tfrac{1}{3}\hat{k} \rightarrow \hat{\imath}-\hat{\jmath}-\hat{k} \quad \therefore \hat{B}_3' = \boxed{\frac{\hat{\imath}-\hat{\jmath}-\hat{k}}{\sqrt{3}}}$$

which has the opposite sign of the original $\hat{B}_3'$. (We had to get either zero or $\pm\hat{B}_3'$ because otherwise our three final vectors would not be orthogonal!) ▨

(b) VS #5

$$\vec{A} = \hat{\imath}+2\hat{\jmath}+\hat{k} \Rightarrow \hat{A} = \boxed{\frac{\hat{\imath}+2\hat{\jmath}+\hat{k}}{\sqrt{6}}}$$

$$\vec{B} = 2\hat{\imath}+\hat{\jmath}+2\hat{k}$$

$$\vec{B}' = \vec{B} - (\vec{B}\cdot\hat{A})\hat{A}$$

$$= (2\hat{\imath}+\hat{\jmath}+2\hat{k}) - (2\hat{\imath}+\hat{\jmath}+2\hat{k})\cdot\left(\frac{\hat{\imath}+2\hat{\jmath}+\hat{k}}{\sqrt{6}}\right)\left(\frac{\hat{\imath}+2\hat{\jmath}+\hat{k}}{\sqrt{6}}\right)$$

$$= 2\hat{\imath}+\hat{\jmath}+2\hat{k} - \cancelto{1}{\tfrac{1}{6}(2+2+2)}(\hat{\imath}+2\hat{\jmath}+\hat{k})$$

$$= \hat{\imath}-\hat{\jmath}+\hat{k} \Rightarrow \hat{B}' = \boxed{\frac{\hat{\imath}-\hat{\jmath}+\hat{k}}{\sqrt{3}}} \quad \text{ck// } \hat{A}\cdot\hat{B}' = 0$$

But $\vec{C} = 2(\hat{\imath}-\hat{\jmath}+\hat{k}) = 2\sqrt{3}\,\hat{B}' \Rightarrow \hat{C}' = \boxed{0}$

$\therefore$ The VS is $\boxed{2D}$. ▨

(c) $|B_5\rangle = x^4$

$$|B_5\rangle - \overbrace{\frac{\langle b_1|B_5\rangle}{\langle b_1|b_1\rangle}}^{\text{EVEN}}|b_1\rangle - \overbrace{\frac{\langle b_2|B_5\rangle}{\langle b_2|b_2\rangle}}^{\text{ODD}=0}|b_2\rangle - \overbrace{\frac{\langle b_3|B_5\rangle}{\langle b_3|b_3\rangle}}^{\text{EVEN}}|b_3\rangle - \overbrace{\frac{\langle b_4|B_5\rangle}{\langle b_4|b_4\rangle}}^{\text{ODD}=0}|b_4\rangle$$

$$= x^4 - \frac{\int_{-1}^{1} x^4\,dx}{\int_{-1}^{1} dx} - \left[\frac{\int_{-1}^{1} \left(\frac{3}{2}x^2 - \frac{1}{2}\right) x^4\,dx}{\int_{-1}^{1} \left(\frac{3}{2}x^2 - \frac{1}{2}\right)^2 dx}\right] \left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

$$= x^4 - \frac{2/5}{2} - \frac{3/7 - 1/5}{9/10 - 1 + 1/2} \left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

$$\therefore |b_5\rangle = \left(x^4 - \frac{6}{7}x^2 + \frac{3}{35}\right) N_5$$

sub $x = 1$ $\Rightarrow$ $\frac{8}{35} N_5 = 1$ $\Rightarrow$ $N_5 = \frac{35}{8}$

$$\Rightarrow |b_5\rangle = \boxed{\frac{1}{8}\left(35x^4 - 30x^2 + 3\right)}$$

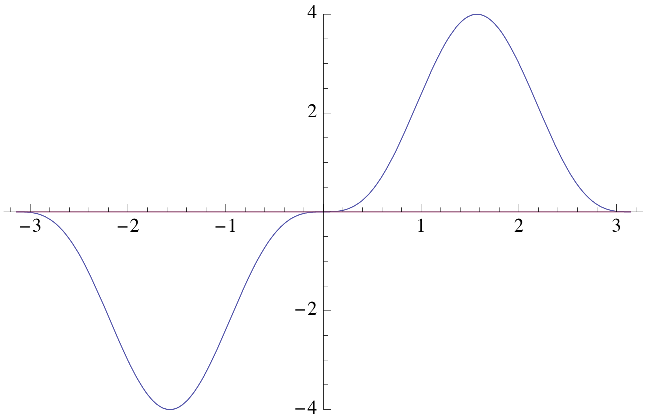
To check, type the following command in Mathematica:

LegendreP[4, x] ☒

(d)

```
In[1]:= a1 = 3; a2 = 0; a3 = -1; a4 = 0
Out[1]= 0
In[2]:= Plot[{4 (Sin[x])^3, 4 (Sin[x])^3 - (a1*Sin[x] + a2*Sin[2 x] + a3*Sin[3 x] + a4*Sin[4 x])},
         {x, -Pi, Pi}, PlotRange → {-4, 4}]
```

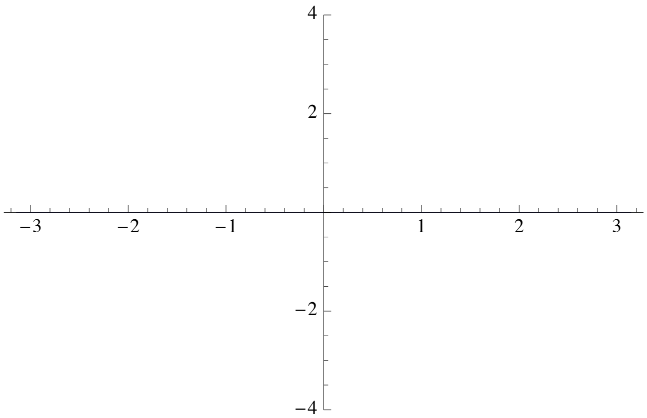
Out[2]=



Not very easy to do! Changing one coefficient tends to mess up the plot for the other coefficients! Takes a fair bit of trial and error to succeed.

```
In[3]:= Plot[{(4 (Sin[x])^3 - (a1*Sin[x] + a2*Sin[2 x] + a3*Sin[3 x] + a4*Sin[4 x]))^2},
         {x, -Pi, Pi}, PlotRange → {-4, 4}]
```

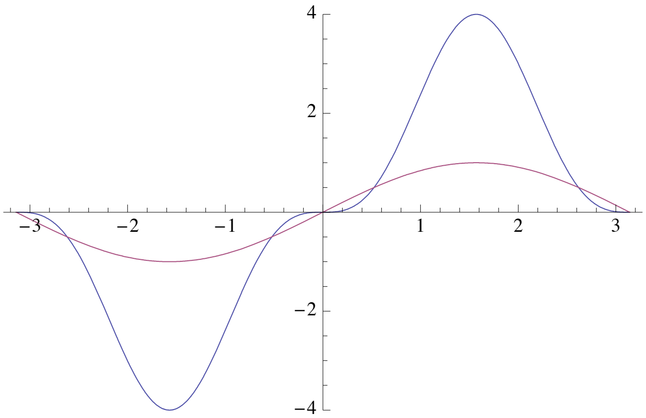
Out[3]=



Slightly easier, but still nontrivial.

```
In[4]:= Plot[{4 (Sin[x])^3, Sin[x]}, {x, -Pi, Pi}, PlotRange → {-4, 4}]
```

Out[4]=



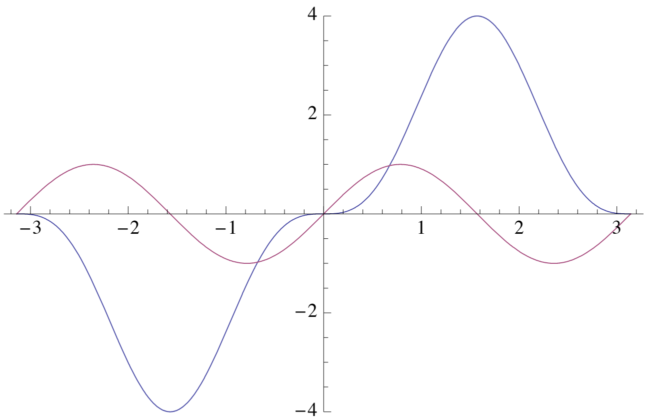
Clearly the pink curve overlaps the blue curve significantly. We expect a sizeable positive coefficient for a1.

```
In[5]:= 1/Pi*Integrate[Sin[x]*(4 (Sin[x])^3), {x, -Pi, Pi}]
Out[5]= 3
```

Nice illustration of the power of inner products!

In[6]:=
```
Plot[{4 (Sin[x])^3, Sin[2 x]}, {x, -Pi, Pi}, PlotRange → {-4, 4}]
```

Out[6]=



Clearly there are positive and negative areas, so we expect cancellations to give a small answer.

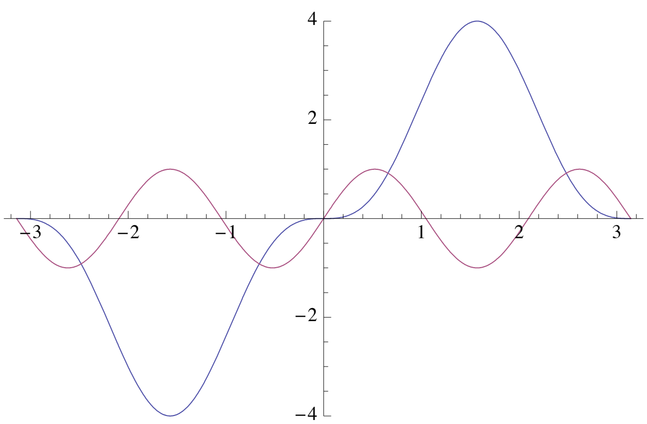
In[7]:=
```
1 / Pi * Integrate[Sin[2 x] * (4 (Sin[x])^3), {x, -Pi, Pi}]
```

Out[7]= 0

In fact, they cancel totally, something you cannot tell by eye alone.

In[8]:=
```
Plot[{4 (Sin[x])^3, Sin[3 x]}, {x, -Pi, Pi}, PlotRange → {-4, 4}]
```

Out[8]=



If we flipped the pink curve over, it would better match the blue curve.

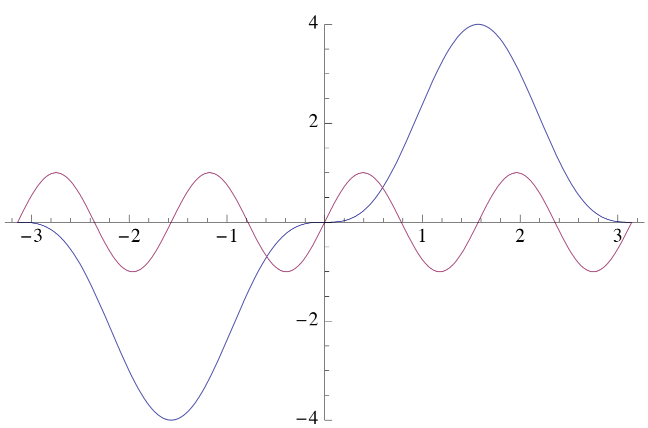
In[9]:=
```
1 / Pi * Integrate[Sin[3 x] * (4 (Sin[x])^3), {x, -Pi, Pi}]
```

Out[9]= -1

It turns out that's all we need to do.

In[10]:=
```
Plot[{4 (Sin[x])^3, Sin[4 x]}, {x, -Pi, Pi}, PlotRange → {-4, 4}]
```

Out[10]=



The pink curve appears to wiggle too fast to be a good representation of the blue curve.

```
In[11]:= 1 / Pi * Integrate[Sin[4 x] * (4 (Sin[x]) ^ 3), {x, -Pi, Pi}]

Out[11]= 0
```

Sure enough the answer is zero.