HW #3

(a) FS #8

$$f(t) = \begin{cases} 2t - t^2 & \text{for } 0 < t < 2 \\ 2t + t^2 & -2 < t < 0 \end{cases} \qquad T = 4$$

$$\Rightarrow \dot{f}(t) = \begin{cases} 2 - 2t & \text{for } 0 < t < 2 \\ 2 + 2t & -2 < t < 0 \end{cases}$$

$\therefore \dot{f}(0^-) = 2 = \dot{f}(0^+) \Rightarrow$ **continuous at $t = 0$**

nb: not x

$$\ddot{f}(t) = \begin{cases} -2 & \text{for } 0 < t < 2 \\ 2 & -2 < t < 0 \end{cases} \Rightarrow$$ **discont. at $t = 0$**

The average value of $f(t)$ is zero $\Rightarrow c_0 = 0$.
Also $f(t)$ is odd about $t = 0 \Rightarrow$ no cosine terms.
Finally $f(t)$ is even about $t = T/4 = 1 \Rightarrow$ only odd sine harmonics — number them by $2m+1$ where $m = 0, 1, 2, \ldots$

$$\therefore f(t) = \sum_{m=0}^{\infty} b_{2m+1} \sin\left[(2m+1)\frac{2\pi}{4} t\right]$$

where $$b_{2m+1} = \frac{2}{T} \int_{-T/2}^{T/2} \sin(\omega_{2m+1} t)\, f(t)\, dt$$

$$= 4\,\frac{2}{4}\int_0^{T/4} \sin\left[(2m+1)\frac{2\pi}{4}t\right] f(t)\,dt$$

$$= 2\int_0^1 \sin\left[(2m+1)\frac{\pi}{2}t\right](2t-t^2)\,dt$$

by parts: $du = \sin\left[(2m+1)\frac{\pi}{2}t\right]dt \Rightarrow u = \dfrac{-\cos\left[(2m+1)\frac{\pi}{2}t\right]}{(2m+1)\frac{\pi}{2}}$

$$v = 2t-t^2 \Rightarrow dv = (2-2t)\,dt$$

$$\therefore b_{2m+1} = -4\,\frac{\cos\left[(2m+1)\frac{\pi}{2}t\right](2t-t^2)}{(2m+1)\pi}\Bigg|_0^1 + 8\int_0^1 \frac{\cos\left[(2m+1)\frac{\pi}{2}t\right](1-t)\,dt}{(2m+1)\pi}$$

$$= \frac{8}{(2m+1)\pi}\int_0^1 \cos\left[(2m+1)\frac{\pi}{2}t\right](1-t)\,dt$$

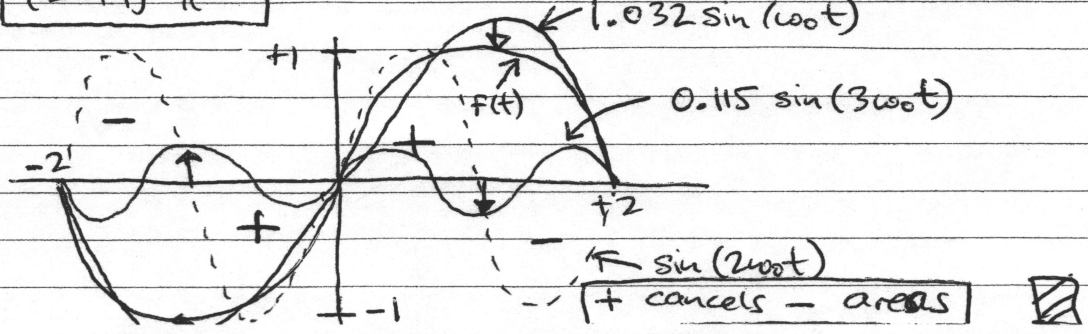
by parts again: $du = \cos\left[(2m+1)\frac{\pi}{2}t\right]dt \Rightarrow u = \dfrac{\sin\left[(2m+1)\frac{\pi}{2}t\right]}{(2m+1)\frac{\pi}{2}}$

$$v = 1-t \Rightarrow dv = -dt$$

$$\therefore b_{2m+1} = \frac{16}{(2m+1)^2\pi^2}\left\{\sin\left[(2m+1)\frac{\pi}{2}t\right](1-t)\Big|_0^1 + \int_0^1 \sin\left[(2m+1)\frac{\pi}{2}t\right]dt\right\}$$

$$= \frac{16}{(2m+1)^2\pi^2}\cdot\frac{-\cos\left[(2m+1)\frac{\pi}{2}t\right]}{(2m+1)\frac{\pi}{2}}\Bigg|_0^1$$

$$= \boxed{\frac{32}{(2m+1)^3\pi^3}}$$ which vanish as the **inverse cube**



(b) $$f(t) = \sum_{m=0}^{\infty} \frac{32}{(2m+1)^3 \pi^3} \sin\left[(2m+1)\omega_0 t\right]$$

$$\Rightarrow \boxed{\dot{f}(t) = \sum_{m=0}^{\infty} \frac{16}{(2m+1)^2 \pi^2} \cos\left[(2m+1)\omega_0 t\right]}$$

$$\boxed{\ddot{f}(t) = \sum_{m=0}^{\infty} \frac{-8}{(2m+1)\pi} \sin\left[(2m+1)\omega_0 t\right]}$$

Compare to the unit amplitude waves:

$$f_{\triangle}^{S}(t) = \sum_{m=0}^{\infty} \frac{8}{(2m+1)^2 \pi^2} \cos\left[(2m+1)\omega_0 t\right]$$

$$f_{\square}^{AS}(t) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} \sin\left[(2m+1)\omega_0 t\right]$$

In our case the amplitudes are:

$$f(t) = 2t - t^2 \quad \text{for } 0 < t < 2$$

$\Rightarrow \dot{f}(0) = \boxed{2}$ amplitude for symm. $\triangle$

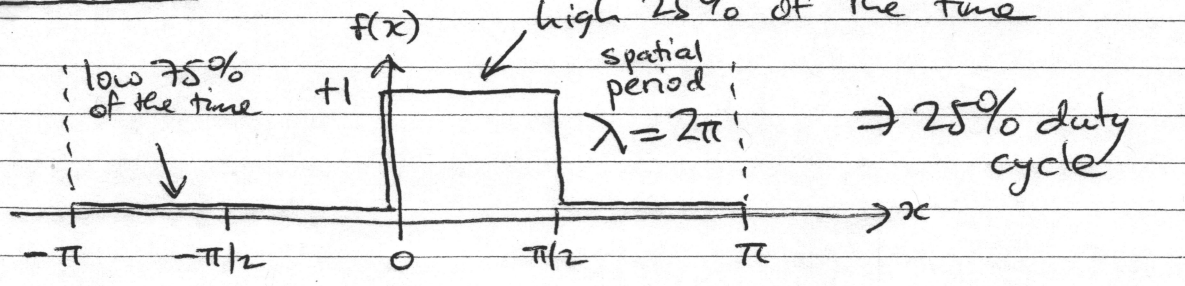
This explains why our series for $\dot{f}(t)$ is twice as large as the series for $f_{\triangle}^{S}(t)$.

and $\ddot{f} = \boxed{-2}$ expected for antisymm. $\square$

This again explains why our series for $\ddot{f}(t)$ is twice as large and opposite sign as that for $f_{\square}^{AS}(t)$.

Notice that we switch from $\boxed{\text{Sine} \leftrightarrow \text{cosine } (\text{odd} \leftrightarrow \text{even}_m)}$ series every time we differentiate. $\boxtimes$

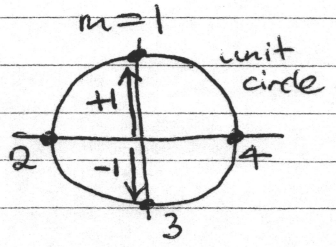
(c) FS #16



average value $= 0.75 * 0 + 0.25 * 1 = 0.25 = c_0$

cosine coeffs:

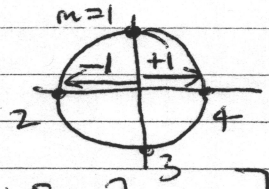
$$a_m = \frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} \cos\left(m \frac{2\pi}{\lambda} x\right) f(x)\, dx$$



$$= \frac{1}{\pi} \int_0^{\pi/2} \cos(mx)\, dx$$

$$= \frac{\sin mx}{m\pi}\Big|_0^{\pi/2} = \frac{1}{m\pi} * \begin{cases} +1 & m = 1, 5, 9, \ldots \\ 0 & 2, 6, 10, \ldots \\ -1 & 3, 7, 11, \ldots \\ 0 & 4, 8, 12, \ldots \end{cases}$$

similarly for sine coeffs:



$$b_m = \frac{1}{\pi} \int_0^{\pi/2} \sin(mx)\, dx$$

$$= \frac{-\cos mx}{m\pi}\Big|_0^{\pi/2} = \frac{1}{m\pi} * \left[\begin{cases} 0 & m = 1, 5, \ldots \\ +1 & 2, 6, \ldots \\ 0 & 3, 7, \ldots \\ -1 & 4, 8, \ldots \end{cases} + 1 \right]$$

$$= \frac{1}{m\pi} * \begin{cases} 1 & m = 1, 5, \ldots \\ 2 & 2, 6, \ldots \\ 1 & 3, 7, \ldots \\ 0 & 4, 8, \ldots \end{cases}$$

$$\therefore \boxed{f(x) = \frac{1}{4} + \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m \cos[(2m+1)x]}{2m+1} + \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{\sin[(2m+1)x]}{2m+1} + \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{\sin[(4m+2)x]}{4m+2}}$$

steps by 4 not by 2

nb: Final representation in the problem statement leaves off $\frac{\sin 10x}{10}$ so that it consistently shows all spatial frequencies $\boxed{\text{up to } 6k_0}$. ☒

(d) FS #17

Sine at all multiples of π equals zero.
Cosine of odd multiples of π equals -1.

$$\Rightarrow f(\pi) = \frac{1}{4} + \frac{1}{\pi}\left[-1 + \frac{1}{3} - \frac{1}{5} + \cdots\right]$$

But from the graph above we see that $f(\pi) = 0$.

$$\therefore \frac{1}{4} = \frac{1}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \cdots\right] \Rightarrow \boxed{\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4}}$$

Next, cosine of odd multiples of $\frac{\pi}{2}$ equals zero.
Sine of even multiples of $\frac{\pi}{2}$ also equals zero.
Sine of odd multiples of $\frac{\pi}{2}$ equals ± 1 from first unit circle above.
And from the graph, there is a discontinuity in $f(x)$ at $x = \pi/2$ so that:

$$f\left(\tfrac{\pi}{2}\right) = \frac{f\left(\frac{\pi}{2}^-\right) + f\left(\frac{\pi}{2}^+\right)}{2} = \frac{1+0}{2} = \frac{1}{2}$$

$$\therefore \frac{1}{2} = \frac{1}{4} + \frac{1}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \cdots\right]$$

which rearranges to the **same answer** as above.

[You also get the same answers for $f(0)$ and $f(-\frac{\pi}{2})$ and $f(-\pi)$. Try them and see for yourself!]

Finally let's apply the Parseval relation:

$$\langle f|f\rangle = c_0^2 + \frac{1}{2}\sum a_m^2 + \frac{1}{2}\sum b_m^2$$

$$\text{LHS} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{1}{2\pi}\int_0^{\pi/2} dx = \frac{1}{4}$$

$$\text{RHS} = \frac{1}{16} + \frac{1}{2\pi^2}\left[\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \cdots\right]$$
$$+ \frac{1}{2\pi^2}\left[\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \cdots\right] + \frac{4}{2\pi^2}\left[\frac{1}{4} + \frac{1}{36} + \cdots\right]$$

$$\therefore \frac{3\pi^2}{16} = \left[1 + \frac{1}{9} + \frac{1}{25} + \cdots\right] + 2\left[\frac{1}{4} + \frac{1}{36} + \cdots\right]$$

$$= \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} + \underbrace{2\sum_{m=0}^{\infty} \frac{1}{(4m+2)^2}}_{\frac{2}{4}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}}$$

$$\Rightarrow \boxed{\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}}$$ as we found in class (AS Δ evaluated at T/4 or Parseval for AS $\square$) ▨

(e) Parseval for AS Δ:

$$LHS = \langle f^{AS}_{\Delta} | f^{AS}_{\Delta} \rangle = \left\{ \begin{array}{c} \text{ave. of parabola } y=x^2 \\ \text{from } x=0 \text{ to } 1 \end{array} \right\} = \frac{1}{3}$$

$$RHS = \frac{1}{2}\frac{64}{\pi^4}\sum_{p=0}^{\infty} \frac{1}{(2p+1)^4}$$

$$\Rightarrow \frac{\pi^4}{96} = \underbrace{\sum_{p=0}^{\infty} \frac{1}{(2p+1)^4}}_{\text{ODDS}} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^4}}_{\text{ALL INTEGERS}} - \underbrace{\sum_{p=1}^{\infty} \frac{1}{(2p)^4}}_{\text{EVENS}}$$

$$\underbrace{\frac{1}{16}\sum_{n=1}^{\infty} \frac{1}{n^4}}$$

$$= \frac{15}{16}\sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\therefore \boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$$ cool!! ▨