$\boxed{\text{HW \#5}}$

/15 + /3 per plot = /33

(a) WUP #1

a. $V = A \sin kx \, e^{-cy}$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = (-k^2 + c^2)V = 0 \Rightarrow c = \pm k$$

assume $c, k > 0 \Rightarrow \boxed{c = k}$

b. $V = A \cos kx \sin kx \, e^{-2ky}$

shortcut

$V = \frac{A}{2} \sin(2kx) e^{-2ky}$

now result follows from part a!

$$\Rightarrow \frac{\partial V}{\partial x} = A e^{-2ky}(-k \sin^2 kx + k \cos^2 kx)$$
$$= kAe^{-2ky}(2\cos^2 kx - 1)$$
$$\Rightarrow \frac{\partial^2 V}{\partial x^2} = kAe^{-2ky}(-4k \cos kx \sin kx)$$
$$= -(2k)^2 V \quad \text{but } \frac{\partial^2 V}{\partial y^2} = (-2k)^2 V$$
$$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \boxed{0}$$

(b) only need (r, θ) parts of Laplacian:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)$$

1. $V = r\cos\theta$

$$\Rightarrow \nabla^2 V = \frac{1}{r^2} 2r\cos\theta - \frac{1}{r^2 \sin\theta} r \underbrace{\frac{\partial}{\partial\theta}\sin^2\theta}_{= 2\sin\theta\cos\theta} = 0$$

2. $V = r^{-2}\cos\theta$

$$\Rightarrow \nabla^2 V = -\frac{1}{r^2}\underbrace{\frac{\partial}{\partial r} 2r^{-1}}_{= -2r^{-2}} \cos\theta - \frac{1}{r^2 \sin\theta} r^{-2} 2\sin\theta\cos\theta = 0$$

3. $V = r^{-1}$

$$\Rightarrow \nabla^2 V = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^{-2}) = 0$$

4. $V = r^{-3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$

$$\Rightarrow \nabla^2 V = -\frac{1}{r^2}\underbrace{\frac{\partial}{\partial r} 3r^{-2}}_{= -6r^{-3}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) - \frac{r^{-3}}{r^2 \sin\theta}\underbrace{\frac{\partial}{\partial\theta}\sin\theta(3\cos\theta\sin\theta)}_{= -3\sin^3\theta + 6\cos^2\theta\sin\theta}$$

$$= \frac{3}{r^5}(3\cos^2\theta - 1) - \frac{1}{r^5}(-3\overbrace{\sin^2\theta}^{1-\cos^2\theta} + 6\cos^2\theta) = 0$$ ✓

(c) SL#1

I iterated thousands of times. Attached are 4 pages:

- spreadsheet of final values $\Rightarrow$ we see that $V = \boxed{0.25}$ at the midpoint as we might have expected (average of the four walls)
- surface plot
- vertical line cut through the center
- horizontal line cut through the center

(d) SL#2

This time I iterated hundreds of thousands of times. Attached are surface plots of both the iterated and Griffiths solution, which agree at every point to at least 3 sigfigs!

(e) SL#12

a. $V=0$ as $x \to \infty \Rightarrow \boxed{A_n = 0}$

b. $V=0$ at $y=0 \Rightarrow \boxed{D_n = 0}$

c. $V=0$ at $y=a \Rightarrow \boxed{k_n = \frac{n\pi}{a}}$

d. $V_0 \sin\left(\frac{\pi y}{a}\right) = V(0,y) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi y}{a}\right)$

match coeffs $\Rightarrow F_1 = V_0$ and $F_n = 0 \;\forall\, n>1$

$$\therefore \boxed{V(x,y) = V_0\, e^{-\pi x/a} \sin(\pi y/a)}$$

e. $\sigma(x,0) = \epsilon_0 E_y(x,0) = -\epsilon_0 \left.\frac{\partial V}{\partial y}\right|_{y=0} = \boxed{-\frac{\pi V_0 \epsilon_0}{a} e^{-\pi x/a}}$

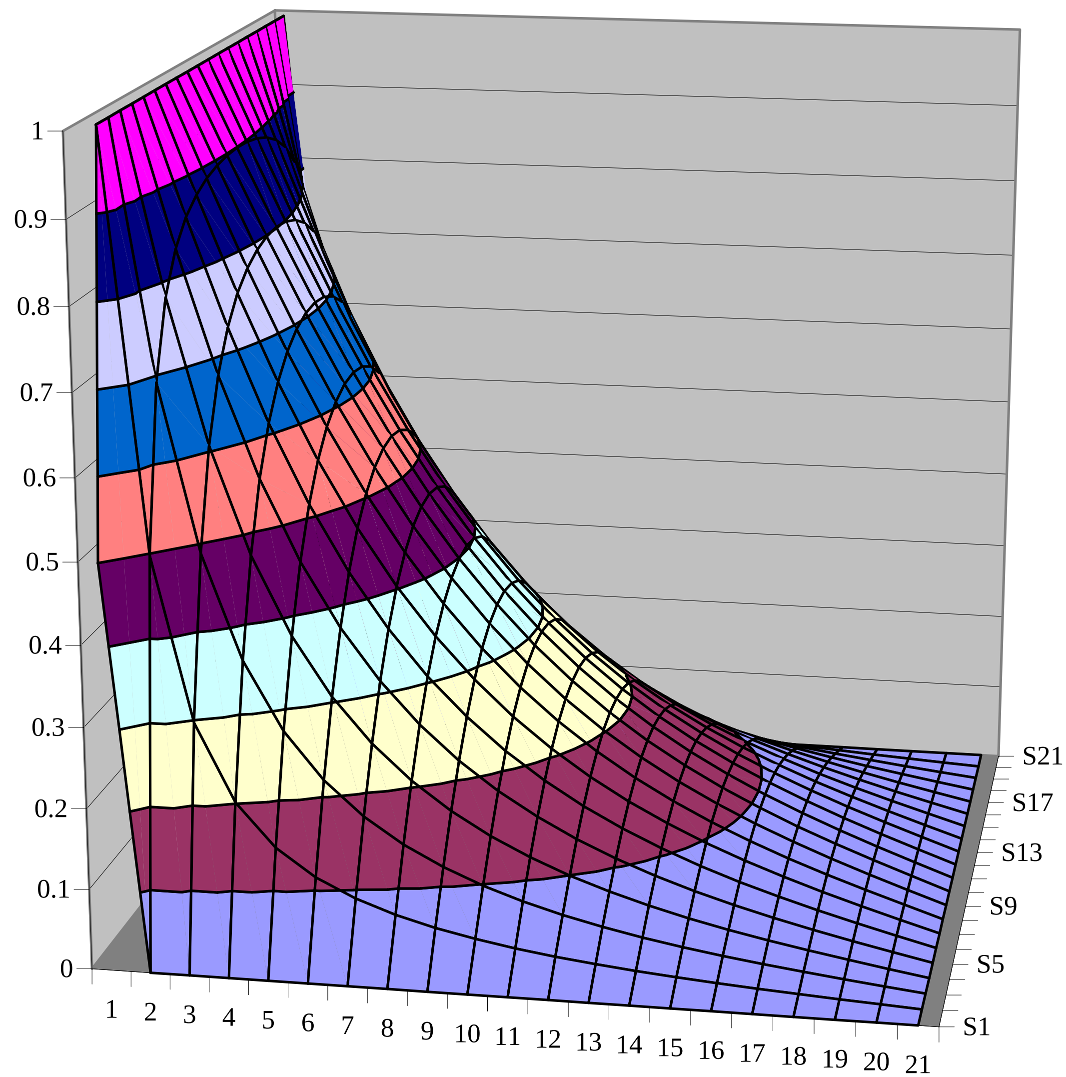
There's (sinusoidal) + charge on left edge, and − charge on top & bottom edges which decays to zero as $x \to \infty$ (right "edge").

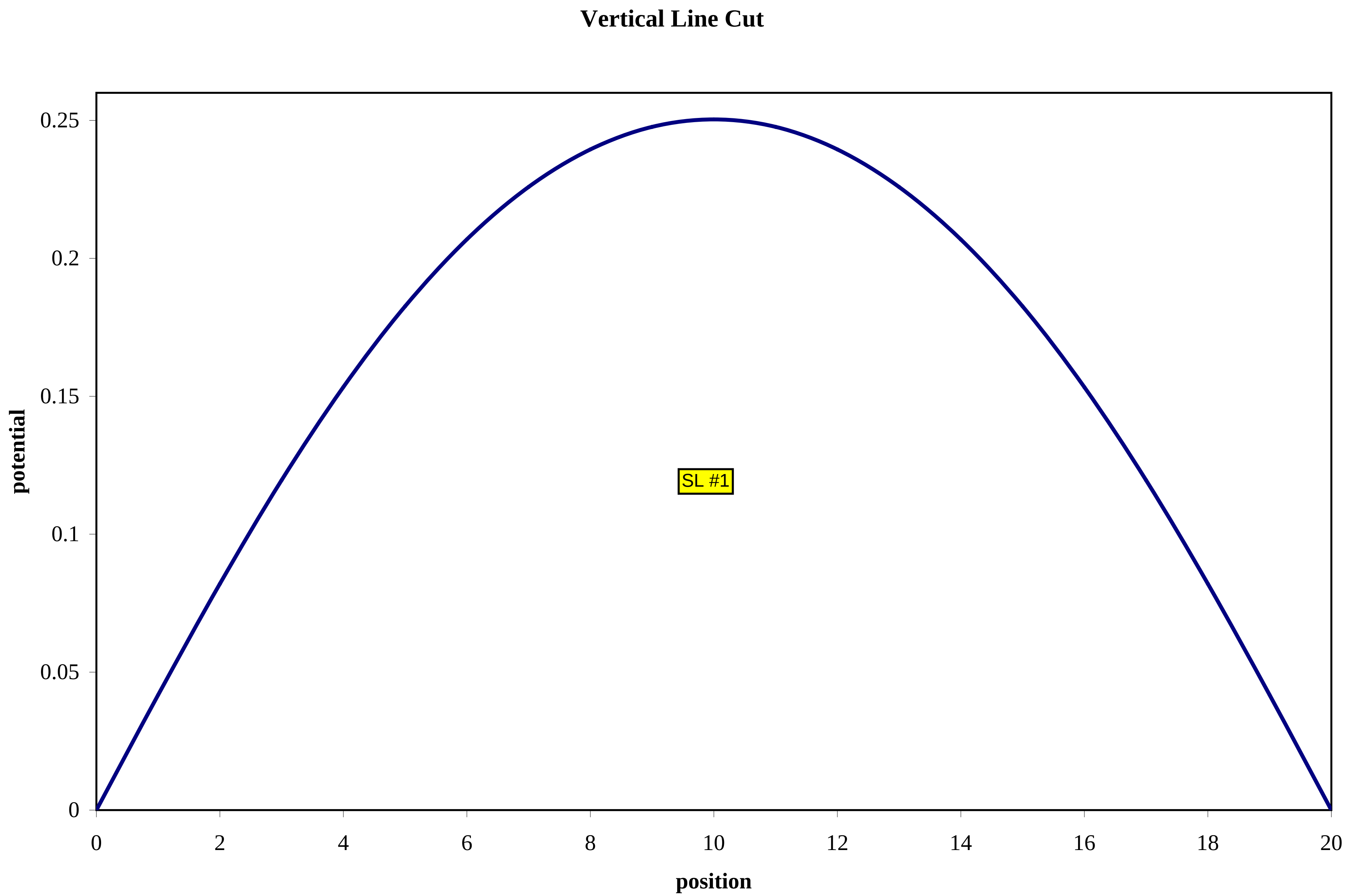
SL #1

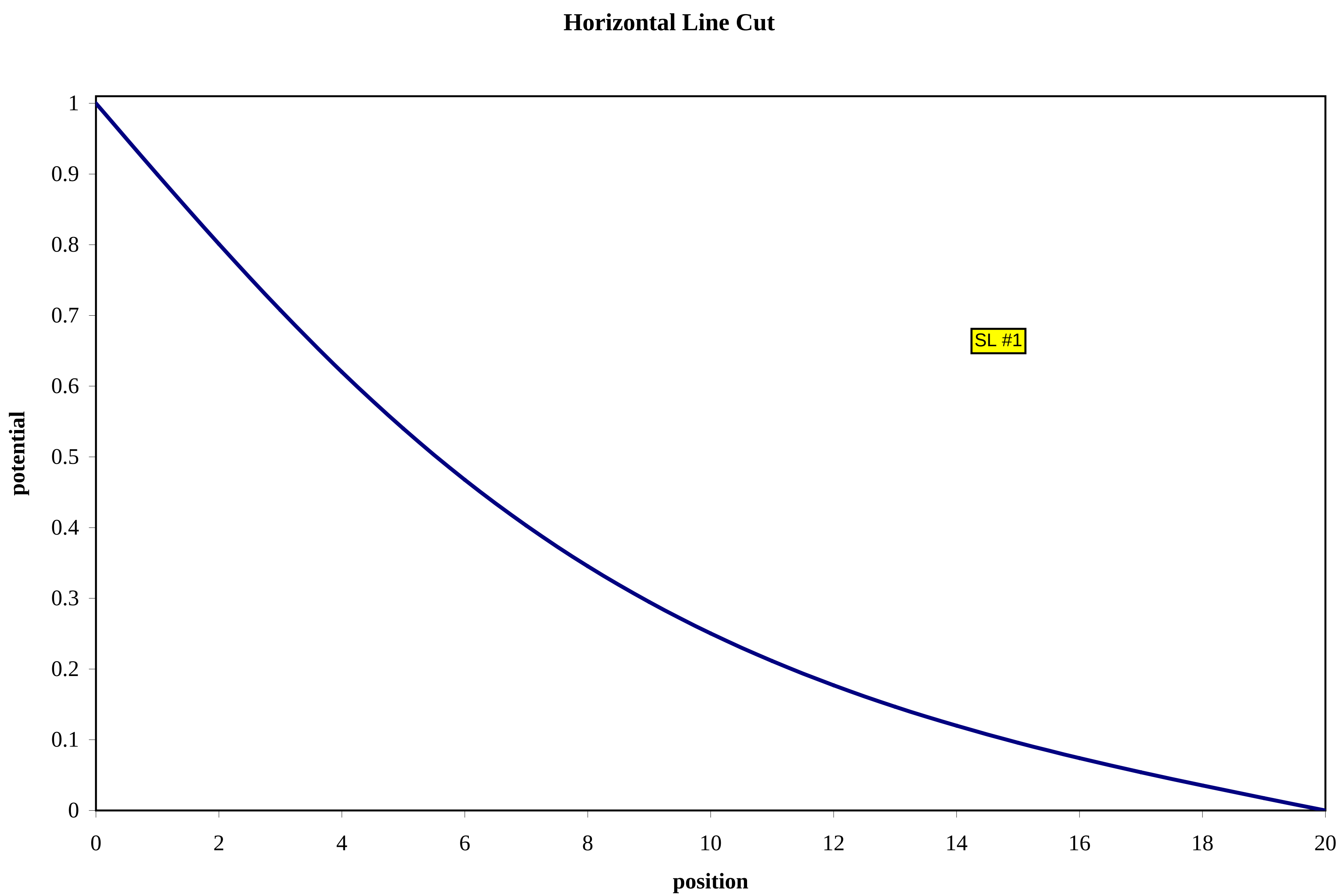
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0.497	0.297	0.201	0.148	0.114	0.091	0.074	0.061	0.050	0.042	0.035	0.029	0.024	0.019	0.015	0.012	0.009	0.006	0.003	0
2	1	0.692	0.489	0.361	0.276	0.218	0.176	0.144	0.119	0.099	0.082	0.068	0.057	0.046	0.038	0.030	0.023	0.017	0.011	0.005	0
3	1	0.782	0.607	0.475	0.379	0.306	0.250	0.207	0.172	0.143	0.120	0.100	0.083	0.068	0.055	0.044	0.034	0.025	0.016	0.008	0
4	1	0.830	0.680	0.556	0.456	0.377	0.313	0.261	0.219	0.183	0.153	0.128	0.107	0.088	0.071	0.057	0.044	0.032	0.021	0.010	0
5	1	0.858	0.727	0.612	0.514	0.432	0.364	0.306	0.258	0.217	0.183	0.153	0.127	0.105	0.086	0.068	0.053	0.038	0.025	0.012	0
6	1	0.876	0.758	0.651	0.556	0.474	0.403	0.342	0.290	0.245	0.207	0.174	0.145	0.120	0.098	0.078	0.060	0.044	0.029	0.014	0
7	1	0.887	0.779	0.678	0.586	0.504	0.432	0.369	0.314	0.267	0.226	0.190	0.159	0.131	0.107	0.086	0.066	0.048	0.032	0.016	0
8	1	0.894	0.792	0.695	0.605	0.524	0.452	0.388	0.332	0.283	0.239	0.202	0.169	0.140	0.114	0.091	0.071	0.051	0.034	0.017	0
9	1	0.898	0.799	0.704	0.617	0.536	0.464	0.399	0.342	0.292	0.248	0.209	0.175	0.145	0.118	0.095	0.073	0.053	0.035	0.017	0
10	1	0.899	0.801	0.708	0.620	0.540	0.468	0.403	0.345	0.295	0.250	0.211	0.177	0.147	0.120	0.096	0.074	0.054	0.035	0.017	0
11	1	0.898	0.799	0.704	0.617	0.536	0.464	0.399	0.342	0.292	0.248	0.209	0.175	0.145	0.118	0.095	0.073	0.053	0.035	0.017	0
12	1	0.894	0.792	0.695	0.605	0.524	0.452	0.388	0.332	0.283	0.239	0.202	0.169	0.140	0.114	0.091	0.070	0.051	0.034	0.017	0
13	1	0.887	0.779	0.678	0.586	0.504	0.432	0.369	0.314	0.267	0.226	0.190	0.159	0.131	0.107	0.086	0.066	0.048	0.032	0.016	0
14	1	0.876	0.758	0.651	0.556	0.474	0.403	0.342	0.290	0.245	0.207	0.174	0.145	0.120	0.098	0.078	0.060	0.044	0.029	0.014	0
15	1	0.858	0.727	0.612	0.514	0.432	0.363	0.306	0.258	0.217	0.183	0.153	0.127	0.105	0.086	0.068	0.053	0.038	0.025	0.012	0
16	1	0.830	0.680	0.556	0.456	0.377	0.313	0.261	0.218	0.183	0.153	0.128	0.106	0.088	0.071	0.057	0.044	0.032	0.021	0.010	0
17	1	0.782	0.607	0.475	0.378	0.306	0.250	0.207	0.172	0.143	0.120	0.100	0.083	0.068	0.055	0.044	0.034	0.025	0.016	0.008	0
18	1	0.692	0.489	0.361	0.276	0.218	0.176	0.144	0.119	0.099	0.082	0.068	0.057	0.046	0.038	0.030	0.023	0.017	0.011	0.005	0
19	1	0.497	0.297	0.201	0.148	0.114	0.091	0.074	0.061	0.050	0.042	0.035	0.029	0.024	0.019	0.015	0.012	0.009	0.006	0.003	0
20	0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

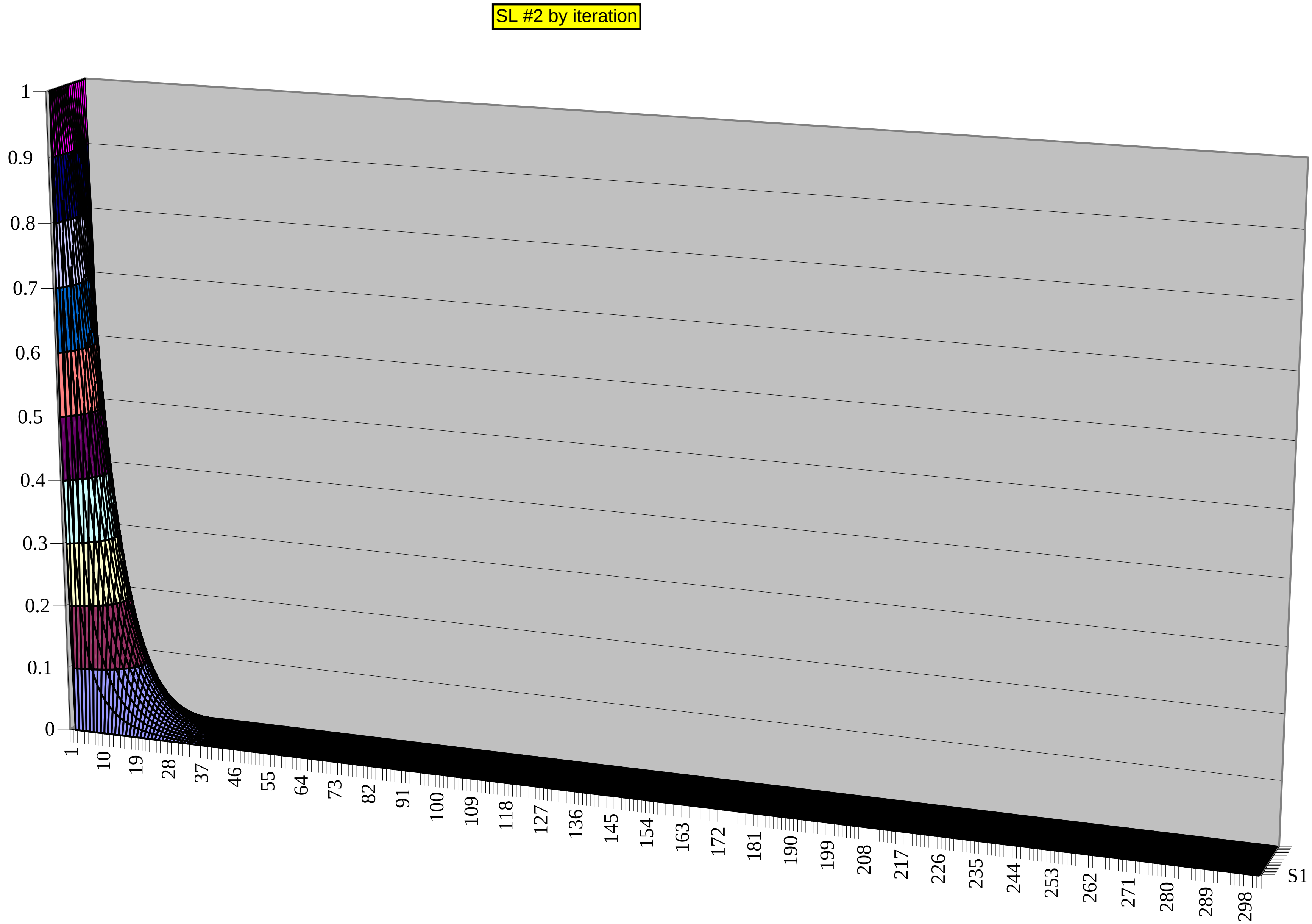
Surface Plot of Potential

SL #1

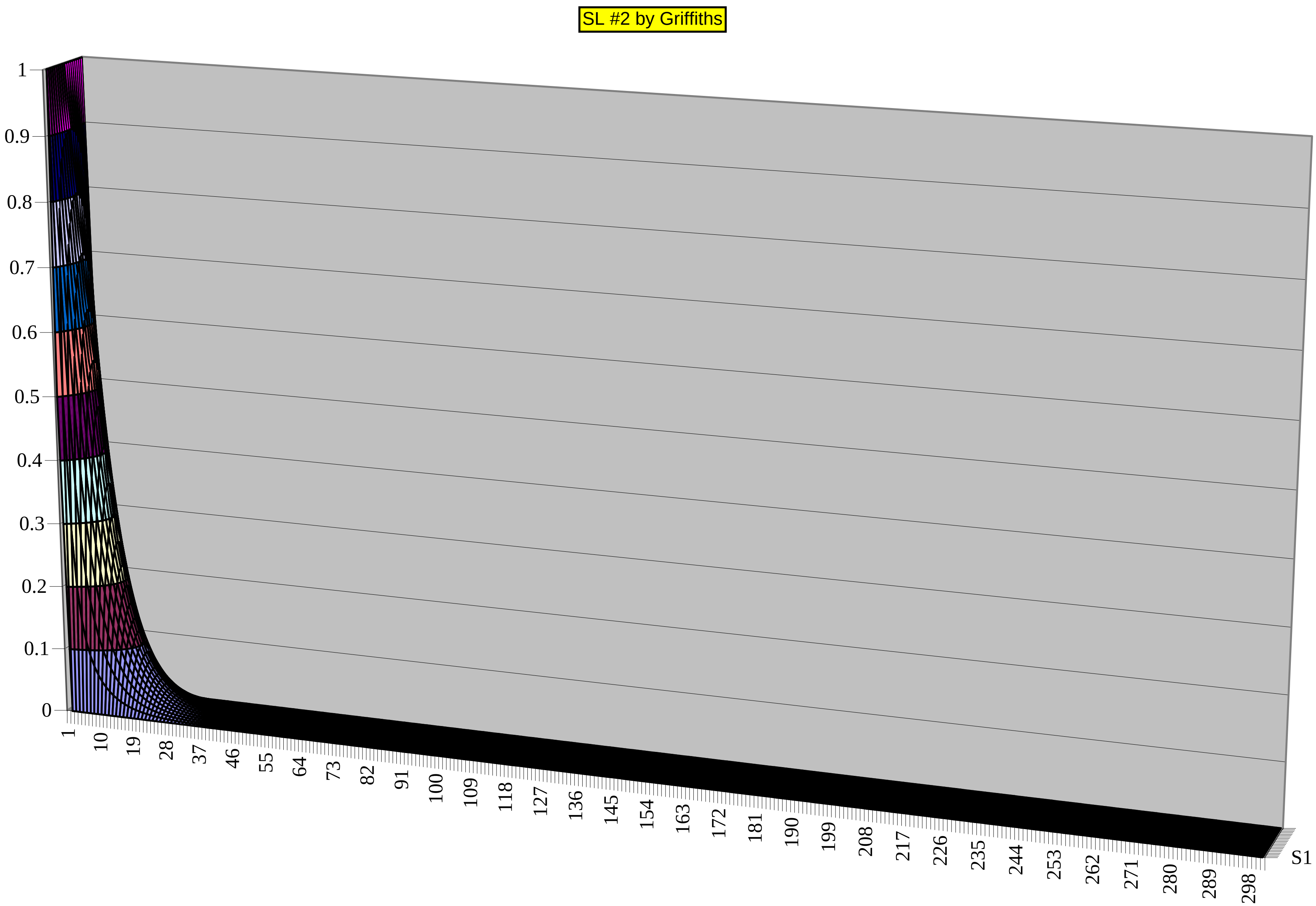



Vertical Line Cut
potential
0.25
0.2
0.15
0.1
0.05
0
SL #1
0
2
4
6
8
10
12
14
16
18
20
position


Horizontal Line Cut
potential
position
SL #1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
12
14
16
18
20



SL #2 by iteration
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
10
19
28
37
46
55
64
73
82
91
100
109
118
127
136
145
154
163
172
181
190
199
208
217
226
235
244
253
262
271
280
289
298
S1


SL #2 by Griffiths
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0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
10
19
28
37
46
55
64
73
82
91
100
109
118
127
136
145
154
163
172
181
190
199
208
217
226
235
244
253
262
271
280
289
298
S1