SP352 Test #1 **Name: _____**

No notes or calculators permitted. Show all work.
Do your work on separate paper and staple to this sheet.
Do not discuss this test with anyone until both sections have completed it!
This exam will be graded out of 17 points, so it is possible to get over 100%.

1. [3 pts] Test the following set of three vectors for linear independence in 3D:

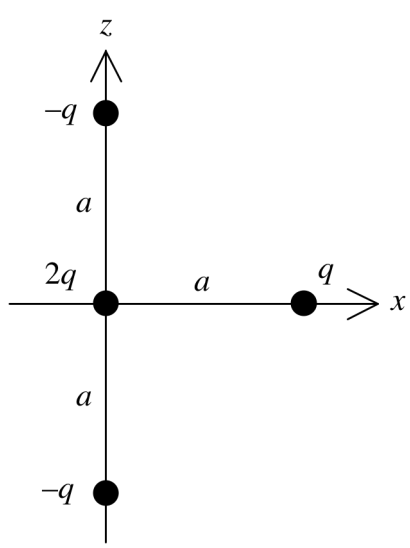
$$\vec{\mathbf{A}} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\vec{\mathbf{B}} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

$$\vec{\mathbf{C}} = 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$$

2. [10 pts] Find the Fourier series in sigma form for a unit-amplitude symmetric triangle wave of period T by explicitly integrating by parts. (For full credit, explicitly write out what you are substituting for u and υ in the integration by parts.) Write the final answer as a sum over m running over EVERY integer from 0 to ∞. Start by drawing a graph and using it to find a formula for $f(t)$ from 0 to $T/2$. Explicitly evaluate any trigonometric functions involving π.

3. Consider the following distribution of two negative and two positive point charges, where the y-axis points into the page:



(a) [6 pts] What are the monopole, dipole, and quadrupole moments? Express the dipole moment in vector form, and the quadrupole moment in the form of a 3×3 matrix.

(b) [4 pts] Write out the expansion of the potential up to and including the second order (quadrupole) term. Use ONLY the following symbols in your final answer: k, q, a, x, y, and z. (Note carefully: do NOT include spherical coordinates such as r in your final answer.)

Test #1

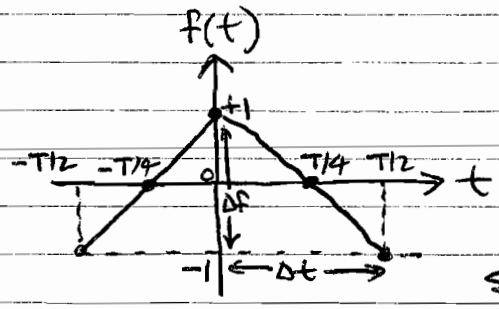
(1) Determinant of the matrix of coefficients in column form is:

$$\begin{vmatrix} 3 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 2 & 0 \end{vmatrix}$$

Take cofactors along row 3 to get:

$$1(6-4) + 2(8-9) = 0 \Rightarrow \boxed{\text{linearly dependent}} \quad \boxtimes$$

(2)



intercept ↓ ↙ slope $\Delta f/\Delta t$ of graph

$$\therefore\ f(t) = 1 - \frac{2}{T/2}\,t \quad \text{for } 0 \le t \le \frac{T}{2}$$

$$= \frac{T-4t}{T}$$

symmetric = even $\Rightarrow$ pure cosine series

$$a_n = \frac{2}{T}\, 2\int_0^{T/2} f(t)\cos\left(n\frac{2\pi}{T}t\right)dt$$

(double $0 \to T/2$ since integrand is even)

$$= \frac{4}{T^2}\int_0^{T/2} (T-4t)\cos\left(\frac{2\pi n}{T}t\right)dt$$

by parts: $u = T-4t \qquad du = -4\,dt$

$$dv = \cos\left(\frac{2\pi n}{T}t\right)dt \quad \Rightarrow \quad v = \frac{T}{2\pi n}\sin\left(\frac{2\pi n}{T}t\right)$$

$$\therefore\ \frac{T^2}{4}a_n = (T-4t)\frac{T}{2\pi n}\sin\left(\frac{2\pi n}{T}t\right)\Big|_0^{T/2} + 4\frac{T}{2\pi n}\int_0^{T/2}\sin\left(\frac{2\pi n}{T}t\right)dt$$

0 at both limits

$$= \frac{2T}{n\pi}\cdot\frac{T}{2\pi n}\underbrace{\cos\left(\frac{2\pi n}{T}t\right)\Big|_{T/2}^{0}}_{1-\underbrace{\cos(n\pi)}_{(-1)^n}} = \begin{cases} 0 & \text{if } n \text{ even} \\ 2 & \text{if } n \text{ odd} \end{cases}$$

(reversed limits to change sign)

$$\Rightarrow a_n = \frac{8}{n^2\pi^2} \quad \text{for odd } n \equiv 2m+1 \quad (m = 0, 1, 2, \ldots)$$

$$\therefore\ f(t) = \boxed{\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{\cos(2m+1)\frac{2\pi}{T}t}{(2m+1)^2}}$$

(As expected, goes like the reciprocal of the SQUARE of the ODD harmonic numbers.) $\boxtimes$

(3) (a) $q_{net} = (-q) + (+2q) + (-q) + (+q) = \boxed{q}$

$$\vec{p} = (-q)(a\hat{k}) + (+2q)(\vec{0}) + (-q)(-a\hat{k}) + (+q)(a\hat{i}) = \boxed{qa\hat{i}}$$

$$Q_{zz} = (-q)\left[3(a)(a) - (a)^2\right] + (+2q)\left[3(0)(0) - (0)^2\right]$$
$$+ (-q)\left[3(-a)(-a) - (a)^2\right] + (+q)\left[3(0)(0) - (a)^2\right] = -5qa^2$$

$$Q_{xx} = (-q)\left[3(0)(0) - (a)^2\right] + (+2q)\left[3(0)(0) - (0)^2\right]$$
$$+ (-q)\left[3(0)(0) - (a)^2\right] + (+q)\left[3(a)(a) - (a)^2\right] = 4qa^2$$

$$Q_{yy} = -Q_{zz} - Q_{xx} = qa^2$$

All off-diagonal quadrupole elements are zero since $\delta_{ij} = 0$ for them and each charge has only ONE nonzero coordinate (so that either $x_{\alpha i} = 0$ or $x_{\alpha j} = 0$ or both).

$$\therefore \boxed{\overleftrightarrow{Q} = qa^2 \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}}$$

(b) $$V \approx V_0 + V_1 + V_2 = \frac{k q_{net}}{r} + \frac{k\vec{p}\cdot\vec{r}}{r^3} + \frac{k}{2r^5}\sum_{i,j=1}^{3} Q_{ij} x_i x_j$$

$$= \frac{kq}{r} + \frac{kqax}{r^3} + \frac{kqa^2}{2r^5}(4x^2 + y^2 - 5z^2)$$

$$= \boxed{\frac{kq}{\sqrt{x^2+y^2+z^2}} + \frac{kqax}{(x^2+y^2+z^2)^{3/2}} + \frac{kqa^2}{2} \cdot \frac{(4x^2+y^2-5z^2)}{(x^2+y^2+z^2)^{5/2}}}$$

(This form fully unpacks the dependence on x, y, z making it easiest to calculate or plot V.) ☒