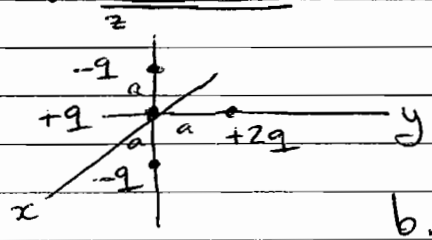
HW #4

(a) MP #1



a. $Q_{net} = (-q) + (+q) + (-q) + (+2q) = \boxed{q}$

b. $\vec{p} = \sum q_\alpha \vec{r}_\alpha$

$= (-q)(0,0,a) + (+q)(0,0,0) + (-q)(0,0,-a) + (+2q)(0,a,0) = \boxed{2qa\hat{j}}$

c. $V_0 + V_1 = \dfrac{kQ_{net}}{r} + \dfrac{k\vec{r}\cdot\vec{p}}{r^3} = \boxed{\dfrac{kq}{r} + \dfrac{2kqay}{r^3}}$

d. $Q_{zz} = \sum q_\alpha \left[3z_\alpha^2 - r_\alpha^2\right]$

$= (-q)\left[3(a)^2 - a^2\right] + (+q)\left[3(0)^2 - 0^2\right] + (-q)\left[3(-a)^2 - a^2\right] + (+2q)\left[3(0)^2 - a^2\right]$

$= \boxed{-6qa^2}$

f. For the off-diagonal terms $Q_{ij} = 3\sum q_\alpha x_{\alpha i} x_{\alpha j}$. But each charge has at most <u>one</u> nonzero coordinate. $\therefore$ Every term in the sum is $\boxed{\text{zero for off-diagonals}}$.

g. $Q_{xx} = \sum q_\alpha \left[3\cancel{x_\alpha^2}^{\,0} - r_\alpha^2\right]$

$= (-q)(-a^2) + (+q)(-0^2) + (-q)(-a^2) + (+2q)(-a^2) = \boxed{0}$

only nonzero for the +2q charge

$Q_{yy} = \sum q_\alpha \left[3y_\alpha^2 - r_\alpha^2\right]$

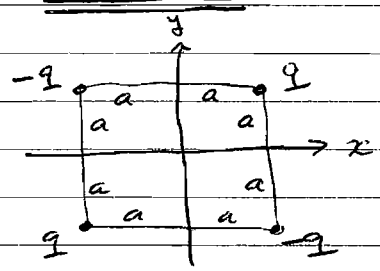
$= (-q)(-a^2) + (+q)(-0^2) + (-q)(-a^2) + (+2q)(3a^2 - a^2) = \boxed{+6qa^2}$

$\Rightarrow Q_{xx} + Q_{yy} + Q_{zz} = \boxed{0}$ ✓

h. $V_2 = \dfrac{k}{2r^5} \sum_{i,j=1}^{3} Q_{ij} x_i x_j = \dfrac{k\left[(6qa^2)(y^2) + (-6qa^2)(z^2)\right]}{2r^5}$

$$= \boxed{\frac{3kqa^2(y^2-z^2)}{r^5}}$$

(b) MP #8



a. $q_{net} = 2q - 2q = \boxed{0}$

b. $\vec{p} = \sum q_\alpha \vec{r}_\alpha$

$= (+q)(a,a) + (-q)(-a,a)$

$+ (-q)(a,-a) + (+q)(-a,-a) = \boxed{\vec{0}}$

c. $Q_{ij} = \sum_\alpha q_\alpha \left[3x_{\alpha i}x_{\alpha j} - r_\alpha^2 \delta_{ij}\right]$

<u>diagonals</u>: all $z_\alpha = 0$ and all $r_\alpha = \sqrt{2}a$

$\Rightarrow Q_{zz} = -\sum q_\alpha r_\alpha^2 = -2a^2 \sum q_\alpha = \boxed{0}$

by symmetry & zero trace $Q_{xx} = Q_{yy} = -\frac{1}{2}Q_{zz} = \boxed{0}$

<u>off-diagonals</u>: $\delta_{ij} = 0$ and all $z_\alpha = 0 \Rightarrow Q_{xz} = Q_{yz} = \boxed{0}$

$Q_{xy} = 3\sum q_\alpha x_\alpha y_\alpha$

$= 3(+q)(a)(a) + 3(-q)(-a)(a)$

$+ 3(-q)(a)(-a) + 3(q)(-a)(-a) = \boxed{12qa^2}$

d. $V_0 + V_1 + V_2 = \frac{k}{2r^5}\sum_{i,j} Q_{ij}x_i x_j = \boxed{\frac{12kqa^2xy}{r^5}}$

(nb: factor of 2 cancels because $Q_{xy} = Q_{yx}$)

e. on x-axis: $y = z = 0 \Rightarrow V = 0$

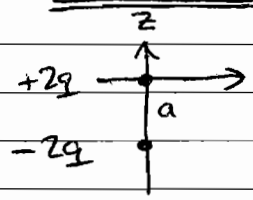
on y-axis: $x = z = 0 \Rightarrow V = 0$

on z-axis: $x = y = 0 \Rightarrow V = 0$

These results on the axes are exact because along any axis you're $\boxed{\text{equidistant}}$ from pairs of $\pm q$ charges.

(c)

MP #13



a. $Q_{net} = (+2q) + (-2q) = \boxed{0}$

b. $\vec{p} = \sum q_\alpha \vec{r}_\alpha$

$= (+2q)(0,0,0) + (-2q)(0,0,-a)$

$= \boxed{2qa\hat{k}}$

c. All charges have $x_\alpha = 0$ and $y_\alpha = 0$

$\Rightarrow Q_{xy} = Q_{xz} = Q_{yz} = \boxed{0}$

$Q_{zz} = \sum q_\alpha \left[3z_\alpha^2 - r_\alpha^2\right]$

$= (+2q)\left[3(0^2) - 0^2\right] + (-2q)\left[3(-a)^2 - a^2\right]$

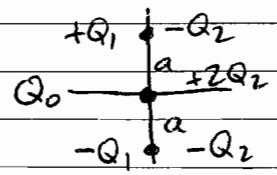
$= \boxed{-4qa^2}$

d. By symmetry and zero trace $Q_{xx} = Q_{yy} = -\frac{1}{2}Q_{zz} = \boxed{+2qa^2}$

$$V_0 + V_1 + V_2 = \frac{kQ_{net}}{r} + \frac{k\vec{r}\cdot\vec{p}}{r^3} + \frac{k}{2r^5}\sum_{i,j} Q_{ij}x_i x_j$$

$$= \boxed{\frac{2kqaz}{r^3} + \frac{kqa^2(x^2+y^2-2z^2)}{r^5}}$$

e & f. Off center we have as given $\boxed{\text{zero monopole} + 2qa\hat{k} \text{ dipole}}$.
The quadrupole moments are nonzero and were calculated in parts c–d.
Here's another technique to get the on-center moments:



monopole of Q_0
dipole of $\pm Q_1$
quadrupole of $-Q_2/+2Q_2/-Q_2$

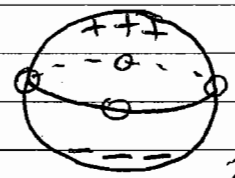
where we sum charges to agree w/ actual charges at each point:

top $\quad Q_1 - Q_2 = 0 \Rightarrow Q_2 = Q_1$
bottom $\quad -Q_1 - Q_2 = -2q \Rightarrow Q_1 = q = Q_2$
middle $\quad Q_0 + 2Q_2 = 2q \Rightarrow Q_0 = 0$

$\therefore$ $\boxed{\begin{array}{l}\text{zero monopole +} \\ \text{dipole of } \pm q \text{ separated by } 2a \Rightarrow 2qa\hat{k} \quad + \\ \text{quadrupole of } -q/2q/-q \Rightarrow Q_{zz} = -4qa^2\end{array}}$

exactly as in parts a–c. The calculations in parts c–d for the other quadrupole moments therefore also give the same answers. According to the italicized statement in the problem, the octupole moment is zero. ▨

(d) <u>MP #4</u>



shell of radius R
$\sigma = \sigma_0 \cos\theta$

a. $$Q_{net} = \int \sigma\, dA = \int_0^{2\pi}\int_0^{\pi} \sigma_0 \cos\theta\, R^2 \sin\theta\, d\theta\, d\phi = \boxed{0}$$

ODD abt. $\theta = \frac{\pi}{2}$ EVEN

b. $\vec{p} = \int \sigma \vec{r}\, dA = \int \sigma (x\hat{i} + y\hat{j} + z\hat{k})\, dA$

but σ is EVEN abt. $x=0$ while x is ODD $\Rightarrow p_x = 0$

likewise $p_y = 0$

$$p_z = \int_0^{2\pi}\int_0^{\pi} \overset{\sigma}{(\sigma_0 \cos\theta)}\overset{z}{(R\cos\theta)}\overset{dA}{(R^2 \sin\theta\, d\theta\, d\phi)}$$

$$= 2\pi \sigma_0 R^3 \tfrac{1}{3}\cos^3\theta \Big|_{\pi}^{0} \Rightarrow \boxed{\vec{p} = \frac{4\pi R^3}{3}\sigma_0 \hat{k}}$$

c. $Q_{zz} = \int \sigma\, dA\, [3z^2 - r^2]$

$$= \int_0^{2\pi}\int_0^{\pi} (\underbrace{\sigma_0 \cos\theta}_{\text{ODD abt. } \theta=\frac{\pi}{2}})(\underbrace{R^2 \sin\theta\, d\theta\, d\phi}_{\text{EVEN}})(\underbrace{3R^2\cos^2\theta - R^2}_{\text{EVEN}})$$

$$= \boxed{0}$$

by symmetry and zero trace $Q_{xx} = Q_{yy} = -\frac{1}{2}Q_{zz} = \boxed{0}$

$$Q_{xy} = \int \underbrace{\sigma}_{\text{EVEN abt } x=0}\, dA\, (3\underbrace{xy}_{\text{ODD}}) = \boxed{0}$$

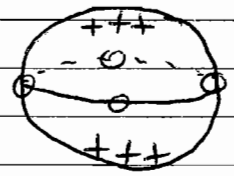
EVEN abt $y=0$; ODD

likewise $Q_{xz} = Q_{yz} = \boxed{0}$ since their integrals contain either odd x or odd y

d. $V = V_0 + V_1 + V_2 = \frac{k\vec{r}\cdot\vec{p}}{r^3} = \boxed{\frac{\sigma_0 z R^3}{3\epsilon_0 r^3}}$

(It turns out this answer is EXACT not approximate, as proven in Griffiths eq. 3.87.) ▨

(e) <u>MP #5</u>



shell of radius R
$\sigma = \sigma_0 \cos^2\theta$

a. $q_{net} = \int_0^{2\pi}\int_0^{\pi} (\sigma_0 \cos^2\theta)(R^2 \sin\theta\, d\theta\, d\phi)$

$$= 2\pi\sigma_0 R^2 \tfrac{1}{3}\cos^3\theta \Big|_{\pi}^{0} = \boxed{\frac{4\pi R^2}{3}\sigma_0}$$

b. $\vec{p} = \int \sigma (x\hat{i} + y\hat{j} + z\hat{k})\, dA$

but σ is EVEN abt. $x=0, y=0, z=0$
while x, y, z are ODD $\Rightarrow$ $\vec{p} = \boxed{\vec{0}}$

c. $$Q_{zz} = \int \sigma \, dA \, (3z^2 - r^2)$$
$$= \int_0^{2\pi}\int_0^{\pi} (\sigma_0 \cos^2\theta)(R^2 \sin\theta \, d\theta \, d\phi)(3R^2\cos^2\theta - R^2)$$
$$= 2\pi \sigma_0 R^4 \left[\frac{3}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta\right]_{\pi}^{0} = \boxed{\frac{16\pi R^4}{15}\sigma_0}$$

by symmetry & zero trace $Q_{xx} = Q_{yy} = -\frac{1}{2}Q_{zz} = \boxed{-\frac{8\pi R^4}{15}\sigma_0}$

$Q_{xy} = Q_{xz} = Q_{yz} = \boxed{0}$ for same reason as MP #4 c.

d. $$V_0 + V_1 + V_2 = \frac{k q_{net}}{r} + \frac{k\,\vec{r}\cdot\vec{p}}{r^3} + \frac{k}{2r^5}\sum_{i,j} Q_{ij} x_i x_j$$
$$= \frac{\sigma_0 R^2}{3\epsilon_0 r} + \frac{\sigma_0 R^4 (2z^2 - x^2 - y^2)}{15\epsilon_0 r^5}$$
$$= \boxed{\frac{\sigma_0 R^2}{3\epsilon_0 r}\left[1 + \frac{2R^2}{5r^2}P_2(\cos\theta)\right]} \quad \blacksquare$$