The Laplace Equation – Solutions & Applications

Background

- **Differential Equations handout**
- **PDE Solution by Separation**

Concepts of primary interest:

- **Absence of internal extrema**
- **Average property**
- **Uniqueness conditions**
- **Product-form trial solution**
- **Separation constants**
- **Sets of orthogonal functions**
- **Locally orthogonal coordinate systems**
- **Discrete and continuous eigenvalue spectra**
- ****** ADD properties of Legendre and $Y_{\ell m}$'s**

Sample calculations:

- **Potential due to a point charge averaged over a sphere**
- **Potential due to a ring of charge at points off axis**
- **Boundary value matching on a spherical surface**

Tools of the trade:

- **Sines and cosines of evenly spaced arguments**

The Laplacian operator appears in a multitude of partial differential equations describing physical situations. Examples of Laplace and Poisson's equations are to be presented primarily in the context of electrostatics. Maxwell's equations, specialized to electrostatics, describe the physics of interest.

$$\oint \vec{E}\cdot d\vec{\ell} = 0 \qquad \vec{\nabla}\times\vec{E} = 0 \qquad \Rightarrow \vec{E} = -\vec{\nabla}V(\vec{r})$$

$$\oint_{\partial V} \vec{E}\cdot\hat{n}\; dA = \tfrac{1}{\varepsilon_o}\int_V \rho\, dV \qquad \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_o \qquad \Rightarrow \vec{\nabla}\cdot\vec{\nabla}V(\vec{r}) = \nabla^2 V(\vec{r}) = -\rho/\varepsilon_o$$

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The conclusion is that the electrostatic potential obeys Poisson's equation which, in charge-free space, reduces to the Laplace equation:

$$\nabla^2 V(\vec{r}) = 0 . \qquad \textbf{[SL.1]}$$

Solutions to the Laplace equation have two related properties that are of interest.

Property one: A function satisfying the Laplace equation in a region does not have a local maximum or minimum at an interior point in that region. Rather, all extrema occur on the boundary of that region. The situation is a generalization of *endpoint extrema* for a function of a single variable.

Property two: A solution to the Laplace equation in a region assumes a value at each point that is the average of its values at (a full set of equally distant) points located symmetrically around the point of interest. The statement of property two is not very precise; its meaning becomes clear as examples are presented.

Laplace in 1D: In Cartesian coordinates, the Laplace equation is:

$$\nabla^2 V(\vec{r}) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Following the little steps approach, we begin with the equation in one dimension:

$$\frac{d^2 V}{dx^2} = \frac{d}{dx}\left(\frac{dV}{dx}\right) = 0 .$$

The plot of $V(x)$ has a constant slope. If $V(x)$ satisfies the equation in the open interval (a, b), then $V(x) = m\,x + b$ and there are no relative maxima or minima in that open interval. The only extrema are the end-point extrema $V(a)$ and $V(b)$. Further, given x_1 and x_2 in the interval (a, b), $V\left(\frac{x_1+x_2}{2}\right) = \frac{V(x_1)+V(x_2)}{2}$, the value of the potential at $\frac{1}{2}(x_1 + x_2)$ is the average of the value of the potential at the two equidistant points x_1 and x_2.

Laplace in 2D: In Cartesian coordinates, the Laplace equation is:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0\,.$$

If the values of *V(x,y)* is to be a local maximum, then both $\frac{\partial^2 V}{\partial x^2}<0$ and $\frac{\partial^2 V}{\partial y^2}<0$ which is not possible for a solution of the Laplace equation: $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$. Maxima of *V(x,y)* can be found only at locations on the boundary of the region in which $V(x,y)$ satisfies the Laplace equation. Clearly, an analogous statement holds for minima. As a motivation for the average value property, approximations for the various derivatives are given in terms of the values of $V(x,y)$ on a square grid of points laid out as $(x+n\Delta,\ y+m\Delta)$ where m and n are integers.

$$\left.\frac{\partial V}{\partial x}\right|_{(x+\Delta/2,y)}\cong\frac{V(x+\Delta,y)-V(x,y)}{\Delta}\qquad\left.\frac{\partial V}{\partial x}\right|_{(x-\Delta/2,y)}\cong\frac{V(x,y)-V(x-\Delta,y)}{\Delta}$$

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{(x,y)}\cong\frac{\left.\frac{\partial V}{\partial x}\right|_{(x+\Delta/2,y)}-\left.\frac{\partial V}{\partial x}\right|_{(x-\Delta/2,y)}}{\Delta}=\frac{V(x+\Delta,y)-2V(x,y)+V(x-\Delta,y)}{\Delta^2}\qquad\text{[SL.2]}$$

After making analogous approximations, the approximate second partial w.r.t. y is:

$$\left.\frac{\partial^2 V}{\partial y^2}\right|_{(x,y)}\cong\frac{\left.\frac{\partial V}{\partial y}\right|_{(x,y+\Delta/2)}-\left.\frac{\partial V}{\partial y}\right|_{(x,y-\Delta/2)}}{\Delta}=\frac{V(x,y+\Delta)-2V(x,y)+V(x,y-\Delta)}{\Delta^2}$$

Substituting these relations into the Laplace equation yields the **Relaxation Method** approximation for the value of the function $V(x,\ y)$ on a 2D *square grid* in terms of its values at the four nearest neighbor points.

$$V(x,y)\cong\frac{V(x+\Delta,y)+V(x-\Delta,y)+V(x,y+\Delta)+V(x,y-\Delta)}{4}$$

The value of the solution $V(x,y)$ is approximately the average value of V at four points that are equally distant from (x,y) along each of the four coordinate directions. This result above is the relaxation method approximation; it is not the precise average of a full set of values at symmetrically-located, equally-distant points.

The full average of a complete set of equidistant points is the average of the values on a circle concentric with the point of interest. That average can be computed as:

$$V(x,y) = \frac{1}{2\pi R}\int_0^{2\pi} V(x + R\cos\theta, y + R\sin\theta)\ R\ d\theta \qquad \text{[SL.3]}$$

(Note: *V(x,y) must satisfy the Laplace equation everywhere within the circle.*)

While not exact, the relaxation method is a useful numerical technique for approximating the solution to the Laplace equation when the values of $V(x,y)$ are given on the boundary of a region.

The 2D Laplace problem solution has an approximate physical model, a uniform elastic membrane held at heights $V(x,y)$ on the boundary of the x-y region in which the Laplace equation is satisfied (with all vertical displacements and slopes kept small). As a model, the elastic membrane facilitates visualizing the absence of local extrema and the average value property.

Laplace in 3D: In Cartesian coordinates, the Laplace equation is:

$$\nabla^2 V(\vec{r}) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0\,.$$

By the arguments above, if one of the second derivatives is positive, then at least one of the others must be negative ruling out the possibility of extrema at an interior point.

The relaxation method approximation for a function evaluated on a cubic grid of points $\{x + k\Delta,\ y + m\Delta,\ z + n\Delta)\}$ can be extended to:

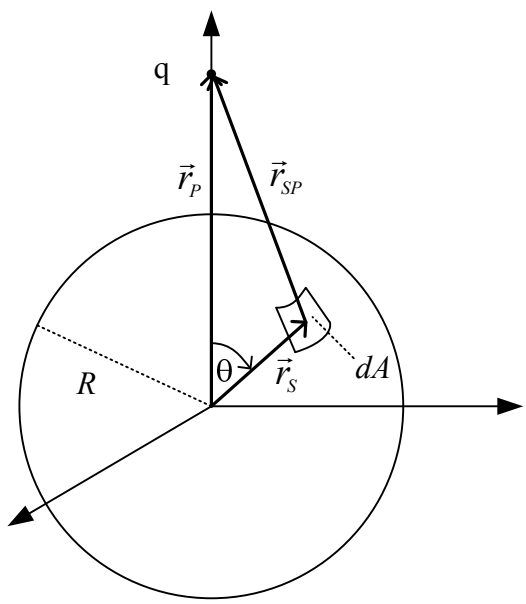
$$V(x,y,z) \cong \frac{V(x+\Delta,y,z)+V(x-\Delta,y,z)+V(x,y+\Delta,z)+V(x,y-\Delta,z)+V(x,y,z+\Delta)+V(x,y,z-\Delta)}{6}$$

This form is an approximate average property with the exact form paralleling that for two dimensions. A full set of equidistant point is a concentric spherical shell in three dimensions. Assuming that *V satisfies* the Laplace equation at all points within the

sphere, the average value of V on the surface of the sphere is equal to the value of V at the center of that sphere.

$$V(x,y,z) = \frac{1}{4\pi R^2}\int_0^{\pi} R^2 \sin\theta\, d\theta \int_0^{2\pi} V(x + R\sin\theta\cos\phi, y + R\sin\theta\sin\phi, z + R\cos\theta)\, d\phi$$

Sample Calculation: The potential due to a point charge averaged over a sphere: For this example, the region inside the sphere is charge-free so the potential satisfies the Laplace equation in that region. Examining the average property for this case:



A point charge is located a distance z up the polar axis. The average value of the potential due to that charge is to be computed over the surface of a sphere of radius R centered on the origin. The distance from the charge to a patch on the surface of the sphere is, by the law of cosines, $\sqrt{R^2 + z^2 - 2zR\cos\theta}$. The potential due to q at the patch is:

$$\frac{q}{4\pi\varepsilon_0\sqrt{R^2 + z^2 - 2zR\cos\theta}}.$$

$$V_{ave} = \frac{1}{A_{sphere}}\int V_{\text{patch}}\, dA_{\text{patch}}$$

The average value of the potential over the sphere is computed as:

$$V_{ave} = \frac{1}{4\pi R^2}\int_0^{\pi}\int_0^{2\pi} \frac{q(R^2 \sin\theta\, d\theta\, d\phi)}{4\pi\varepsilon_0\sqrt{R^2 + z^2 - 2zR\cos\theta}} = \frac{1}{2}\int_0^{\pi} \frac{q\, \sin\theta\, d\theta}{4\pi\varepsilon_0\sqrt{R^2 + z^2 - 2zR\cos\theta}}$$

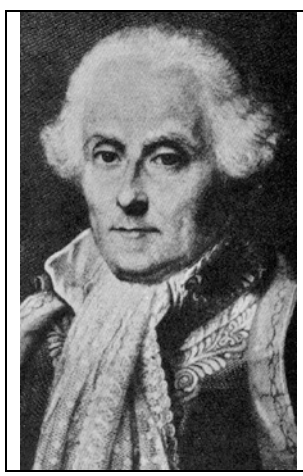
The integral yields to the change of variable $u = R^2 + z^2 - 2zR\cos\theta$ and $du = 2zR\sin\theta d\theta$. Making the corresponding changes to the limits of integration,

$$V_{ave} = \frac{q}{2(4\pi\varepsilon_0)2zR}\int_{[z-R]^2}^{[z+R]^2} \frac{du}{u^{1/2}} = \frac{q}{2(4\pi\varepsilon_0)2zR}\left[2u^{1/2}\right]_{[z-R]^2}^{[z+R]^2}$$

The square root represents a distance requiring that the positive root be chosen leading to distinct algebraic forms for $z > R$ and for $z < R$, that is: for charges outside the sphere and for charges inside the sphere.

$$V_{ave} = \frac{q}{(4\pi\varepsilon_0)2zR}\left[\,|z+R|-|z-R|\,\right] = \begin{cases} \dfrac{q}{(4\pi\varepsilon_0)\,2zR}(2R) = \dfrac{q}{4\pi\varepsilon_0 z} & \textit{for } z>R;\ q \textit{ outside} \\ \dfrac{q}{(4\pi\varepsilon_0)\,2zR}(2z) = \dfrac{q}{4\pi\varepsilon_0 R} & \textit{for } z<R;\ q \textit{ inside} \end{cases}$$

The 'q outside' result is of immediate interest as the potential is to satisfy the Laplace equation inside the sphere requiring that the region inside is charge-free. In this case, the average of the potential due to q over a complete set of points equidistant from the center of the sphere is $V_{\text{ave}} = (^{q}/4\pi\varepsilon_o z)$, which is exactly the potential at the center of the sphere due to the charge q. It follows from superposition that the potential at the center of the sphere due to an arbitrary charge distribution <u>outside</u> the sphere satisfies: $V_{ave\ on\ shell} = V_{center}$.



Pierre Laplace: French physicist and mathematician who put the final capstone on mathematical astronomy by summarizing and extending the work of his predecessors in his five volumes: *Mécanique Céleste (Celestial Mechanics)* (1799-1825). This work was important because it translated the geometrical study of mechanics used by Newton to one based on calculus, known as physical mechanics. He invented gravitational potential and showed that it obeyed the Laplace equation in empty space.

Eric W. Weisstein http://scienceworld.wolfram.com/biography/Laplace.html, a Wolfram site

Linearity and Uniqueness:

The solution to a problem in *classical* physics should exist and should be completely determined by an adequate specification of initial or boundary conditions. Establishing conditions under which this assertion can be proven for the Laplace equation is simplified because the Laplacian is a linear differential operator.

An operator $\mathcal{L}$ that operates on a space of vectors is linear if, $\mathcal{L}\,[\,a|v_1\rangle + b|v_2\rangle\,] = a\,\mathcal{L}\,[\,|v_1\rangle\,] + b\,\mathcal{L}\,[\,|v_2\rangle\,]$ where a and b are scalars and $|v_1\rangle$ and $|v_2\rangle$ are vectors .

Our two vectors are the two differentiable functions $V_1(\vec{r})$ and $V_2(\vec{r})$.

$$\nabla^2\left[a V_1(\vec{r}) + b V_2(\vec{r})\right] = a\nabla^2 V_1(\vec{r}) + b\nabla^2 V_2(\vec{r}) \qquad \text{[SL.4]}$$

Clearly any linear combination of solutions to the Laplace equation $\nabla^2 V(\vec{r}) = 0$ is also a solution to the Laplace equation. The discussion can be extended to cases with source charges and Poisson's equation $\nabla^2 V(\vec{r}) = -\rho/\varepsilon_0$ in which case the ***difference*** between any two solutions to Poisson's equation *for the same charge distribution* is a region is a solution to the Laplace equation in that same region.

The celebrated principle of superposition (for linear operations) states that $V_1(\vec{r})$ is a solution to Poisson's equation with a source charge density $\rho_1(\vec{r})$ and $V_2(\vec{r})$ is a solution to Poisson's equation with a source charge density $\rho_2(\vec{r})$ then $a V_1(\vec{r}) + b V_2(\vec{r})$ is a ***particular*** solution to Poisson's equation with a source charge density: $a\,\rho_1(\vec{r}) + b\,\rho_2(\vec{r})$. The boundary conditions are to be met by adding a solution to the Laplace equation. This approach can be based on the known solution to Poisson's equation.

$$\vec{\nabla}^2 V(\vec{r}) = -s(\vec{r}) \Rightarrow V(\vec{r}) = \int_{all\ \vec{r}'} \frac{s(\vec{r}')\,d^3r'}{4\pi\left|\vec{r}-\vec{r}'\right|} \qquad \text{[SL.5]}$$

Exercise: Give the expression for the Laplacian of the electrostatic potential. Replace the source term $s(\vec{r})$ by its electrostatic value in equation [SL.5].

Simple uniqueness given boundary values: If the value of the solution to the Laplace equation is given at every point on a closed surface bounding the region of interest then the solution in the region is unique. A proof by contradiction (*reductio ad absurdum*) follows. Assume that there are two distinct solutions $V_1(\vec{r})$ and $V_2(\vec{r})$ that satisfy the equation and the boundary conditions. Define the function:

$U(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r})$. Clearly, $U(\vec{r})$ satisfies the Laplace equation and is equal to zero everywhere of the closed surface that bounds the region of interest. Then $U(\vec{r})$ must vanish everywhere as a solution to the Laplace equation has all its extrema on the boundaries. That is: $U(\vec{r})$ is both less than or equal to 0 and greater than or equal to zero everywhere in the enclosed region. The conclusion is that $V_2(\vec{r}) - V_1(\vec{r}) = 0$ throughout the enclosed region, and hence there is no second distinct solution.

The region of interest may be multiply connected with a bounding surface consisting of two or more parts. Examples are the region between to concentric spheres or the region inside a Hershey's with almonds but outside the almonds. The bounding surface is the union of all the bounding surface segments.

More general conditions for uniqueness follow by assuming that there are two solutions $V_1(\vec{r})$ and $V_2(\vec{r})$ that satisfy the Laplace equation and the boundary conditions. Defining the function: $U(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r})$. A product rule states that:

$$\vec{\nabla} \cdot \left[U(\vec{r}) \vec{\nabla} U(\vec{r}) \right] = \vec{\nabla} U(\vec{r}) \cdot \vec{\nabla} U(\vec{r}) + U(\vec{r}) \nabla^2 U(\vec{r})$$

Using Gauss's theorem relating the surface integral of the normal component to the volume integral of the divergence of the vector field,

$$\oint_{\partial V} U(\vec{r}) \vec{\nabla} U(\vec{r}) \cdot \hat{n}\, dA = \int_V \vec{\nabla} \cdot \left[U(\vec{r}) \vec{\nabla} U(\vec{r}) \right] dV = \int_V \vec{\nabla} U(\vec{r}) \cdot \vec{\nabla} U(\vec{r})\, dV = \int_V \left| \vec{\nabla} U(\vec{r}) \right|^2 dV \geq 0$$

where the last term vanishes as U satisfies the Laplace equation. It follows that:

$$\oint_{\partial V} U(\vec{r}) \vec{\nabla} U(\vec{r}) \cdot \hat{n}\, dA = \int_V \left| \vec{\nabla} U(\vec{r}) \right|^2 dV \geq 0$$

The right hand side is positive definite, and it can vanish only if the gradient is everywhere zero. That is: $\oint_{\partial V} U(\vec{r}) \vec{\nabla} U(\vec{r}) \cdot \hat{n}\, dA = 0$ requires that $U(\vec{r})$ is constant.

Boundary Condition Cases:

Dirichlet: The solutions $V_1(\vec{r})$ and $V_2(\vec{r})$ have the same value at each point on the boundary. Conclusion: $V_1(\vec{r}) = V_2(\vec{r})$; the solution is unique.

Neumann: The solutions $V_1(\vec{r})$ and $V_2(\vec{r})$ have the same normal derivative $\frac{dV}{dn} = \vec{\nabla} V(\vec{r}) \cdot \hat{n}$ at each point on the boundary. Conclusion: $V_1(\vec{r}) = V_2(\vec{r}) + c$; the solution is unique *up to an additive constant* as $U(\vec{r})$ must be constant.

Mixed: The solutions $V_1(\vec{r})$ and $V_2(\vec{r})$ have the same value at some points on the boundary and have the same normal derivative at all other points on the boundary. Conclusion: $V_1(\vec{r}) = V_2(\vec{r}) + c$; the solution is at least unique *up to an additive constant*. The solution is ***unique*** if the value of the solution is specified at any point on the boundary.

For any of the conditions listed, $\oint_{\partial V} U(\vec{r}) \vec{\nabla} U(\vec{r}) \cdot \hat{n}\, dA = 0$ which requires that $U(\vec{r})$ is constant. Here, ∂V represents the closed surface that bounds the volume V.

Conductors with known net charge: Physical insight provides an additional situation in which uniqueness can be established. Suppose that mixed conditions hold on all segments of the boundary except for those bounding embedded conductors with known net charge. The contributions to $\oint_{\partial V} U(\vec{r}) \vec{\nabla} U(\vec{r}) \cdot \hat{n}\, dA$ vanish for surface elements on which the mixed conditions are met. Equilibrium requires that the potential is constant ($\oint_{S_i} V_1(\vec{r}) \vec{\nabla} V_1(\vec{r}) \cdot \hat{n}\, dA = V_{1S_i} \oint_{S_i} \vec{\nabla} V_1(\vec{r}) \cdot \hat{n}\, dA$) on the surface of each conductor, and Gauss's Law specifies $\oint_{S_i} \vec{\nabla} V_1(\vec{r}) \cdot \hat{n}\, dA = \oint_{S_i} \vec{\nabla} V_2(\vec{r}) \cdot \hat{n}\, dA = -q_i/\varepsilon_0$ for each conductor surface segment S_i ensuring that:

$$\oint_{S_i} U(\vec{r})\vec{\nabla}U(\vec{r})\cdot\hat{n}\,dA = U_{S_i}\oint_{S_i}\vec{\nabla}U(\vec{r})\cdot\hat{n}\,dA = \left(V_{1S_i}-V_{2S_i}\right)\left(\oint_{S_i}\vec{\nabla}V_1(\vec{r})\cdot\hat{n}\,dA - \oint_{S_i}\vec{\nabla}V_2(\vec{r})\cdot\hat{n}\,dA\right)$$

$$= \left(V_{1S_i}-V_{2S_i}\right)\left(-\oint_{S_i}\vec{E}_1\cdot\hat{n}\,dA + \oint_{S_i}\vec{E}_2\cdot\hat{n}\,dA\right) = \left(V_{1S_i}-V_{2S_i}\right)\left(\left[-q_i/\varepsilon_0\right]+\left[q_i/\varepsilon_0\right]\right) = 0$$

Recalling that $\oint_{\partial V} U(\vec{r})\vec{\nabla}U(\vec{r})\cdot\hat{n}\,dA = \int_V \left|\vec{\nabla}U(\vec{r})\right|^2 dV \geq 0$ and that we have shown that $\oint_{\partial V} U(\vec{r})\vec{\nabla}U(\vec{r})\cdot\hat{n}\,dA = 0$, we conclude that the difference between two solutions for the potential due to an array of conductors each with a specified net charge is at most a constant.

Uniqueness Summary: Thus $\oint_{\partial V} U(\vec{r})\vec{\nabla}U(\vec{r})\cdot\hat{n}\,dA$ vanishes on the boundary segments where either the potential or its normal derivative is specified, and $\oint_{\partial V} U(\vec{r})\vec{\nabla}U(\vec{r})\cdot\hat{n}\,dA$ vanishes on the boundaries of embedded conductors with specified total charge under electrostatic conditions. Again, any two solutions $V_1(\vec{r})$ and $V_2(\vec{r})$ can differ by at most a constant. We say, "the solution is unique *up to a possible additive constant*".

Property three: A solution to the Laplace equation is unique (*up to a possible additive constant*) is any one of the condition sets discussed above is met.

Conditions have been established under which the solutions to the Laplace equation are unique at least up to an additive constant. For many physical applications such as the problems in electrostatics, the additive constant does not change the measured fields and hence is of no consequence. That is: the physics of the problem is unique if any one of the four condition criteria is met.

Partial Differential Equations: Plan of attack

We solve partial differential equations by separating them into collections of ordinary differential equations (ODEs). Then the established methods for attacking ordinary differential equations are unleashed on the ODEs. As an example, the Laplace equation is to be studied in Cartesian, cylindrical and spherical coordinates.

$$\nabla^2 G(\vec{r}) = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} = 0 \qquad \text{Laplace equation} \qquad [SL.6]$$

Basically, it is the sum of the second derivatives with respect to distance of the function in *three orthogonal directions*. This form is open to separation in any coordinate system in which the three coordinate directions at any point in space are mutually orthogonal (perpendicular). The requirement is that locally (at each point) the coordinate directions are orthogonal. Cartesian coordinates is the only system with coordinate directions that are globally orthogonal.

The Helmholtz equation arises when time dependence is added to the mix. When it arises, work through the separation steps carefully and track the changes.

$$\nabla^2 G(\vec{r}) = \frac{\partial^2 G(\vec{r})}{\partial x^2} + \frac{\partial^2 G(\vec{r})}{\partial y^2} + \frac{\partial^2 G(\vec{r})}{\partial z^2} = -k^2\ G(\vec{r}) \qquad \text{Helmholtz Equation} \qquad [SL.7]$$

Cartesian Coordinates: Step-by-step solution of the Laplace equation

Step One: Assume that the solution can be written as a product of three functions each of which depends on only one of the three coordinates.

$$G(x, y, z) = X(x)\ Y(y)\ Z(z)$$

Step Two: Allow the differential operator to act on the assumed form.

$$\nabla^2 \left[X(x)\ Y(y)\ Z(z) \right] = Y(y)\ Z(z) \frac{\partial^2 X(x)}{\partial x^2} + X(x)\ Z(z) \frac{\partial^2 Y(y)}{\partial y^2} + X(x)\ Y(y)\ \frac{\partial^2 Z(z)}{\partial z^2} = 0$$

The partial derivatives may be replaced by totals as each function depends on a single coordinate variable.

Step Three: Divide by: $X(x)\ Y(y)\ Z(z)$, the form assumed for $G(x,y,z)$.

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}+\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=0 \qquad \text{[SL.8]}$$

Step Four: Assign separation constants. Each term above is independent of the coordinates upon which the other two terms depend. These conditions require that each term equals a constant, a value independent of (x, y, z). That is:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=C_x \qquad \text{[SL.9]}$$

where C_x is a separation constant independent of x, y and z. The set becomes:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=C_x \qquad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}=C_y \qquad \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=C_z \quad \text{or}$$

$$\frac{d^2X(x)}{dx^2}=C_x\ X(x) \qquad \frac{d^2Y(y)}{dy^2}=C_y\ Y(y) \qquad \frac{d^2Z(z)}{dz^2}=C_z\ Z(z)$$

A supplemental requirement is that: $C_x + C_y + C_z = 0$ [SL.10]. The condition [SL.10] ensures that the sum of the terms in step three is zero as required by the Laplace equation. The condition requires that if one separation constant is positive then at least one is negative. A negative separation constant leads to sine and cosine solutions while a positive separation constant leads to real (growing and decaying) exponentials. ***A point of special interest is the relationship between the allowed values of the separation constants and the separation of the boundary points.*** Boundary conditions at points separated by a finite interval set a **discrete spectrum** of allowed values for the associated separation constants that are sometimes called eigenvalues. Boundaries separated by an infinite interval admit a **continuous spectrum** of separation values. The transition from discrete to continuous spectra follows smoothly as the discrete values become more closely spaced as the linear extent of the interval increases.

Exercise: There is a very special case in which $C_x = C_y = C_z = 0$. In this case show that $X(x)$ has the form $m\,x + b$ where m and b are constants. Give the general form of $V(x, y, z)$ in this special case. Give the form of the electric field described by this $V(x, y, z)$. Find the specific form for an electrostatic potential that corresponds to a uniform electric field: $\vec{E} = E_0\,\hat{j}$. Recall: $\vec{E} = -\vec{\nabla} V$

CAUTION: The zero valued separation constant solutions are often ignored. Be sure to consider them whenever you begin the study of a new problem.

Cartesian Sample: 2D Electrostatics.

A problem can be defined to be in two dimensions, or, equivalently, it can be stated that there is no z-dependence. In either case $\nabla^2 [\, G(x,y)\,] = \nabla^2 [\, X(x)\; Y(y)\,] = 0$ leads to:

$$\frac{d^2 X(x)}{dx^2} = C_x \;\; X(x) \qquad \frac{d^2 Y(y)}{dy^2} = C_y \;\; Y(y) \qquad C_x + C_y = 0$$

It is convenient to set $C_x = \pm\, k^2$ and $C_y = \mp\, k^2$ to enforce the condition $C_x + C_y = 0$.

$$\frac{d^2 X(x)}{dx^2} + \left(\pm k^2\right) \;\; X(x) = 0 \qquad \frac{d^2 Y(y)}{dy^2} + \left(\mp k^2\right) \;\; Y(y) = 0$$

The $+k^2$ form leads to sines and cosines and is chosen for the coordinate for boundary conditions that require *matching the form of a function* as that can be accomplished using a Fourier series expansion. The $-k^2$ form leads to real (growing and decaying) exponentials and is often the correct choice for a coordinate that has an infinite range plus the requirement that the solution vanish for arguments that are large positive (or negative).

Advance Notice: In the limit that the period for the Fourier series becomes infinite, the spectrum of eigen-frequencies becomes continuous and the solution series morphs into a Fourier transform.

Typical terms in a solution have the forms: $\left[A_k \sin(ky) + B_k \cos(ky)\right]\left[C_k e^{kx} + D_k e^{-kx}\right]$, $\left[A_k \sin(kx) + B_k \cos(kx)\right]\left[C_k e^{ky} + D_k e^{-ky}\right]$ and $\left[E + F\,x\right]\left[G + H\,y\right]$ in two dimensions. The spectrum of separation constants may be different depending on whether the terms have trigonometric functions for the *x* direction or for the *y* direction. The spectra are set usually by adjusting arguments of the trigonometric functions to meet boundary conditions. The spatial frequencies *k* usually are set so the spatial extent in a direction is a multiple of either one-half the period or the full period of the trigonometric functions. Alternative representations arise if complex exponentials are used to represent the sines and cosines. The dependence on the other variable is expressed using e^{kx} and e^{-kx} or, for problem that are symmetric or anti-symmetric, the corresponding combinations of the real exponentials, cosh(*kx*) and sinh(*kx*), are chosen.

General 2D Cartesian Solution:

$$\begin{aligned} G(x,y) &= [A + B\,x][C + D\,y] \\ &\quad + \sum_{m=1}^{\infty}\left[E_m \sin(k_m x) + F_m \cos(k_m x)\right]\left[G_m e^{k_m y} + H_m e^{-k_m y}\right] \\ &\quad + \sum_{n=1}^{\infty}\left[S_n e^{k_n x} + T_n e^{-k_n x}\right]\left[U_n \sin(k_n y) + V_n \cos(k_n y)\right] \end{aligned} \qquad \text{[SL.11]}$$

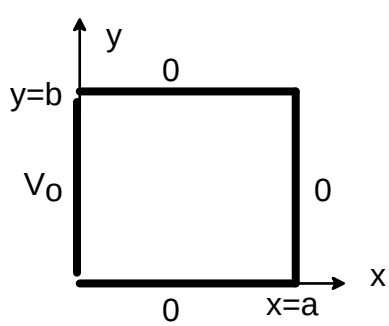
The game is to find reasons to exclude as many of the terms in the general solution as possible and then to try to fit the boundary conditions with the remaining terms. The terms $[A + B\,x][C + D\,y]$ are to be called the *zero frequency* terms.

Every term in the general solution satisfies the Laplace equations. So any linear combination of them satisfies the Laplace equation. If a combination of terms can be found the satisfies a *set of boundary conditions* sufficient to meet the requirements for *uniqueness* is meet, THE SOLUTION has been found. The solution of the Laplace

equation is unique given Dirichlet, Neumann or mixed conditions. By unique, we mean unique except for the possibility of an additive constant. The point is that no matter how ad hoc the matching process seems, it generates THE SOLUTION if the boundary conditions can be matched.

Exercise: Generate the general form of the solution to the Laplace equation in three dimensions and Cartesian coordinates. Include the cases in which one separation constant is zero.

The standard first problem is a (2D) rectangular box* with conducting sides. Three sides are held at zero potential and the fourth (insulated from the others) is held at V_o. In the context of the uniqueness criteria of Dirichlet, the potential has been specified everywhere on the boundary so the potential is uniquely determined inside that boundary. If a solution is found that satisfies the Laplace equation in the interior and the required values on the boundary, it is the unique solution for the problem. (* The corresponding 3D problem is an infinitely long conducting channel of rectangular cross section with no z dependence.)



The y dependence is distinct and runs over a finite range which suggests using Fourier methods. Choose $Y(y)= A_k \sin(ky)+B_k\cos(ky)$. A typical solution **<u>term</u>** *:

$$V_k(x,y)=\left[A_k \sin(ky)+B_k \cos(ky)\right]\left[C_k e^{kx}+D_k e^{-kx}\right]$$

*As the Laplace equation is linear, a **<u>sum of terms</u>** which are individually solutions is a solution. The next problem is to find the spectrum of k-values that is appropriate for the problem. The ***boundary values*** must be matched.

*The condition $Y(0) = Y(b) = 0$ eliminates the zero separation constant solution. The trigonometric forms are chosen for the y dependence because we wish to fit $V_o(0,y)$, a function of y, and we know that Fourier series methods are up to that task.

$y = 0$: $V=0$. $B_k = 0$. Can't have $\cos(ky)$ terms; not zero for $y = 0$.

$y = b$: $V=0$. $\sin(kb) = 0 \quad \Rightarrow k = m\,(\pi/b)$ where m is an integer index

$x = a$: $V=0$. $C_m e^{(m\pi a/b)} + D_m e^{-(m\pi a/b)} = 0 \qquad \Rightarrow +D_m = -C_m e^{(m\pi[2a]/b)}$

$$\text{use: } X_m(x) = C_m\left[e^{-(m\pi x/b)} - e^{-(2m\pi a/b)} e^{(m\pi x/b)}\right]$$

$$x = 0 : V=V_o . \quad V_0 = \sum_{m=1}^{\infty} a_m \sin(m\pi y/b) = \sum_{m=1}^{\infty} A_m C_m \left[1 - e^{-(2m\pi a/b)}\right] \sin(m\pi y/b)$$

The condition in line two above $\Rightarrow k = m\,(\pi/b)$ illustrates that a finite range for a coordinate leads to a discrete eigenvalue spectrum. The smaller the range b of the variable y is, the more widely spaced are the eigenvalues (separation constants) for the y dependence differential equation. (In quantum mechanics, a particle confined in a narrower well has more widely spaced energy eigenvalues.)

The last condition directs that the Fourier sine series for a constant between 0 and b be computed *using only sines* ($B_k = 0$) which are anti-symmetric functions. Equivalently, the Fourier series for the anti-symmetric square wave is to be computed ($V(0,y) = -V_o$ for $-b < y < 0$ and $= +V_o$ for $< y < b$) . (See the Fourier series handout for details.) The point is that the set of all sines of the arguments ($m\pi y/b$) is a complete set of orthogonal functions for the representation of *well-behaved* functions over the interval [0, b]. (One must add a constant plus all the cosines to represent *even* and *odd* over the interval [-b, b].) The orthogonality provides the means to ***project out*** the coefficient of any one of those functions from the sum by pre-multiplying by that function and integrating over the range (applying the inner product procedure).

To isolate a_n, one multiplies by $\sin({}^{n\pi y}/_{b})$; integrates with respect to y over its full range; and uses $\int_0^b \sin({}^{n\pi y}/_{b})\,\sin({}^{m\pi y}/_{b})\,dy = \left({}^{b}/_{2}\right)\delta_{mn}$. This result is a statement of orthogonality of the sines of ${}^{n\pi y}/_{b}$ and that the average value of sine squared is ½.

$$\int_0^b \sin({}^{n\pi y}/_{b})\,V_0\,dy = \int_0^b \sum_{m=1}^{\infty} a_m \sin({}^{n\pi y}/_{b})\,\sin({}^{m\pi y}/_{b})\,dy = \sum_{m=1}^{\infty} a_m \left({}^{b}/_{2}\right)\delta_{mn}$$

$$a_n = \tfrac{2}{b}\int_0^b \sin({}^{n\pi y}/_{b})\,V_0\,dy = \tfrac{2}{b}V_0\left(\tfrac{b}{n\pi}\right)\left[-\cos(u)\right]_0^{n\pi} = \begin{cases} \frac{4V_0}{n\pi} & \textit{for } n \textit{ odd} \\ 0 & \textit{for } n \textit{ even} \end{cases}$$

Matching at $x = 0$.

$$V(x,0) = V_0 = \sum_{m=0}^{\infty} \frac{4V_0}{[2m+1]\pi}\ \sin({}^{[2m+1]\pi y}/_{b}) = \sum_{k=1}^{\infty} (A_k C_k)\left[1 - e^{-(2k\pi a/b)}\right]\sin({}^{k\pi y}/_{b})$$

$$(A_k C_k)\left[1 - e^{-(2k\pi a/b)}\right] = \begin{cases} \dfrac{4V_0}{[2n+1]\pi} & \textit{for } k = 2m+1 \quad \text{odd integers} \\ 0 & \textit{for } k = 2m \quad \text{even integers} \end{cases}$$

One could replace $A_k C_k$ by a single constant such as $\mathcal{A}_k$. Plugging in the values of $A_k C_k$ and k, we find:

$$V(x,y) = \sum_{m=0}^{\infty} \frac{4V_0}{[2m+1]\pi}\ \frac{\left[e^{-(n\pi x/b)} - e^{-(2n\pi a/b)}e^{(n\pi x/b)}\right]}{\left[1 - e^{-(2n\pi a/b)}\right]}\sin({}^{[2m+1]\pi y}/_{b})$$

$$= \sum_{m=0}^{\infty} \frac{4V_0}{[2m+1]\pi}\ \frac{e^{-(n\pi a/b)}\left[e^{(n\pi a/b)}e^{-(n\pi x/b)} - e^{-(n\pi a/b)}e^{(n\pi x/b)}\right]}{e^{-(n\pi a/b)}\left[e^{(n\pi a/b)} - e^{-(n\pi a/b)}\right]}\sin({}^{[2m+1]\pi y}/_{b})$$

Note: $n = 2\,m + 1$ in the lines above. The $2\,m + 1$ represents the odd integers

$$V(x,y) = \sum_{m=0}^{\infty} \frac{4V_0}{[2m+1]\pi}\ \frac{\sinh\left[{}^{(2m+1)\pi[a-x]}/_{b}\right]}{\sinh\left[{}^{(2m+1)\pi a}/_{b}\right]}\sin({}^{[2m+1]\pi y}/_{b}) \qquad \text{[SL.12]}$$

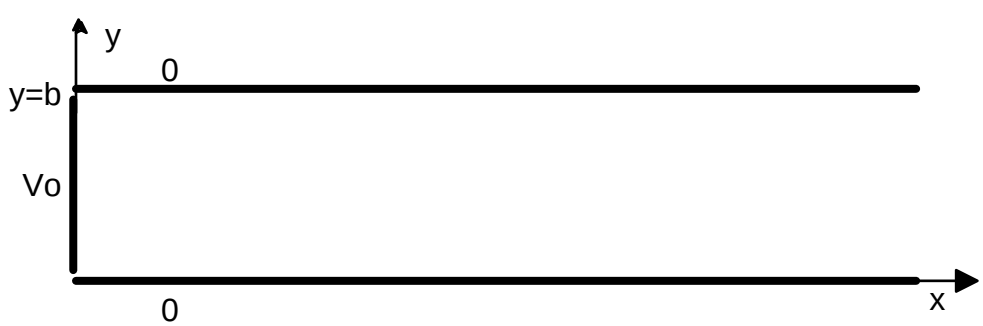
The full calculation may seem challenging, but it is mostly bookkeeping. Be disciplined. Organize your work.

Exercise: The general solution to the 2D Laplace equation has the form:

$$G(x,y)=[A+B\,x][C+D\,y]+\sum_{m=1}^{\infty}\left[E_m\sin(k_m x)+F_m\cos(k_m x)\right]\left[G_m e^{k_m y}+H_m e^{-k_m y}\right]$$
$$+\sum_{n=1}^{\infty}\left[S_n e^{k_n x}+T_n e^{-k_n x}\right]\left[U_n\sin(k_n y)+V_n\cos(k_n y)\right]$$

The solution chosen used only $\sum_{n=1}^{\infty}\left[S_n e^{k_n x}+T_n e^{-k_n x}\right]\left[U_n\sin(k_n y)+V_n\cos(k_n y)\right]$. Test the other terms to see if they could fit the boundary conditions.

The slot problem follows when is a is made infinite. Adding the physical requirement that $V = 0$ for $x \to \infty$ and $0 < y < b$ the potential is specified over the complete boundary of the slot. In the context of the uniqueness criteria of Dirichlet, the potential has been specified everywhere on the boundary so the potential is uniquely determined inside that boundary. If a solution is found that satisfies the Laplace equation in the interior and the required values on the boundary, it is the unique solution for the problem.



$V_k(x,y)=\left[A_k\sin(ky)+B_k\cos(ky)\right]\left[C_k e^{kx}+D_k e^{-kx}\right]$; ***boundary-value*** match.

$y = 0$: $V=0$. $B_k = 0$. Can't have cosine terms; not zero for $y = 0$.

$y = b$: $V=0$. $\sin(kb) = 0 \Rightarrow kb = m\pi \Rightarrow k_m = {}^{m\pi}/_{b}$ where m is an integer index

$x \to \infty$: The solution should be well-behaved (finite). $C_k = 0$.

Hence $V_m(x,y) \to A_m \sin({}^{m\pi y}/_{b})\, e^{-m\pi x/b}$

$$x = 0 : V = V_o . \quad V_0 = \sum_{m=1}^{\infty} a_m \sin(m\pi y/b) = \sum_{m=1}^{\infty} A_m \sin(m\pi y/b)$$

With somewhat less grief, it follows that, in the slot,

$$V(x,y) = \sum_{m=0}^{\infty} \frac{4V_0}{[2m+1]\pi} e^{-([2m+1]\pi x/b)} \sin([2m+1]\pi y/b) \qquad \text{[SL.13]}$$

This result can be summed to yield (see Griffiths 3rd Ed., Eqn 3.37, page 132):

$$V(x,y) = \frac{2V_0}{\pi} \tan^{-1}\left[\frac{\sin(\pi y/b)}{\sinh(\pi x/b)}\right]$$

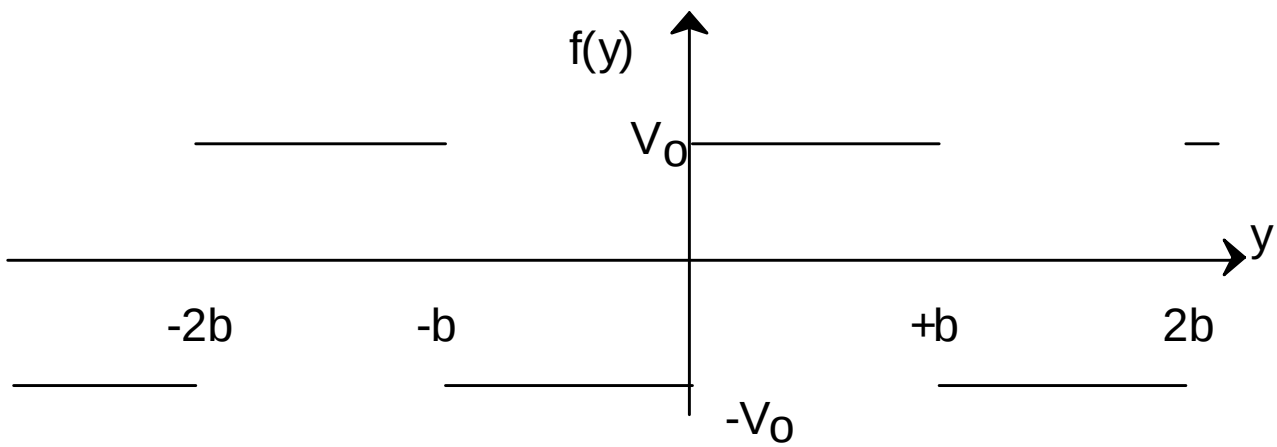
The Rest of the Story:

Fourier expansions are made in a rather cavalier fashion when one solves a Laplace boundary value problem. In this *optional* section, a couple of the details that justify the application of the Fourier methods are discussed. The fundamental frequency for the expansion as set by the boundary conditions is π/b which should be $2\pi/T$ where T is the period. The conclusion is that a function with period $2b$ is being expanded and that it is anti-symmetric as only sines appear in its expansion. The expansion coefficients could be computed using the standard Fourier series equations:

$$b_p = \frac{2}{2b}\int_{-b}^{b} \sin\left[p\,\frac{\pi}{b}\,y\right] f(y)\,dy \equiv \frac{2}{b}\int_{0}^{b} \sin\left[p\,\frac{\pi}{b}\,y\right] f(y)\,dy$$

The last step follows as the product of a sine and $f(y)$ is an even function.

Square waves come in several forms. As an example, an odd square wave with period T stepping from -1 to +1 is presented.



The square wave $f(t)$ has a period of T and is extended beyond the base period to display the discontinuities at each multiple of $^{T}/_{2}$. The function is represented as:

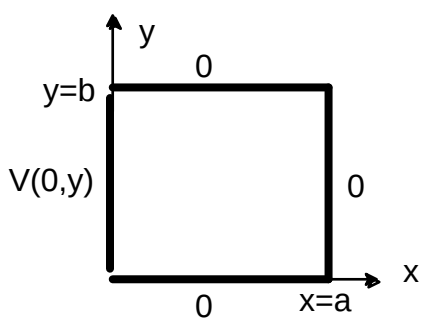
$$f(y) = \begin{cases} +V_0 & for \quad 0 < y < b \\ -V_0 & for \quad -b < y < 0 \end{cases}$$

Exercise: Consider the form of each term as m increases.

$$e^{-([2m+1]\pi x/b)} \sin([2m+1]\pi y/b)$$

Motivate the result the potential contributions associated with the faster y variations (higher m) decay more rapidly as x increases. Appeal to a property of solutions of the Laplace equation discussed several pages earlier.

More general box: A rectangular box with three conducting sides held at zero potential and the fourth insulated side held at $V(0,y)$.

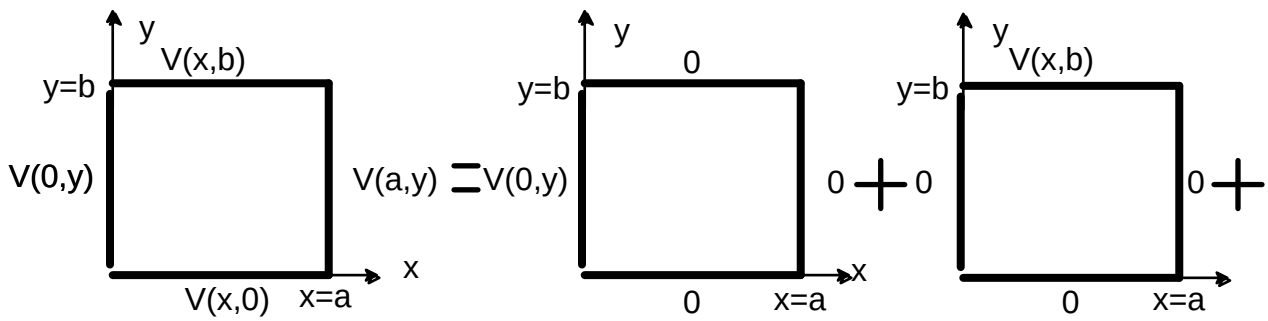


The problem develops as before, but ends with a more general Fourier series problem.

$$V(0,y) = \sum_{m=1}^{\infty} a_m \sin(m\pi y/b) = \sum_{m=1}^{\infty} A_m D_m \left[1 - e^{-(2m\pi a/b)}\right] \sin(m\pi y/b)$$

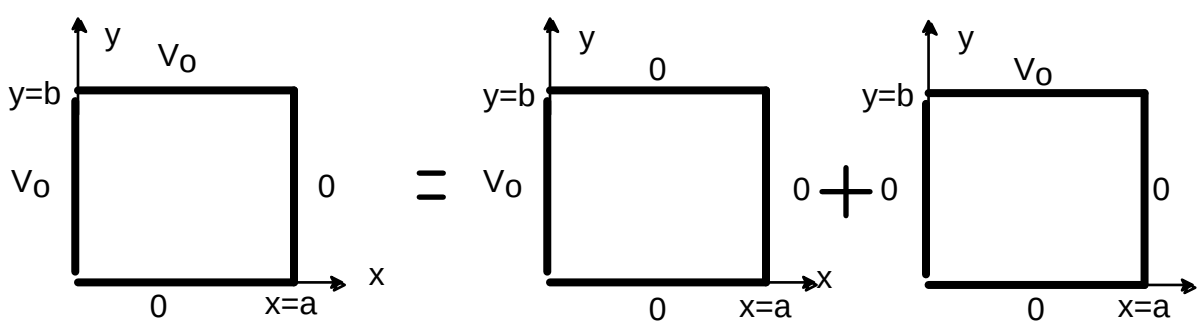
leading to $a_n = \frac{2}{b}\int_0^b \sin(n\pi y/b)\, V(0,y)\, dy$.

An even more general box might specify a different functional form on each side of the box.



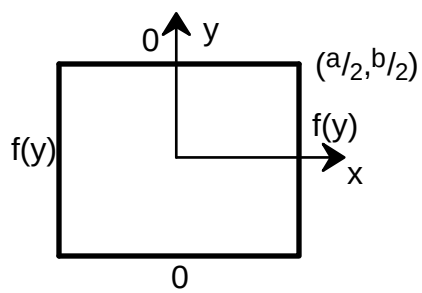
As suggested by the drawing, attack the problem by superposing the results obtained for each of the four sides held at the prescribed function and the others at zero. The sine and cosine forms are used for the coordinate/side with the functional form. This approach is critical as the exponentials lack a convenient orthogonality relation. One can project out the coefficients if the expansion functions obey an orthogonality relation.

Even a problem as *simple* as



is best approached using superposition as illustrated.

Boxes with symmetric boundary conditions:



This problem requires fitting a function of y on the boundaries with $x = \pm a/2$, so the sinusoids are to be used for the y-dependence. The problem is symmetric in x so rather than choose the bare real exponentials, their symmetric and anti-symmetric combinations, cosh and sinh are to be employed. Using the even with respect to x character, the x dependence is reduced to hyperbolic cosines only. The sinusoid selection is somewhat problematical. All sinusoids that vanish at $y = \pm b/2$ are required. They may be non-zero at $y = 0$. The required forms are:

$$\sin[\tfrac{m\pi}{b}\left(y+\tfrac{b}{2}\right)] = \sin[\tfrac{m\pi y}{b} + m\left(\tfrac{\pi}{2}\right)] = \begin{cases} \sin[\tfrac{m\pi y}{b}] & for \quad m = 4n \\ \cos[\tfrac{m\pi y}{b}] & for\ m = 4n+1 \\ -\sin[\tfrac{m\pi y}{b}] & for\ m = 4n+2 \\ -\cos[\tfrac{m\pi y}{b}] & for\ m = 4n+3 \end{cases} \qquad \text{[SL.14]}$$

Adopting the leftmost form, a sine series is the answer with the function $f(y)$ being extended to $-3b/2$ anti-symmetrically about $y = -b/2$. The allowed k_m values are all positive integer multiples of π/b.

$$V(x,y) = \sum_{m=1}^{\infty} A_m \cosh(\tfrac{m\pi x}{b}) \sin[\tfrac{m\pi}{b}\left(y+\tfrac{b}{2}\right)]$$

$$\text{where } f(y) = V(x, a/2) = \sum_{m=1}^{\infty} A_m \cosh(\tfrac{m\pi a}{2b}) \sin[\tfrac{m\pi}{b}\left(y+\tfrac{b}{2}\right)] \ .$$

The orthogonality relation for the shifted sines becomes:

$$\int_{-b/2}^{b/2} \sin[\tfrac{m\pi}{b}\left(y+\tfrac{b}{2}\right)] \sin[\tfrac{\ell\pi}{b}\left(y+\tfrac{b}{2}\right)] dy = \tfrac{b}{2}\delta_{\ell m}$$

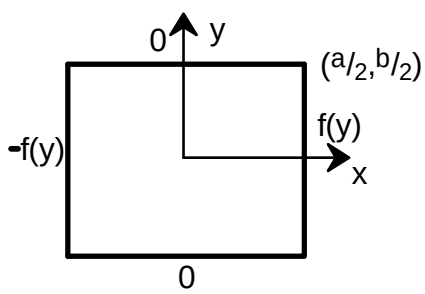
$$f(y) = \sum_{m=1}^{\infty} C_m \sin[\tfrac{m\pi}{b}\left(y+\tfrac{b}{2}\right)]$$

Assuming that the Fourier coefficients C_m for $f(y)$ has been found, $V(x,y)$ is:

$$V(x,y) = \sum_{m=1}^{\infty} C_m \frac{\cosh(\frac{m\pi x}{b})}{\cosh(\frac{m\pi a}{2b})} \sin[\tfrac{m\pi}{b}\left(y+\tfrac{b}{2}\right)].$$

Summary: This example is inelegant! A more satisfying approach is to map or shift the problem to one in $y' = y + ½\, b$ defined for $0 < y' < b$. A normal Fourier sine series would then arise naturally. At the end, replace y' with $y + ½\, b$. *Do remember* that cosh and sinh can be used rather than the bare real exponentials to simplify the solution steps if the problem has even or odd symmetry.

Boxes with anti-symmetric boundary conditions:



This problem requires fitting a function of y on the boundaries with $x = \pm a/2$, so the sinusoids are to be used for the y-dependence. The problem is anti-symmetric in x so rather than choose the bare real exponentials, their anti-symmetric combination, sinh, should be employed. Compare with the symmetric case just above.

$$V(x,y) = \sum_{m=1}^{\infty} A_m \sinh\left(\frac{m\pi x}{b}\right) \sin\left[\frac{m\pi}{b}\left(y + \frac{b}{2}\right)\right]$$

Three dimensional boxes: Separation in three dimension leads to solution terms that have either one oscillating factor and two real exponential factors or two oscillating factors and one real exponential factor. The later case is probably the first that is encountered leading to a general term of the form:

$$V_{k_x k_y}(x,y,z) = \left[A_{k_x} \sin(k_x x) + B_{k_x} \cos(k_x x)\right]\left[C_{k_y} \sin(k_y y) + D_{k_y} \cos(k_y y)\right]\left[E_{k_x k_y} e^{\kappa z} + F_{k_x k_y} e^{-\kappa z}\right]$$

The separation constant condition $C_x + C_y + C_z = 0$ leads to $k_x^2 + k_y^2 = \kappa^2$ or $\kappa = \sqrt{k_x^2 + k_y^2}$.

A box problem with x-extent a and y-extent b leads to $k_x = m\left(\frac{\pi}{a}\right)$, $k_y = n\left(\frac{\pi}{b}\right)$ and $\kappa_{mn} = \pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$. Problems of this complexity are too time and space consuming to be treated here and are reserved for homework.

Cylindrical Coordinates: Separation Steps

Step One: Assume that the solution can be written as a product of three functions each of which depends on only one of the three coordinates.

$$G(r,\phi,z) = R(r)\ \Phi(\phi)\ Z(z) \qquad \text{[SL.15]}$$

Step Two: Allow the differential operator to act on the assumed form.

$$\nabla^2\left[R(r)\ \Phi(\phi)\ Z(z)\right] = \Phi(\phi)\,Z(z)\left(\frac{1}{r}\right)\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) + R(r)\,Z(z)\left(\frac{1}{r^2}\right)\frac{\partial^2\Phi(\phi)}{\partial\phi^2} + R(r)\,\Phi(\phi)\frac{\partial^2 Z(z)}{\partial z^2} = 0$$

The partial derivatives may be replaced by total derivatives as each function depends on a single variable.

Step Three: Divide by $G(r, \theta, z) = R(r)\ \Phi(\phi)\ Z(z)$.

$$\frac{1}{R(r)}\left(\frac{1}{r}\right)\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) + \frac{1}{\Phi(\phi)}\left(\frac{1}{r^2}\right)\frac{\partial^2\Phi(\phi)}{\partial\phi^2} + \frac{1}{Z(z)}\ \frac{\partial^2 Z(z)}{\partial z^2} = 0$$

Step Four: Assign separation constants. Separation is a staged process this time. The last term above is independent of r and ϕ upon which the other two terms depend. That is: (**Straighten the derivatives for functions of one variable**)

$$\frac{1}{Z(z)}\ \frac{d^2 Z(z)}{dz^2} = C_z \quad \text{and} \quad \frac{1}{R(r)}\left(\frac{1}{r}\right)\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + \frac{1}{\Phi(\phi)}\left(\frac{1}{r^2}\right)\frac{d^2\Phi(\phi)}{d\phi^2} = -C_z$$

Multiplying by r^2

$$\frac{1}{R(r)}(r)\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + \frac{1}{\Phi(\phi)}\ \frac{d^2\Phi(\phi)}{d\phi^2} = -C_z\ r^2 \qquad \text{[SL.16]}$$

so that:

$$\frac{1}{\Phi(\phi)}\ \frac{d^2\Phi(\phi)}{d\phi^2} = +C_\phi \qquad \frac{1}{R(r)}(r)\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + C_z\ r^2 - C_\phi = 0 \qquad \text{[SL.17]}$$

The requirement of the Laplace equation on the sum of the separation constants has been incorporated, and C_r is expressed in terms of C_ϕ and C_z in [SL.17]. The boundary conditions at points separated by a finite increment set a discrete spectrum of allowed values for the associated separation constants that are sometimes called eigenvalues.

For the $\Phi(\phi)$, the most common range for ϕ is from 0 to 2 π and the solution satisfies the boundary condition $\Phi(2\pi) = \Phi(0)$ as the two values of ϕ correspond to the same point in space. This periodicity requirement leads to $C_\phi = -m^2$ where m is an integer. The solutions are:

$$\Phi_m(\phi) = a_m \sin(m\phi) + b_m \cos(m\phi) \equiv c_m e^{im\phi} + d_m e^{-im\phi}$$

Note that it is the finite range of the ϕ variable leads to a discrete rather than a continuous spectrum for the ϕ separation eigenvalues.

A special class of problems identifies the z separation constant as n^2 leading to:

$$(r)\frac{d}{dr}\left(r\frac{dR_{nm}(r)}{dr}\right) + \left(n^2 r^2 - m^2\right)R_{nm}(r) = 0 \quad \text{and} \quad \frac{d^2 Z_n(z)}{dz^2} = n^2 Z_n(z)$$

The radial equation is Bessel's equation, and the $Z(z)$ equation has real exponentials as solutions.

A general solution is of the form:

$$G(\vec{r}) = \sum_{n,m} \alpha_{nm} R_{nm}(r) \Phi_m(\phi) Z_n(z)$$

$$G(r,\phi,z) \approx \sum_m \sum_n \left[A_{mn} J_n(k_{nm}r) + B_{mn} N_n(k_{nm}r)\right] e^{\pm im\phi} e^{\pm k_m z} \qquad \text{[SL.18]}$$

where J_n and N_n are Bessel functions of the first and second kind.
The determination of the allowed values k_{nm} is more complicated that was the case for trig functions. Refer to the Tools of the trade discussion around equation [SL.35]. See Wikipedia for a discussion of the orthogonality relations

$$\int_0^1 xJ_\alpha(xu_{\alpha,m})J_\alpha(xu_{\alpha,n})dx = \frac{\delta_{m,n}}{2}[J_{\alpha+1}(u_{\alpha,m})]^2 = \frac{\delta_{m,n}}{2}[J_{\alpha'}(u_{\alpha,m})]^2$$

and other properties.

The techniques required for matching boundary conditions to evaluate the constants such as A_{mn} are somewhat different than for the Cartesian and spherical problems. Review the procedures in detail in

the Special Functions Applications handout or a reference such as Marion's **Classical Electromagnetic Radiation** when you first encounter a cylindrical Laplace or Helmholtz problem.

Exercise: Consider the Laplace equation in cylindrical coordinates for a case with no ϕ or z dependence. Find $R(r)$ for this restricted case. There should be two independent solutions. To find them, integrate $\left(\frac{1}{r}\right)\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)=0$ directly. Only one of the solutions follows easily from the standard trial solution: $R(r) = A\, r^n$. See the Differential Equations handout for details.

Polar Coordinates: Reducing the problem to 2D by specifying *no z-dependence* leads to polar coordinate form of the Laplace equation.

$$\frac{1}{R(r)}(r)\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right)+\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=0 \qquad \text{[SL.19]}$$

This equation separates to yield:

$$(r)\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)=+m^2R(r) \quad \text{and} \quad \frac{d^2\Phi(\phi)}{d\phi^2}=-m^2\,\Phi(\phi)$$

The ϕ equation is trivial leading to the requirement that m be an integer to insure that $\Phi(0) = \Phi(2\,\pi)$. The radial equation is solved by assuming the trial solution $A\, r^n$ with the result that $n = \pm\, m$ except for the case that $m^2 = 0$ for which a direct integration of the radial equation gives $A + B\,\ln(r)$. The ϕ equation is trivial leading to the form of the general solution to the Laplace equation in 2D polar coordinates.

A general solution that is periodic in ϕ has the form:

$$V(r,\phi)=A_0+B_0\,\ln(r)+\sum_{m=1}^{\infty}\left[A_m r^m+B_m\,r^{-m}\right]\left[C_m\sin(m\phi)+D_m\cos(m\phi)\right] \qquad \text{[SL.20]}$$

or equivalently:

$$V(r,\phi)=A_0+B_0\,\ln(r)+\sum_{m=1}^{\infty}\left[A_m r^m+B_m\,r^{-m}\right]\left[E_m\,e^{im\phi}+F_m\,e^{-im\phi}\right] \qquad \text{[SL.21]}$$

REVIEW THIS SECTION

If the range of ϕ is restricted to a wedge [ϕ_1 to ϕ_2] so that there is no physical requirement that $\Phi(\phi_1) = \Phi(\phi_2)$. In this case, terms like $\left[A_0 + B_0 \ln(r)\right]\left[C_0 + D_0 \phi\right]$ and $\sum_{m=1}^{\infty}\left[A_m r^m + B_m r^{-m}\right]\left[C_m \sin(m\phi) + D_m \cos(m\phi)\right]$ are allowed with m chosen to be an integer times $2\pi/(\phi_2-\phi_1)$.

Note that for each distinct value of the separation constant, there are two independent solutions to the second order differential equation for each coordinate. Physical considerations may exclude one of these solutions as divergent or multi-valued functions are unlikely to represent an allowed behavior. The $m^2 = 0$ ***solution*** $\Phi(\phi)=\phi$ ***is omitted above in the case of that the angle runs through its full range 0 to*** 2π.

Exercise: Consider the Laplace equation in cylindrical coordinates and examine the ϕ equation for separation constant $m^2 = 0$. There should be two independent solutions. For non-zero m^2, the periodicity condition $\Phi(0) = \Phi(2\pi)$ requires that m is an integer leading to the sines and cosines.

The Laplace equation inside a circle of radius R centered on the origin.

A standard requirement is that a physical solution be well-behaved (has defined values) in the region of interest. Inside the circle includes $r = 0$. Note that the function Φ is not well-behaved (is multi-valued) at $r = 0$ and must therefore only appear multiplied by a factor that vanishes at $r = 0$. Excluding radial functions that diverge at $r = 0$, the form of an acceptable solution is reduced to:

$$V(r,\phi) = A_0 + \sum_{m=1}^{\infty} r^m \left[A_m \cos(m\phi) + B_m \sin(m\phi)\right]$$

The boundary of the region is the circle $r = R$. Assume that the values on the boundary is specified by an arbitrary function $f(\phi)$. Following standard Fourier techniques, the expression for the values on the boundary is multiplied in turn by each

of the Fourier basis functions, and the corresponding coefficient is *projected out* for evaluation.

$$V(R,\phi) = f(\phi) = A_0 + \sum_{m=1}^{\infty} R^m \left[A_m \cos(m\phi) + B_m \sin(m\phi)\right]$$

$$A_0 = \tfrac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi = \tfrac{1}{2\pi R}\int_0^{2\pi} V(R,\phi)\ R\ d\phi; \text{ the average value on the circle}$$

$$A_m = \tfrac{R^{-m}}{\pi}\int_0^{2\pi} \cos(m\phi)\ f(\phi)\,d\phi \qquad B_m = \tfrac{R^{-m}}{\pi}\int_0^{2\pi} \sin(m\phi)\ f(\phi)\,d\phi$$

The average value principle in 2D: At the center of the circular region,

$$V(r,\phi) = A_0 + \sum_{m=1}^{\infty} r^m \left[A_m \cos(m\phi) + B_m \sin(m\phi)\right] \rightarrow$$

$$V(0,\phi) = A_0 = \tfrac{1}{2\pi R}\int_0^{2\pi} V(R,\phi)\ R\ d\phi$$

The value at the center is the average of the values on the circle of radius R concentric with that point (the average over a complete set of symmetrically located equally distant points).

Spherical Coordinates

Step One: Assume that the solution can be written as a product of three functions each of which depends on only one of the three coordinates.

$$G(r,\theta,\phi) = R(r)\ \Theta(\theta)\ \Phi(\phi) \qquad \text{[SL.22]}$$

Step Two: Allow the Laplacian operator to act on the assumed form.

$$\nabla^2\left[R(r)\ \Theta(\theta)\ \Phi(\phi)\right] = \Theta(\theta)\ \Phi(\phi)\left(\frac{1}{r^2}\right)\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right) + R(r)\ \Phi(\phi)\left(\frac{1}{r^2\sin\theta}\right)\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)$$
$$+ R(r)\ \Theta(\theta)\left(\frac{1}{r^2\sin^2\theta}\right)\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = 0$$

Step Three: Divide by $R(r)\ \Theta(\theta)\ \Phi(\phi)$, the form assumed for G.

$$\frac{1}{R(r)}\left(\frac{1}{r^2}\right)\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{1}{\Theta(\theta)}\left(\frac{1}{r^2\sin\theta}\right)\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)+\frac{1}{\Phi(\phi)}\left(\frac{1}{r^2\sin^2\theta}\right)\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=0$$

Step Four: Assign separation constants. Separation is a staged process this time.

$$\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{1}{\Theta(\theta)}\left(\frac{1}{\sin\theta}\right)\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)+\frac{1}{\Phi(\phi)}\left(\frac{1}{\sin^2\theta}\right)\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=0$$

Multiply by r^2 to separate r from θ and ϕ.

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)=C_{\ell m}\qquad\frac{1}{\Theta(\theta)}\left(\frac{1}{\sin\theta}\right)\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)+\frac{1}{\Phi(\phi)}\left(\frac{1}{\sin^2\theta}\right)\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=-C_{\ell m}$$

Multiplying by $\sin^2\theta$ to separate θ and ϕ (*Straighten the derivatives when equations are in one variable.*)

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}=-m^2;\qquad\frac{1}{\sin\theta\ \Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)-\frac{m^2}{\sin^2\theta}=-C_{\ell m}\qquad\text{[SL.23]}$$

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)-C_{\ell m}R(r)=r^2\frac{d^2R(r)}{dr^2}+2r\frac{dR(r)}{dr}-C_{\ell m}R(r)=0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+\left[C_{\ell m}-\frac{m^2}{\sin^2\theta}\right]\Theta(\theta)=0\qquad\text{[SL.24]}$$

$$\frac{d^2\Phi(\phi)}{d\phi^2}+m^2\,\Phi(\phi)=0$$

The boundary conditions at points separated by a finite increment set a discrete spectrum of allowed values for the associated separation constants that are sometimes called eigenvalues. For the ϕ equation, a standard situation is that ϕ runs from 0 to 2 π and that Φ(2 π) = Φ(0) as the two values of correspond to the same point in space. This periodicity requirement leads to separation constants - m^2 where m is an integer. The solutions are:

$$\Phi_m(\phi)=a_m\,\sin(m\phi)+b_m\,\cos(m\phi)\equiv c_m\,e^{im\phi}+d_m\,e^{-im\phi}$$

Note that: the finite range of the ϕ variable leads to a discrete rather than a continuous spectrum for the ϕ separation eigenvalues. More over as the range of θ is also finite,

the overall separation constant $C_{\ell m}$ also has a discrete spectrum of allowed values: $\ell(\ \ell+1)$ where ℓ is a non-negative integer. (The values of are discussed in the orthogonal polynomials handout.)

The θ equation is that for the associated Legendre polynomials, the $P_\ell^m(\cos\theta)$. The normalized combinations of the θ and ϕ solutions form the spherical harmonics (the YLMs). *There is a second solution to the* θ *equation that exhibits singular behavior* (*blows up*; *infinite*) *and thereby fails the test to have the character of physically allowed solutions. We will pretend that we never heard of it or thought about it.*

$$Y_{\ell m}(\theta,\phi) = (-1)^m \sqrt{\frac{(2\ell+1)\ \ (\ell-m)!}{4\pi\ \ \ (\ell+m)!}}\ P_\ell^m(\cos\theta)\ e^{im\phi} \qquad \text{[SL.25]}$$

The world famous YLMs are the eigenfunctions for orbital angular momentum and appear as the angular dependence of the hydrogen atom eigenfunctions (in the standard physics representation).

> The spherical harmonics or *YLMs* play a special role in physics because they are appropriate for all spherically symmetric problems. The separation-constant spectrum is determined by the finite ranges of the angular coordinates. That is: Spherically symmetric means that the angles range freely through their full domains each of which is finite leading to a universal set of discrete spectrum of separation constants for the angular part of all spherically symmetric problems.

The general form of a solution to the Laplace equation in spherical coordinates is:

$$G(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\ \sum_{m=-\ell}^{\ell}\ \left[a_{\ell m}\ r^{\ell} + b_{\ell m}\ r^{-(\ell+1)}\right] Y_{\ell m}(\theta,\phi) \qquad \text{[SL.26]}$$

If the is no ϕ dependence, then $m^2 = 0$. This yields:

$$\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right) = \alpha_\ell = \ell(\ell+1) \quad \text{and} \quad \frac{1}{\sin\theta\ \ \Theta(\theta)}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right) = -\alpha_\ell = -\ell(\ell+1)$$

The θ equation has as its solutions, the Legendre polynomials of cosθ, $P_\ell(\cos\theta)$, and to ensure that the solutions are defined (don't blow up) sets the eigenvalue spectrum as $\alpha_\ell = \ell(\ell + 1)$ where ℓ can assume non-negative integer values.

In the absence of ϕ dependence and restricting to solutions that are finite for θ = 0 and π, the form of a solution to the Laplace equation in spherical coordinates is:

$$G(r,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell\ r^{\ell} + b_\ell\ r^{-(\ell+1)}\right] P_\ell(\cos\theta) \qquad \text{[SL.27]}$$

***Warning**:* If cones of space around θ = 0 and π are removed from the region of interest, the $Q(\theta)$, the irregular solutions to the Legendre equations, must be included in the general forms of the solutions. These cases are not to be treated here.

Note that for each distinct value of the separation constant, there are two independent solutions to the second order differential equation for each coordinate. Physical considerations may exclude one of these solutions as divergent or multi-valued functions are unlikely to represent allowed behavior.

Exercise: Consider the Laplace equation in spherical coordinates in a case with no θ or ϕ dependence. Find $R(r)$ for this restricted case. There should be two independent solutions. Use the standard trial solution: $A\ r^n$. .

A Few Properties of the Legendre Polynomials

Orthogonality:

$$\int_{-1}^{+1} P_m(x)\ P_\ell(x)\ dx = \left(\frac{2}{2\ell+1}\right)\delta_{\ell m} \qquad \text{[SL.28]}$$

$$\int_0^{\pi} P_m(\cos\theta)\ P_\ell(\cos\theta)\ \sin\theta\ d\theta = \left(\frac{2}{2\ell+1}\right)\delta_{\ell m}$$

Normalization: $P_\ell(1) = 1\ \forall$ integers $\ell \ \Rightarrow\ P_\ell(\cos[0]) = 1\ \forall$ integers ℓ

ℓ	$P_\ell(x)$	$P_\ell(\cos\theta)$
0	1	1
1	x	$\cos\theta$
2	$\frac{3}{2}x^2 - \frac{1}{2}$	$\frac{3}{2}\left[\cos\theta\right]^2 - \frac{1}{2}$

$$P_3(x) = {}^5/_2\ x^3 - {}^3/_2\ x\ ; \qquad P_4(x) = {}^{35}/_8\ x^4 - {}^{15}/_4\ x^2 + {}^3/_8$$

Note that the nth Legendre polynomial has symmetry $(-1)^n$ and has x^n as its highest power. Each polynomial is a sum of either odd or of even powers.

Allowed Characteristic Behaviors:

$$G(r,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell\ r^\ell + b_\ell\ r^{-(\ell+1)}\right] P_\ell(\cos\theta)$$

Each term in the expansion of $G(r, \theta)$ represents an *independent characteristic behavior* solutions to the Laplace equation in spherical coordinates of azimuthally symmetric (no ϕ dependence) solutions with θ ranging from 0 to π. The positive power terms $a_\ell\ r^\ell\ P_\ell(\cos\theta)$ are examined first.

$\boldsymbol{a_o\ r^0\ P_o(\cos\theta)} = a_o$, a constant: One characteristic behavior of the potential is to be a constant in a region of space such as inside a conductor. (Recall that the electrostatic potential satisfies the Laplace in *charge-free* space.)

$\boldsymbol{a_1\ r^1\ P_1(\cos\theta)} = a_1\ r\ \cos\theta = a_1\ z$, a linearly varying potential: Equivalently, the potential describes a uniform electric field in the z direction ($\vec{E} = -a_1\,\hat{k}$). A region of uniform electric field is also an allowed characteristic behavior.

Attacking the inverse power $b_\ell\ r^{-(\ell+1)}\ P_\ell(\cos\theta)$ terms, $\boldsymbol{b_o\ r^{-1}\ P_o(\cos\theta)} = b_o\ r^{-1}$. This ℓ = 0 potential term is just that of a ***monopole***, a point charge or of any region exterior to a spherically symmetric charge distribution in otherwise empty space. (Again, it must be exterior to the charge distribution because the potential only satisfies the Laplace equation in charge-free regions.)

$\boldsymbol{b_1 r^{-2} P_1(\cos\theta)} = b_1 r^{-2} \cos\theta$: The $\cos\theta$ factor is appears in vector dot products. This $\ell = 1$ potential term is just that of a ***dipole*** oriented along the z - direction.

$$\frac{p_0 \hat{k}\cdot\vec{r}}{4\pi\,\varepsilon_0\, r^3} = \frac{p_0 \cos\theta}{4\pi\,\varepsilon_0\, r^2} = \frac{p_0}{4\pi\,\varepsilon_0} r^{-2}\, P_1(\cos\theta) = b_1\, r^{-2}\, P_1(\cos\theta)$$

$\boldsymbol{b_2 r^{-3} P_2(\cos\theta)} = b_2 r^{-3} [\, {}^3/_2 \cos^2\theta - {}^1/_2] = ½\, b_2 r^{-5} [\, 3 z^2 - r^2 \,]$: This $\ell = 2$ potential term is just that of a ***zz Quadrupole*** ($\Rightarrow Q_{xx} = Q_{yy} = -\,½\, Q_{zz}$). The potential due to equal and opposite dipoles that lie on the z-axis and that are displaced from one another along the z-axis.

The multipole expansion was a large r expansion valid in the *charge-free* region outside the charge distribution, the same region in which the $\sum_{\ell=0}^{\infty} b_\ell\, r^{-(\ell+1)}\, P_\ell(\cos\theta)$ form is the allowed representation for a potential that vanishes at infinity. The ℓth term in the expansion represents the 2^ℓ- multipole with azimuthal symmetry (no ϕ dependence). The z – dipole and the zz – quadrupole are azimuthally symmetric (unchanged by rotations about the polar axis; independent of ϕ).

Sample Calculations for Problems with Azimuthal Symmetry:

The trial solution: $G(r,\theta) = \sum_{\ell=0}^{\infty} \left[a_\ell\, r^\ell + b_\ell\, r^{-(\ell+1)} \right] P_\ell(\cos\theta)$

The first two examples match boundary conditions along the line θ = *constant*. In the case of three-dimensional problems, matching on a boundary surface is the norm. With azimuthal symmetry, the problems have two free variables, and the boundaries become lines in r - θ space. These θ = constant examples are usually presented second. They are treated first only to ensure that they are not lost in the almost exhaustive list of examples with a single spherical shell boundary or r = *constant* examples.

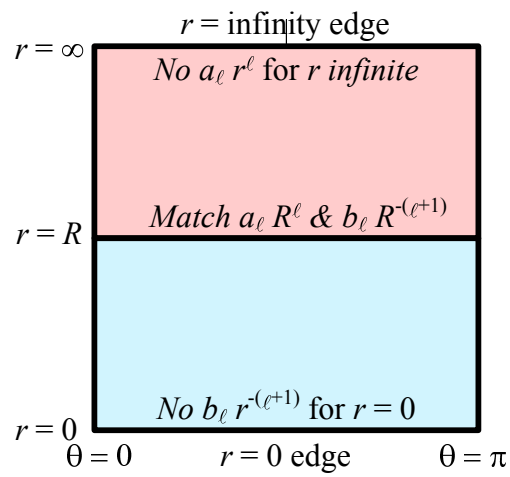
Coefficient Matching:

(1) On an $r = R$ boundary, the θ dependence is in play, and the procedure is to note that the Legendre polynomials $P_\ell(\cos\theta)$ are mutually orthogonal ***so the coefficient of each Legendre polynomial must match across the boundary***.

(2) Along a θ = *constant* boundary, the procedure is to find cause to restrict the solution either to only *non-negative* or to only *negative* powers of r. The various powers of r are linearly independent (*not as good as orthogonal, but still good enough*) so the coefficients of *each power of r* must match across a constant θ boundary. In the case of a θ = 0 boundary, use $P_\ell(\cos\theta) = 1$ for all ℓ.

The argument that only positive powers of r can appear in an inside solution $V_<(r,\theta)$, is that the solution must be well-behaved (not divergent) at $r = 0$. The argument that only negative powers can appear in an outside solution $V_>(r,\theta)$ is equivalent to applying the matching condition $V = 0$ at $r = \infty$. The two pieces of the solution $G(r,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell\ r^\ell + b_\ell\ r^{-(\ell+1)}\right] P_\ell(\cos\theta)$ require matching the value at two distinct surfaces or matching the value plus the normal derivative on one surface. Again, surfaces are lines in r - θ space for problems with azimuthal symmetry.

r - θ Boundary Value Matching Map	
A boundary value matching map for problems with azimuthal symmetry ⇒ match along a line in r - θ space. Infinity is drawn at a finite distance for convenience. Matching a finite behavior at $r = 0$ means that inverse powers are excluded ⇒ all $b_\ell = 0$. Matching a vanishing behavior at $r = \infty$ excludes positive powers of r in the large r region. The point charge potential example matches along large and small r lines for	

$\theta = 0$. **Use the *well-behaved* pieces in each region.**	 $\sum_{\ell=0}^{\infty}\left[a_\ell r^\ell + b_\ell r^{-(\ell+1)}\right]P_\ell(\cos\theta)$

The point: All the terms in $G(r,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell\ r^\ell + b_\ell\ r^{-(\ell+1)}\right] P_\ell(\cos\theta)$ are allowed solutions of the Laplace equation in spherical coordinates with the angle θ running over its full range. All of the terms in this expression must be included *unless a reason can be found to exclude them*. The *inside region* includes the point $r = 0$ at which all the $r^{-(\ell+1)}$ diverge and hence are not physically acceptable. In the *outside region*, $r \to \infty$ leading the usual expectation that the potential should approach zero; the terms like r^ℓ do not meet this expectation and are excluded in the usual cases. Those terms may be present if there is an external applied field.

Comment on Boundary Value Matching: The normal requirement to ensure uniqueness is that the boundary values be matched on a surface that encloses the region of interest. For the case of azimuthal symmetry, surfaces in three dimensions are represented by lines in r - θ space. For problems in which the potential or a charge density is specified on an r = constant surface, the potential satisfies the Laplace equation at points infinitesimally distant from the surface on which the charge exists. As long as the charge density is no more singular than a surface density, continuity can be used to argue that we essential match the values of the *inside* ($r < R$) and

outside ($r > R$) solution values on the $r = R$ surface. The resulting solution does not, in general, continue analytically[1] across the surface, but it is well-behaved in the connected regions bounded by constant r and constant θ surfaces in which the Laplace equation is obeyed($\Rightarrow$in which the charge density $\rho = 0$). For our examples that region is bounded by the $r = R$ surface and the surface at infinity of the point at the origin. The point at the origin is a *degenerate surface*, the limit of a concentric spherical surface as $r \to 0$. The inverse powers cannot appear in the *small r* region as they correspond to having multipole sources at the origin, a region assumed to be charge free. Another choice for a separating surface would be θ = constant. As $\theta \to 0$, this surface devolves into a degenerate case, a line along the z – axis.

<u>Examples matching along a constant θ line in *r* - θ space</u>:

Matching on a constant θ line (or surface) is less demanding because we have already eliminating the irregular angular functions, the ones that diverge at $\theta = 0$ and at $\theta = \pi$.

No $N_\ell(\cos\theta)$'s. Not this: $\sum_{\ell=0}^{\infty}\left[a_\ell r^\ell + b_\ell r^{-(\ell+1)}\right]\left[c_\ell P_\ell(\cos\theta) + d_\ell N_\ell(\cos\theta)\right]$

Only this: $\sum_{\ell=0}^{\infty}\left[a_\ell r^\ell + b_\ell r^{-(\ell+1)}\right]P_\ell(\cos\theta)$

Sample Calculation: The potential due to a point charge on the z-axis

The potential due to a point charge q located at $d\hat{k}$ is given by:

$$V(x,y,z) = \frac{q}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z-d)^2}} \qquad \text{[SL.29]}$$

For points on the $+z$-axis, [SL.29] becomes

$$V(0,0,z) = \frac{q}{4\pi\varepsilon_0\sqrt{(z-d)^2}} \to \frac{q}{4\pi\varepsilon_0(z-d)} \quad for\ z > d$$

---

[1] The radial derivative of the potential may not be defined at $r = R$. A function is analytic at a point if it has a convergent Taylor's series representation in a neighborhood of that point. $\Rightarrow$ in an open interval (R-ε_1, R+ε_2).

The form for $z > d$ can be rewritten as:

$$V(0,0,z) \to \frac{q}{4\pi\varepsilon_0 z}\left[1 - {}^{d}\!/_{z}\right]^{-1} \quad for\ z > d$$

Pushing onward, a binomial expansion is invoked.

$$\begin{aligned}[1-x]^{-1} &= 1 + \tfrac{-1}{1!}(-x)^1 + \tfrac{(-1)(-2)}{2!}(-x)^2 + \ldots + \tfrac{(-1)(-2)\ldots(-n)}{n!}(-x)^n + \ldots \\ &= 1 + x^1 + x^2 + \ldots + x^n + \ldots\end{aligned}$$

Next, ${}^{d}\!/_{z}$ replaces x to yield a series expansion for the potential; along the z-axis.

$$V(0,0,z) = \frac{q}{4\pi\varepsilon_0 z}\left[1 - {}^{d}\!/_{z}\right]^{-1} = \frac{q}{4\pi\varepsilon_0 z}\left[1 + {}^{d}\!/_{z} + \left({}^{d}\!/_{z}\right)^2 + \ldots + \left({}^{d}\!/_{z}\right)^n + \ldots\right] for\ z > d$$

Here, the small parameter in the expansion is ${}^{d}\!/_{z}$. This expansion must agree with the general solution of the Laplace equation with azimuthal symmetry in the charge-free region $r > d$ for $\theta = 0$.

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left[b_\ell\ r^{-(\ell+1)}\right] P_\ell(\cos\theta) \to \sum_{\ell=0}^{\infty} b_\ell\ r^{-(\ell+1)} \quad for\ \theta = 0$$

Recall that all the P_ℓ (cos[0]) = $P_\ell(1) = 1$ for all ℓ.

[To match along the negative z-axis, use $P_\ell(-1) = (-1)^\ell$.]

Along the z-axis, $z = r\cos\theta = r$ so the two series can be compared as:

$$V(0,0,z=r) = \frac{q}{4\pi\varepsilon_0 r}\left[1 + {}^{d}\!/_{r} + \left({}^{d}\!/_{r}\right)^2 + \ldots + \left({}^{d}\!/_{r}\right)^n + \ldots\right] = \sum_{\ell=0}^{\infty}\frac{q}{4\pi\varepsilon_0}\left(\frac{d^n}{r^{n+1}}\right) = \sum_{\ell=0}^{\infty} b_\ell\ r^{-(\ell+1)}$$

The two series are equal if $b_\ell = \dfrac{q\,d^\ell}{4\pi\varepsilon_0}$ for all ℓ which leads to the expression for the potential valid everywhere in the region $r > d$.

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\right)\frac{d^\ell}{r^{(\ell+1)}}\ P_\ell(\cos\theta) \qquad \text{[SL.30]}$$

This result is unremarkable and clearly inferior to the exact closed form solution [SL.29] that was the basis for the original expansion. The example can be used to show that the expansion approach does work.

Numerical validation: Consider $V(x,y,z)=\dfrac{q}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z-d)^2}}$ with d = 1, r = 10, θ = 45° and ϕ = 0. $V(x,y,z)=\dfrac{q}{4\pi\varepsilon_0\sqrt{(10/\sqrt{2})^2+0^2+(10/\sqrt{2}-1)^2}}=1.072989\,\dfrac{q}{4\pi\varepsilon_0}$. Using the series expansion, $V(r,\theta)=\sum_{\ell=0}^{\infty}\left(\dfrac{q}{4\pi\varepsilon_0}\right)\dfrac{d^\ell}{r^{(\ell+1)}}\,P_\ell(\cos\theta)=\left(\dfrac{q}{4\pi\varepsilon_0}\right)\sum_{\ell=0}^{\infty}\dfrac{1^\ell}{10^{(\ell+1)}}\,P_\ell(2^{-1/2})$. Stop at 6.

Mathematica: N[Sum[LegendreP[m,1/Sqrt[2]]/10^(m+1),{m,0,6}],10] = 0.1072989371.

An important result can be developed if the final expression is made portable, put in coordinate independent form or vector notation. The source charge is at position $\vec{r}'$ and the point at which the potential is to be evaluated is $\vec{r}$. The angle θ is between the direction from the origin to the source and the direction from the origin to the field point so: $\cos\theta=\hat{r}\cdot\hat{r}'$. The portable form for $r>r'$ is:

$$V(r,\theta)=\frac{q}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|}=\sum_{\ell=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\right)\frac{(r')^\ell}{r^{(\ell+1)}}\,P_\ell(\hat{r}\cdot\hat{r}'). \qquad \text{[SL.31]}$$

This result establishes the identity used in Griffiths (Eq^n 3.94) as the basis for multipole expansions. $\dfrac{1}{|\vec{r}-\vec{r}'|}=\sum_{\ell=0}^{\infty}\dfrac{(r')^\ell}{r^{(\ell+1)}}\,P_\ell(\hat{r}\cdot\hat{r}')$ for $r'<r$ [SL.32]

Vector Space Concept: The set of functions z^{-m} or equivalently r^{-m} are linearly independent and a basis for functions that vanish as $r\to\infty$. This independence means the expansion coefficients in the representation of any (vector) function are unique so that the coefficients must match term-by-term. $b_\ell=\dfrac{q\,d^\ell}{4\pi\varepsilon_0}$ for all ℓ

Problem VS10.) A vector is expanded in terms of a basis set as $|I\rangle=a_i|e_1\rangle+b_i|e_2\rangle+c_i|e_3\rangle$ where the constants a_i, b_i, and c_i are scalar values and $\{|e_1\rangle,|e_2\rangle,|e_3\rangle\}$ is a basis set. Show that the components of the vector, the values a_i, b_i, and c_i, are unique. HINT: Review the definition of a basis set.

Sample Calculation: The potential due to a ring of charge on the z-axis

A useful example follows from the expression for the potential of a uniform ring of charge of radius a concentric with the origin in the x-y plane at points along the z-axis.

$$V(0,0,z) = \frac{Q}{4\pi\varepsilon_0\sqrt{a^2+z^2}} \rightarrow \frac{Q}{4\pi\varepsilon_0 z}\left[1+\left(a/z\right)^2\right]^{-1/2} \text{ for } z > a$$

$$V(0,0,z) = \frac{Q}{4\pi\varepsilon_0 z}\left[1+\left(a/z\right)^2\right]^{-1/2} \approx$$

$$\frac{Q}{4\pi\varepsilon_0 z}\left[1+\left(\frac{-1/2}{1!}\right)\left(a/z\right)^2+\left(\frac{(-1/2)(-3/2)}{2!}\right)\left(a/z\right)^{2*2}+\ldots+\left(\frac{(-1)(-3)\ldots(2n-1)}{2^n\,n!}\right)\left(a/z\right)^{2*n}+\ldots\right]$$

Matching along the z-axis for $z > a$:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left[b_\ell\, r^{-(\ell+1)}\right]P_\ell(\cos\theta) \rightarrow \sum_{\ell=0}^{\infty} b_\ell\, r^{-(\ell+1)} \;\; for\ \theta = 0$$

$$V(r,\theta) = \frac{Q}{4\pi\varepsilon_0 r}P_0(\cos\theta) + \sum_{\ell=1}^{\infty}\left(\frac{Q}{4\pi\varepsilon_0}\right)\left((-1)^\ell\frac{(2\ell-1)!!}{2^\ell\;\ell!}\right)\frac{a^{2\ell}}{r^{(2\ell+1)}}\;P_{2\ell}(\cos\theta)$$

The form includes the ***double factorial*** notation which is defined as:

$$n!! = (n)(n-2)\,)(n-4)\;\ldots\; terminating\ at\ 2\ or\ 1.$$

Much has been gained in this case. The original expression is only valid on axis, but the final expression, an infinite series, is convergent at points on and off the axis in the entire region $r > a$.

Exercise: Explain why the expansion for the potential of the ring only includes Legendre polynomials of even index. What is $P_0(\cos\theta)$?

It is possible to form expansions valid for r < a for the two sample calculation problems above using the same methods. It is not possible to join the solutions smoothly at r = a however. There is singular charge density* at r = a, and thus the potential does not satisfy the Laplace equation at all points for which r = a. A Laplace solution just does not work at r = a. Note that the convergence of each

series becomes problematic as r approaches a from above or below. The potential is continuous across a sheet of surface charge density although the normal component of the electric field is not. Neither the potential are the field is continuous at the location of a linear charge density or a point charge.

<u>Examples matching along a constant *r* line in *r* - θ space</u>:

Typical Applications: Electrostatics problems with concentric spherical boundaries.

Note that the general solution has the form: $G(r,\theta)=\sum_{\ell=0}^{\infty}\left[a_\ell\ r^{\ell}+b_\ell\ r^{-(\ell+1)}\right]P_\ell(\cos\theta)$. For a fixed r, it is a linear combination of the Legendre polynomials of cosθ, the $P_\ell(\cos\theta)$. For this reason, the procedure is to express any functional dependence on θ as a ***linear combination*** first of powers of cosθ and then of the $P_\ell(\cos\theta)$.

(⇒ use linear vector space methods.) $f(\theta)=\sum_{\ell=0}^{\infty}c_\ell\ P_\ell(\cos\theta)$

Avoid representations that involve products of the $P_\ell(\cos\theta)$ unless you know they are desired. However, you can always multiply or divide terms by $P_o(\cos\theta)$ as it is just 1.

Match the value at *r* = *R*: A problem may have a boundary at $r = R$ leading to solutions for *inside* solutions which are to be valid for $0 < r < R$ and solutions for *outside* which are to be valid for $R < r < \infty$. The solutions are to be matched (*set equal*) at $r = R$. The values of the potential must match at the boundary because a discontinuity in the potential corresponds to an infinite electric field which is can only occur for a charge density more singular that a surface charge density[1].

inside: $V^{<}(r,\theta)=\sum_{\ell=0}^{\infty}\ a_\ell\ r^{\ell}\ P_\ell(\cos\theta)$ *and outside:* $V^{>}(r,\theta)=\sum_{\ell=0}^{\infty}\ b_\ell\ r^{-(\ell+1)}\ P_\ell(\cos\theta)$

Matching at $r = R$:

[1] As models, dipole surface layers give rise to potentials that are discontinuous across the layer. The lipid membrane enclosing a cell in the body may have a potential difference of several tens of millivolts across its thickness which is modeled as small.

$$V^{<}(R,\theta)=\sum_{\ell=0}^{\infty} a_\ell\ R^\ell\ P_\ell(\cos\theta)=V(R,\theta)=V^{>}(R,\theta)=\sum_{\ell=0}^{\infty} b_\ell\ R^{-(\ell+1)}\ P_\ell(\cos\theta)$$

Matching the inside and outside solutions at $r = R$, requires that: $b_\ell = a_\ell\ R^{2\ell+1}$.

> If two expansions in terms of the Legendre polynomials are equal over the domain $0 < \theta < \pi$, then the coefficients in the expansions are equal index value by index value.
>
> $$\sum_{\ell=0}^{\infty} A_\ell\, P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} B_\ell\, P_\ell(\cos\theta) \quad \text{for } 0<\theta<\pi \Leftrightarrow A_\ell = B_\ell\ \forall\,\ell$$
>
> The coefficients in two linear combinations of the same set of linearly independent (in this case: mutually orthogonal) functions must be equal term-by-term. See the Vector Spaces handout.

If the potential $V(R, \theta)$ is specified then simple matching at $r = R$ sets the values of the a_ℓ and b_ℓ. The well-behaved requirement at $r = 0$ ($r = \infty$) restricts the solution to $a_\ell\, r^\ell$ ($b_\ell\, r^{-(\ell+1)}$) in the inside (outside) region.

For example, if $V(R, \theta) = \sum_{\ell=0}^{\infty} v_\ell\ P_\ell(\cos\theta)$, then $a_\ell = v_\ell\, R^{-\ell}$ because the coefficients of the $P_\ell(\cos\theta)$ must match term-by-term as they are linearly independent (in fact, they are orthogonal). If the expansion of $V(R, \theta)$ is not known, one must ***project out*** the expansion coefficients. To find v_m, first multiply by $P_m(\cos\theta)$ and then exercise the inner product.

$$\int_0^\pi P_m(\cos\theta)V(R,\theta)\sin\theta\, d\theta=\sum_{\ell=0}^{\infty} v_\ell \int_0^\pi P_m(\cos\theta)\,P_\ell(\cos\theta)\sin\theta\, d\theta=\sum_{\ell=0}^{\infty} v_\ell\left(\frac{2}{2m+1}\right)\ \delta_{m\ell}$$

$$\Rightarrow v_m=\left(\frac{2m+1}{2}\right)\int_0^\pi P_m(\cos\theta)\ V(R,\theta)\sin\theta\, d\theta=\frac{\langle P_m|V\rangle}{\langle P_m|P_m\rangle} \qquad \text{[SL.33]}$$

Boundary Value Matching with thick shell regions *D* < *r* < *L*:

Matching across thick shells is much more tedious than matching on a single spherical surface. Expect one problem per semester of this type.

$$V^{<}(r,\theta)=\sum_{\ell=0}^{\infty} a_\ell\ r^\ell\ P_\ell(\cos\theta) \qquad \text{for } r<D$$

$$V^{shell}(r,\theta)=\sum_{\ell=0}^{\infty}\left(c_\ell\ r^\ell+d_\ell\ r^{-(\ell+1)}\right)P_\ell(\cos\theta)\qquad \text{for } D<r<L$$

$$V^{>}(r,\theta)=\sum_{\ell=0}^{\infty}b_\ell\ r^{-(\ell+1)}\ P_\ell(\cos\theta)\qquad \text{for } r>L$$

Match at D:

$$V^{<}(D,\theta)=\sum_{\ell=0}^{\infty}a_\ell\ D^\ell\ P_\ell(\cos\theta)=V^{shell}(D,\theta)=\sum_{\ell=0}^{\infty}\left(c_\ell\ D^\ell+d_\ell\ D^{-(\ell+1)}\right)P_\ell(\cos\theta)$$

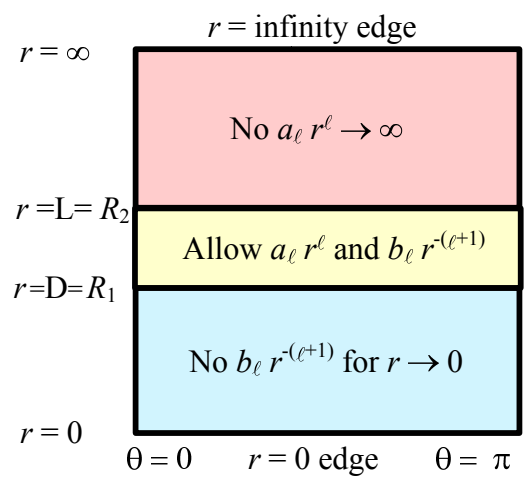
Match at L:

$$V^{shell}(L,\theta)=\sum_{\ell=0}^{\infty}\left(c_\ell\ L^\ell+d_\ell\ L^{-(\ell+1)}\right)P_\ell(\cos\theta)=V^{>}(L,\theta)=\sum_{\ell=0}^{\infty}b_\ell\ L^{-(\ell+1)}\ P_\ell(\cos\theta)$$

The conditions for each ℓ are:

$$\begin{aligned} a_\ell\ D^\ell &= c_\ell\ D^\ell+d_\ell\ D^{-(\ell+1)} \\ c_\ell\ L^\ell+d_\ell\ L^{-(\ell+1)} &= b_\ell\ L^{-(\ell+1)} \end{aligned} \qquad \text{[SL.34]}$$

It may be necessary to match the normal (radial) derivatives as well to complete the problem.



$$D=R_1<r<R_2=L$$

The point: All the terms in $G(r,\theta)=\sum_{\ell=0}^{\infty}\left[a_\ell\ r^\ell+b_\ell\ r^{-(\ell+1)}\right]P_\ell(\cos\theta)$ are allowed solutions of the Laplace equation in spherical coordinates with the angle θ running over its full range. All of the terms in this expression must be included *unless a reason can be found to exclude them*. The *inside region* includes the point $r = 0$ at which all the $r^{-(\ell+1)}$ diverge and hence are not physically acceptable.

In the *outside region*, $r \to \infty$ leading the usual expectation that the potential should approach zero; the terms like r^{ℓ} do not meet this expectation and are excluded in the usual cases. Those terms may be present if there is an external applied field.

Conditions on the normal derivative: The normal derivative on the surface a sphere is the partial derivative with respect to r. Further, it is the negative of the radial component of the electric field. It is the normal component of the electric field that appears in the integral representation of Gauss's Law and that has a boundary-value matching requirement linked to the local surface charge density. Allowing for differing dielectric (*constant* ε_r) properties in the interior and exterior regions, the possible sets of matching condition include:

$$E_r^> - E_r^< = \varepsilon_0^{-1}\,\sigma_{total} \quad ; \quad \varepsilon_r^> E_r^> - \varepsilon_r^< E_r^< = 0 \quad , \quad \varepsilon_r^> E_r^> - \varepsilon_r^< E_r^< = \varepsilon_0^{-1}\,\sigma_{free}$$

$$\frac{\partial V^<}{\partial r} - \frac{\partial V^>}{\partial r} = \varepsilon_0^{-1}\,\sigma_{total} \quad ; \quad \varepsilon_r^< \frac{\partial V^<}{\partial r} - \varepsilon_r^> \frac{\partial V^>}{\partial r} = 0 \quad , \quad \varepsilon_r^< \frac{\partial V^<}{\partial r} - \varepsilon_r^> \frac{\partial V^>}{\partial r} = \varepsilon_0^{-1}\,\sigma_{free}\,.$$

These boundary conditions are developed in another section of this note set.

Boundary Matching Conditions:

BC1.) At each constant R boundary, match the value of the inside solution to the value of the outside solution. As the solutions approach the boundary, the inside coefficient of each Legendre polynomial must approach the same value as does the outside coefficient in the outside solution.

BC2.) Apply the conditions on the normal derivative of the potential (which is the negative of the radial component of the electric field) if dielectric properties are specified or a surface charge density is specified or sought. A conducting material specified that the electric field vanished inside and hence specifies a boundary condition on the normal derivative of the potential.

$$E_r^> - E_r^< = \varepsilon_0^{-1}\,\sigma_{total} \quad ; \quad \varepsilon_r^> E_r^> - \varepsilon_r^< E_r^< = 0 \quad , \quad \varepsilon_r^> E_r^> - \varepsilon_r^< E_r^< = \varepsilon_0^{-1}\,\sigma_{free}$$

$$\frac{\partial V^<}{\partial r} - \frac{\partial V^>}{\partial r} = \varepsilon_0^{-1}\,\sigma_{total} \quad ; \quad \varepsilon_r^< \frac{\partial V^<}{\partial r} - \varepsilon_r^> \frac{\partial V^>}{\partial r} = 0 \quad , \quad \varepsilon_r^< \frac{\partial V^<}{\partial r} - \varepsilon_r^> \frac{\partial V^>}{\partial r} = \varepsilon_0^{-1}\,\sigma_{free}$$

Sample Calculations with an *r* =*R* Boundary:

SC1. A spherical shell at potential V_o:

An insulating spherical shell has a surface charge that makes the shell an **equipotential with potential V_o**. Space is charge-free for $r < R$ and for $r > R$.

The electrostatic potential satisfies the Laplace equation in charge-free space so:

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left[a_\ell\ r^\ell + b_\ell\ r^{-(\ell+1)} \right] P_\ell(\cos\theta)$$

The potential should remain finite in charge-free space and should approach zero at large distances from a finite charge distribution.

$$\text{inner region:} \quad V(r,\theta) \to V^<(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell\ r^\ell\ P_\ell(\cos\theta) \qquad (r < R)$$

$$\text{outer region:} \quad V(r,\theta) \to V^>(r,\theta) = \sum_{\ell=0}^{\infty} b_\ell\ r^{-(\ell+1)}\ P_\ell(\cos\theta) \qquad (r > R)$$

The two solution must agree at $r = R$. A discontinuous potential would require an infinite electric field which is not physical.

$$V^<(R,\theta) = \sum_{\ell=0}^{\infty} a_\ell\ R^\ell\ P_\ell(\cos\theta) = V_0 = V^>(R,\theta) = \sum_{\ell=0}^{\infty} b_\ell\ R^{-(\ell+1)}\ P_\ell(\cos\theta)$$

The $P_\ell(\cos\theta)$ form an orthogonal set of functions so the coefficients must match term-by-term. ($\Rightarrow$ $a_\ell R^\ell = b_\ell R^{-(\ell+1)}$ for all ℓ.) As $V_o = V_o P_o(\cos\theta)$, all the a_ℓ and b_ℓ vanish except for $a_o = V_o$ and $b_o = V_o R$. Substituting,

$$\text{inner region:} \quad V(r,\theta) \to V^<(r,\theta) = V_0\ P_0(\cos\theta) \equiv V_0 \qquad (r < R)$$

$$\text{outer region:} \quad V(r,\theta) \to V^>(r,\theta) = V_0 \left(R/r \right) P_0(\cos\theta) \qquad (r > R)$$

The potential is V_o everywhere on the spherical surface bounding the region $r < R$ and the potential has all its extrema on the boundary. Hence the potential has the constant

value V_o at all points inside the shell. In the exterior region, the potential is that of a point charge $q = 4\pi\, \varepsilon_o\, R\, V_o$. The potential is between the values $V(R) = V_o$ and $V(\infty) = 0$ in the region $r > R$. Note that neither the potential imposed on the shell or the shape of the shell has any angular dependence. As a result the solution throughout space is independent of angle. The solution only has $P_o(\cos\theta)$ character. It's spherically symmetric.

Exercise: Use the solution to SC1. Compute the electric field inside and outside the shell $r = R$. Use the form of the field outside and Gauss's Law to compute the net charge on the shell.

SC2. A spherical shell at potential $V_o \cos^2\theta$:

An insulating spherical shell has a surface charge that makes the shell and equipotential with potential V_o. Space is charge-free for $r < R$ and for $r > R$.
The electrostatic potential satisfies the Laplace equation in charge-free space so:

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left[a_\ell\, r^\ell + b_\ell\, r^{-(\ell+1)} \right] P_\ell(\cos\theta)$$

The potential should remain finite in charge-free space and should approach zero at large distances from a finite charge distribution.

$$\text{inner region:} \quad V(r,\theta) \to V^<(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell\, r^\ell\, P_\ell(\cos\theta) \qquad (r < R)$$

$$\text{outer region:} \quad V(r,\theta) \to V^>(r,\theta) = \sum_{\ell=0}^{\infty} b_\ell\, r^{-(\ell+1)}\, P_\ell(\cos\theta) \qquad (r > R)$$

The two solutions must be equal at $r = R$. A discontinuous potential would require an infinite electric field. Point charges and lines of charge can cause localized infinite fields. A finite surface charge density cannot.

$$V^<(R,\theta) = \sum_{\ell=0}^{\infty} a_\ell\, R^\ell\, P_\ell(\cos\theta) = V_0 \cos^2\theta = V^>(R,\theta) = \sum_{\ell=0}^{\infty} b_\ell\, R^{-(\ell+1)}\, P_\ell(\cos\theta)$$

The $P_\ell(\cos\theta)$ form an orthogonal set of functions so the coefficients must match term-by-term. As $V_o \cos^2\theta = V_o\,[{}^1/_3\,P_o(\cos\theta) + {}^2/_3\,P_2(\cos\theta)]$, all the a_ℓ and b_ℓ vanish except for a_o, a_2, b_o and b_2. Matching the coefficients term-by-term:

$$\sum_{\ell=0}^{\infty} a_\ell\ R^\ell\ P_\ell(\cos\theta) = V_0 \cos^2\theta = V_0\left[\tfrac{1}{3}P_0(\cos\theta) + \tfrac{2}{3}P_2(\cos\theta)\right] = \sum_{\ell=0}^{\infty} b_\ell\ R^{-(\ell+1)}\ P_\ell(\cos\theta)$$

$$a_o = {}^1/_3\ V_o;\ a_2 = {}^2/_3\ V_o\ R^{-2};\ b_o = {}^1/_3\ V_o\ R;\ b_2 = {}^2/_3\ V_o\ R^3$$

$$\text{inner region:}\quad V^<(r,\theta) = \tfrac{1}{3}V_0\,P_0(\cos\theta) + \tfrac{2}{3}V_0\left(r/R\right)^2 P_2(\cos\theta) \qquad (r < R)$$

$$\text{outer region:}\quad V^>(r,\theta) = \tfrac{1}{3}V_0\left(R/r\right)P_0(\cos\theta) + \tfrac{2}{3}V_0\left(R/r\right)^3 P_2(\cos\theta) \qquad (r > R)$$

Note that the shape of the shell (spherical) has no angular dependence so all the angular behavior of the potential is imposed by the surface charge distribution. That distribution has $P_o(\cos\theta)$ and $P_2(\cos\theta)$ characters, and, as a result, the solution throughout space has $P_o(\cos\theta)$ character and $P_2(\cos\theta)$ character only.

Exercise: For the case of SC2, the potential is between 0 and V_o (a positive value) on the surface $r = R$. Show that the potential is between 0 and V_o for $r < R$.

SC3. A spherical shell with surface charge density $\sigma_o \cos\theta$:

An insulating spherical shell has a radius R and a surface charge density $\sigma_o \cos\theta$. Space is charge-free for $r < R$ and for $r > R$. The electrostatic potential satisfies the Laplace equation in charge-free space so:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell\ r^\ell + b_\ell\ r^{-(\ell+1)}\right] P_\ell(\cos\theta)$$

The potential should remain finite in charge-free space and should approach zero at large distances from a finite charge distribution.

$$\text{inner region:}\quad V(r,\theta) \to V^<(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell\ r^\ell\ P_\ell(\cos\theta) \qquad (r < R)$$

$$\text{outer region:} \quad V(r,\theta) \to V^{>}(r,\theta) = \sum_{\ell=0}^{\infty} b_\ell \; r^{-(\ell+1)} \, P_\ell(\cos\theta) \qquad (r > R)$$

The two solution must be equal at $r = R$. A discontinuous potential would require an infinite electric field. A finite surface charge density does not cause an infinite field.

$$V^{<}(R,\theta) = \sum_{\ell=0}^{\infty} a_\ell \; R^{\ell} \, P_\ell(\cos\theta) = V^{>}(R,\theta) = \sum_{\ell=0}^{\infty} b_\ell \; R^{-(\ell+1)} \, P_\ell(\cos\theta)$$

Matching the coefficients of the Legendre polynomials term-by-term, $a_\ell = b_\ell \, R^{-(2\ell+1)}$ for all ℓ. This result is general and reflects only the requirement that $V(r, \theta)$ must be continuous at $r = R$. The charge is the source of the potential, and its distribution sets the values of the various b_ℓ. Gauss's Law provides the connection between the normal component of the electric field and the surface charge density.

$$E_r^{>}(R,\theta) - E_r^{<}(R,\theta) = \frac{\sigma(\theta)}{\varepsilon_0} \quad \text{or} \quad \left.\frac{\partial V^{<}}{\partial r}\right|_{r=R} - \left.\frac{\partial V^{>}}{\partial r}\right|_{r=R} = \frac{\sigma(\theta)}{\varepsilon_0} \quad \text{GAUSS}$$

Computing the derivatives:

$$V^{<}(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell \; r^{\ell} \, P_\ell(\cos\theta) \qquad \frac{\partial V^{<}}{\partial r} = \sum_{\ell=0}^{\infty} \ell \, a_\ell \; r^{\ell-1} \, P_\ell(\cos\theta)$$

$$\left.\frac{\partial V^{<}}{\partial r}\right|_{r=R} = \sum_{\ell=0}^{\infty} \ell \, a_\ell \; R^{\ell-1} \, P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} \ell \, b_\ell \; R^{-(\ell+2)} \, P_\ell(\cos\theta)$$

$$V^{>}(r,\theta) = \sum_{\ell=0}^{\infty} b_\ell \; r^{-(\ell+1)} \, P_\ell(\cos\theta) \qquad \left.\frac{\partial V^{>}}{\partial r}\right|_{r=R} = \sum_{\ell=0}^{\infty} [-(\ell+1)] \, b_\ell \; R^{-(\ell+2)} \, P_\ell(\cos\theta)$$

Evaluating the difference normal derivatives:

$$\left.\frac{\partial V^{<}}{\partial r}\right|_{r=R} - \left.\frac{\partial V^{>}}{\partial r}\right|_{r=R} = \sum_{\ell=0}^{\infty} (2\ell+1) \, b_\ell \; R^{-(\ell+2)} \, P_\ell(\cos\theta) = \varepsilon_0^{-1} \, \sigma(\theta) = \varepsilon_0^{-1} \, \sigma_0 \; P_1(\cos\theta)$$

Matching the coefficients of the $P_\ell(\cos\theta)$ term-by-term,

$$3 \, b_1 \, R^{-3} = \varepsilon_o^{-1} \, \sigma_o \text{ and } a_1 = b_1 \, R^{-3} \text{ or } \; a_1 = \frac{\sigma_0}{3 \, \varepsilon_0} \; ; \; b_1 = \frac{\sigma_0 \, R^3}{3 \, \varepsilon_0}.$$

$$V^{<}(r,\theta) = \frac{\sigma_0 \, r}{3 \, \varepsilon_0} P_1(\cos\theta) = \frac{\sigma_0 \, r \cos\theta}{3 \, \varepsilon_0} ; \quad V^{>}(r,\theta) = \frac{\sigma_0 \, R^3}{3 \, \varepsilon_0 r^2} P_1(\cos\theta) = \frac{\sigma_0 \, R^3 \cos\theta}{3 \, \varepsilon_0 r^2}$$

The potential inside corresponds to a uniform electric field: $\vec{E}^{<} = -\frac{\sigma_0}{3\,\varepsilon_0}\hat{k}$. The exterior solution has the dipole potential form for a dipole $\vec{p} = p_0\,\hat{k}$.

$$V_{dipole} = \frac{\vec{p}\cdot\vec{r}}{4\,\pi\,\varepsilon_0 r^3} \rightarrow \frac{p_0\,\cos\theta}{4\,\pi\,\varepsilon_0 r^2} \quad \text{for } \vec{p} = p_0\,\hat{k} = \frac{4\,\pi\,R^3\,\sigma_0}{3}\hat{k}\,.$$

The potentials in the two regions only have $P_1(\cos\theta)$ character. The spherical shell does not contribute any angular dependence, and the charge distribution is pure $P_1(\cos\theta)$ in angular character. Hence only $P_1(\cos\theta)$ character occurs.

Sanity checks do not prove that a result is correct. They merely establish that the proposed result is not as horribly wrong as it could be.

Sanity Check 0: Check the dimensions of the results. The relation

$$E_r^{>}(R,\theta) - E_r^{<}(R,\theta) = \frac{\sigma(\theta)}{\varepsilon_0}$$

shows that the field has the same dimensions as charge density divided by the permittivity of free space. Our result for the field inside the shell $\vec{E}^{<} = -\frac{\sigma_0}{3\,\varepsilon_0}\hat{k}$ has the correct dimensions. Examining the potential expressions, $V^{<}(r,\theta) = \frac{\sigma_0\,r\cos\theta}{3\,\varepsilon_0}$ and $V^{>}(r,\theta) = \frac{\sigma_0\,R^3\cos\theta}{3\,\varepsilon_0 r^2}$, both have the dimensions of length times electric field (length * charge density divided by the permittivity of free space) as expected. (✓)

Sanity Check 1: Verify that the dipole moment of the charge distribution has the value claimed.

$$\vec{p} = \int dq\,\vec{r} = \int \sigma\,dA\,\vec{r} = \int_0^{\pi}\int_0^{2\pi}\left(\sigma_0\,\cos\theta\right)\left(R^2\sin\theta\,d\theta\,d\phi\right)R\left(\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\right)$$

$$\Rightarrow \quad \vec{p} = \frac{4\,\pi\,R^3\,\sigma_0}{3}\hat{k} \qquad (✓)$$

Sanity Check 2: Verify that the electric field due to the charge distribution has the value claimed at the center of the shell. (It would be difficult to show that the field is uniform with that value everywhere inside!) Thin $d\theta$ slices of the spherical partition the source into a stack of rings perpendicular to the z axis.

The field due to a uniform circular ring of charge with radius a at a point a distance d along a perpendicular to its plane at its center is:

$$\vec{E}_{ring} = \frac{q_{ring}\ \ d}{4\pi\,\varepsilon_0\,[a^2+d^2]^{3/2}}(a\hat{w}ay)$$

Slicing the charged spherical surface into $d\theta$ rings, the charge of each ring is:

$$q_{ring} = \sigma(\theta)\,dA = \sigma_0\,\cos\theta\,2\pi\,R^2\,\sin\theta\,d\theta$$

The factor $\sqrt{a^2+d^2}$ is the distance from the source charge to field point. All the sources are on the spherical shell and the field point is its center $\Rightarrow \sqrt{a^2+d^2} = R$. A sketch shows that $d = R\,|\cos\theta|$ and that the direction normally away from the ring is $-\frac{\cos\theta}{|\cos\theta|}\hat{k}$. Assembling the mess:

$$\Rightarrow\ \vec{E}(0\hat{k}) = \int_0^{\pi}\frac{\sigma_0\,\cos\theta\,2\pi\,R^2\,\sin\theta\,d\theta\ \ R\cos\theta}{4\pi\,\varepsilon_0\ \ R^3}\left(-\hat{k}\right) = -\frac{\sigma_0}{3\,\varepsilon_0}\hat{k} \qquad (\checkmark)$$

$$\int_0^{\pi} P_m(\cos\theta)\frac{\sigma(\theta)}{\varepsilon_0}\sin\theta\ d\theta = \sum_{\ell=0}^{\infty}(\ell+1)\,b_\ell\ R^{-(\ell+2)}\int_0^{\pi} P_m(\cos\theta)\ P_\ell(\cos\theta)\ \sin\theta\ d\theta$$

$$b_m = \frac{(2m+1)\,R^{(m+2)}}{(2m+2)\,\varepsilon_0}\int_0^{\pi}\sigma(\theta)\ P_m(\cos\theta)\ \sin\theta\ d\theta$$

Exercise: A spherical conductor is an equipotential. Show that the relation:

$$V^{>}(R,\theta) = \sum_{\ell=0}^{\infty} b_\ell\ R^{-(\ell+1)}\ P_\ell(\cos\theta) = V_0$$

requires that all the $b_\ell = 0$ except for $b_0 = V_0\,R$. Alternatively, in the absence of an applied field, the symmetry is completely spherical so $\sigma(\theta)\to\sigma_0$, a constant. Using $\sigma(\theta)=\sigma_0$ show that $V^{>}(r,\theta) = \frac{Q}{4\pi\varepsilon_0 r}$. Matching to find the interior solution shows that $V^{>}(R,\theta) = V_o$. Why is this result expected?

Thin Insulating Charged Shell: (No externally applied field.)

If $r = R$ is a thin insulating spherical shell with an azimuthally symmetric surface charge density σ(θ), then $E_r^> - E_r^< = \left(-\frac{\partial V^>}{\partial r}\right)\Big|_{r=R} + \left(-\frac{\partial V^<}{\partial r}\right)\Big|_{r=R} = \frac{\sigma(\theta)}{\varepsilon_0}$. The continuity condition still applies so that: $b_\ell = a_\ell \; R^{2\ell+1}$. Computing the derivatives term by term, applying the continuity equation and finally projecting out the coefficients a_m leads to the result

$$a_m = \frac{1}{2\,\varepsilon_0\,R^{(m-1)}} \int_0^\pi \sigma(\theta)\; P_m(\cos\theta)\,\sin\theta\; d\theta \qquad \textbf{\textit{this result appears to valid}}$$

The proof of the relation is a problem below.

Dielectric Sphere: (No externally applied field.)

If the region $r < R$ is filled with a uniform linear dielectric with constant ε_{r1} and the region $r > R$ is filled with a uniform linear dielectric with constant ε_{r2} , then $\varepsilon_{r2}\; E_r^> - \varepsilon_{r1} E_r^< = -\varepsilon_{r2}\,\frac{\partial V^>}{\partial r}\Big|_{r=R} + \varepsilon_{r1}\,\frac{\partial V^<}{\partial r}\Big|_{r=R} = \frac{\sigma_{free}(\theta)}{\varepsilon_0}$. Here, σ_{free} is the free surface charge density on the insulating dielectric sphere. The continuity condition still applies so that $b_\ell = a_\ell \; R^{2\ell+1}$. Computing the derivatives term by term, applying the continuity equation and finally projecting out the coefficients a_m leads to the result

$$a_m = \frac{2m+1}{2\left[\left(\varepsilon_{r1}+\varepsilon_{r2}\right)m+\varepsilon_{r2}\right]\varepsilon_0\,R^{(m-1)}} \int_0^\pi \sigma_{free}(\theta)\; P_m(\cos\theta)\,\sin\theta\; d\theta$$

this result has not been verified

The proof of this result is a problem below.

Applied External Fields: Consider problems with spheres immersed in what had been a uniform applied electric field $E_0\,\hat{k}$. The immersed materials alter the field locally, but, in the limit of large distances, the potential and electric fields approach the externally applied value of $V_o - E_o\,z$ for the potential plus terms that vanish as $r \to \infty$.

$$V_{ext}(r,\theta) = V_0 - E_0\, z = V_0 - E_0\, r\cos\theta = V_0 - E_0\, r\, P_1(\cos\theta)\,.$$

The constant V_0 is set to zero corresponding to zero potential at $r = 0$. The allowed exterior ($r>R$) solution becomes:

$$V^{>}(r,\theta) = -E_0\, r\cos\theta + \sum_{\ell=0}^{\infty} b_\ell\; r^{-(\ell+1)}\; P_\ell(\cos\theta)\,.$$

Potential due to the distant external charges plus that due to induced charges on the shell.

Conducting Shell: (In a uniform applied field)

If $r = R$ is the outer surface of a conducting sphere, then $V(R) = 0$. Applying this condition,

$$V^{>}(R,\theta) = b_0\; R^{-1} P_0(\cos\theta) + \left(b_1\; R^{-2} - E_0\, R\right) P_1(\cos\theta) + \sum_{\ell=2}^{\infty} b_\ell\; R^{-(\ell+1)}\, P_\ell(\cos\theta) = 0 = \sum_{\ell=0}^{\infty} [0]\, P_\ell(\cos\theta)$$

Hence $b_\ell = 0$ except for $b_1 = E_0\, R^3$. The final result becomes:

$$V^{>}(r,\theta) = E_0\left[\frac{R^3}{r^2} - r\right]\cos\theta = E_0\left[\frac{R^3}{r^2} - r\right] P_1(\cos\theta)$$

This result is just the external potential plus the additional potential due to the electric dipole moment induced on the sphere.

Note that each $P_\ell(\cos\theta)$ represents a distinct, orthogonal angular behavior. Absent the applied field, the problem was perfectly spherical (only $P_0(\cos\theta)$ behavior). The applied field imposed a $P_1(\cos\theta)$ disturbance. Only $P_1(\cos\theta)$ behavior was induced as a result. If the initial problem had possessed a $P_2(\cos\theta)$ character, then imposing a $P_1(\cos\theta)$ disturbance could have linked to characters $P_1(\cos\theta)$, $P_2(\cos\theta)$ and $P_3(\cos\theta)$ following the same rules as the addition of angular momenta under the vector model. In the previous examples without an applied field, the angular dependence was imposed by the charge distribution.

Vector Model: If vectors of magnitude A and B are added, the resultant has a magnitude in the range |A-B| to A+B. In the case of Legendre polynomials, only integer index values are meaningful.

$$\left|\ell_{initial} - \ell_{disturb}\right| \le \ell_{answer} \le \ell_{initial} + \ell_{disturb} \quad \textbf{or}$$

$$\ell_{answer} = \left|\ell_{initial} - \ell_{disturb}\right|; \left|\ell_{initial} - \ell_{disturb}\right| + 1;\; \left|\ell_{initial} - \ell_{disturb}\right| + 2;\; \ldots\; ;\; \ell_{initial} + \ell_{disturb}$$

Dielectric Sphere: (In a uniform applied field)

If the region $r < R$ is filled with a uniform linear dielectric with constant ε_{r1} and the region $r > R$ is filled with a uniform linear dielectric with constant ε_{r2} , then $\varepsilon_{r2}\ E_r^{>} - \varepsilon_{r1} E_r^{<} = -\varepsilon_{r2} \frac{\partial V^{>}}{\partial r}\Big|_{r=R} + \varepsilon_{r1} \frac{\partial V^{<}}{\partial r}\Big|_{r=R} = \frac{\sigma_{free}(\theta)}{\varepsilon_0}$. Here, σ_{free} is the free surface charge density on the insulating dielectric sphere. The continuity condition still applies so: $b_\ell = a_\ell\ R^{2\ell+1}$. Computing the derivatives term by term, applying the continuity equation and finally projecting out the coefficients a_m leads to the result (for $m \neq 1$):

$$a_m = \frac{2m+1}{2\left[\left(\varepsilon_{r1}+\varepsilon_{r2}\right)m+\varepsilon_{r2}\right]\varepsilon_0\, R^{(m-1)}} \int_0^{\pi} \sigma_{free}(\theta)\ P_m(\cos\theta)\ \sin\theta\ d\theta \qquad (m>1)$$

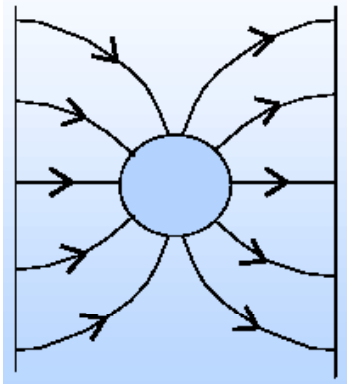
this result has not been verified

The result for a_1 can be derived using the external potential representation: $V^{>}(r,\theta) = -E_0\, r\cos\theta + \sum_{\ell=0}^{\infty} b_\ell\ r^{-(\ell+1)}\ P_\ell(\cos\theta)$. The proof of the relation is a problem for which σ_{free} =0. The result gives the addition to a_1 due to the external field. The result with a free charge density σ_{free} would follow from superposition of the external field, zero-charge problem and the non-zero free charge problem in absence of an external applied field.

Sample Calculation: The Complete Conducting Sphere in a Uniform External Field.

A prefect conducting sphere at the origin by itself is spherically symmetric; it only has $P_o(\cos\theta)$ character. A uniform field in the z-direction $E_0\ \hat{k}$ corresponds to a potential $V_0^{ext} - E_0\ z = V_0^{ext} - E_0\ r\ \cos\theta = V_0^{ext} - E_0\ r\ P_1(\cos\theta)$. Using the freedom to adjust the potential by an additive constant, the potential at the origin is chosen to be zero eliminating the constant and with it the $P_o(\cos\theta)$ character of the external potential.

	Placing the finite (radius R) sphere in the external field perturbs the potential, but it only adds terms that vanish



for large r. Hence the total potential must be of the form: $V^{>}(r,\theta) = -E_0\ r\ P_1(\cos\theta) + \sum_{\ell=0}^{\infty} b_\ell\ r^{-(\ell+1)}\ P_\ell(\cos\theta)$

Matching the solution to the equipotential conductor at $r = R$.

$$V^{>}(R,\theta) = 0 = -E_0\ R\ P_1(\cos\theta) + \sum_{\ell=0}^{\infty} b_\ell\ R^{-(\ell+1)}\ P_\ell(\cos\theta)$$

Matching the coefficients of the Legendre polynomials on both sides of the equation yields $b_\ell = 0\ \forall\ \ell \neq 1$. The coefficient equation for $P_1(\cos\theta)$ is: $0 = -E_o R + b_1 R^{-2}$ leading to $b_1 = E_o R^3$ The potential becomes:

$$V^{>}(r,\theta) = -E_0\ r\ P_1(\cos\theta) + \frac{E_0 R^3}{r^2}\ P_1(\cos\theta) = \left[-E_0\ r + \frac{E_0 R^3}{r^2}\right]\cos\theta$$

The charge density of the surface of a conductor is related to the field just outside the conductor by Gauss's Law.

$$\sigma = \varepsilon_0\ E_n \rightarrow \varepsilon_0\ E_r \rightarrow -\varepsilon_0\ \partial V^{>}/\partial r\Big|_{r=R}$$

$$\sigma(\theta) = -\varepsilon_0\ \partial V^{>}/\partial r\Big|_{r=R} = \varepsilon_0 E_0\ P_1(\cos\theta) + 2\ \varepsilon_0 E_0\ P_1(\cos\theta) = 3\,\varepsilon_0 E_0\ P_1(\cos\theta) = 3\ \varepsilon_0\ E_0\ \cos\theta$$

This charge density corresponds to a separation of positive and negative charge and hence indicates that the sphere has an electric dipole moment.

$$\vec{p} = \sum_{charges\ i} q_i\,\vec{r}_i = \int \rho(\vec{r})\,\vec{r}\,d^3r = \int \sigma(\vec{r})\,\vec{r}\,d^2r$$

For the case of the sphere above:

$$\begin{aligned}\vec{p} &= \int \sigma(\vec{r})\,\vec{r}\,d^2r \\ &= \int_0^{\pi}\int_0^{2\pi} 3\,\varepsilon_0\ E_0\left[R\sin\theta\cos\phi\,\hat{i} + R\sin\theta\sin\phi\,\hat{j} + R\cos\theta\,\hat{k}\right]R^2\sin\theta\ d\theta\ d\phi\end{aligned}$$

$$\vec{p} = 4\ \pi\ \varepsilon_0\,E_0\ R^3\ \hat{k}$$

The potential due to this dipole would be:

$$V_{dip} = \frac{\vec{p}\cdot\vec{r}}{4\pi\,\varepsilon_0\,r^3} = \frac{4\,\pi\,\varepsilon_0\,E_0\,R^3\,\hat{k}\cdot\vec{r}}{4\pi\,\varepsilon_0\,r^3} = \frac{E_0\,R^3\,\cos\theta}{r^2}$$

or exactly the addition to the external potential due to the presence of the induced charge distribution on the conducting sphere.

The electrostatic field is caused by charge. The next problem is to compute the potential that would be caused by the surface charge density induced on the conductor. The potential is due to just the charge density with no thought given to the conductor (which has been dissolved by some strong acid).

The solutions templates inside and out are:

inside: $V^{<}(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell\, r^\ell\, P_\ell(\cos\theta)$ and outside: $V^{>}(r,\theta) = \sum_{\ell=0}^{\infty} b_\ell\, r^{-(\ell+1)}\, P_\ell(\cos\theta)$

Matching the values of the solutions at $r = R$, requires that $b_\ell = a_\ell\, R^{2\ell+1}$. The next step is to match the discontinuity in the normal component of the electric field to the surface charge density divided by the permittivity of free space.

$$E_r^{>} - E_r^{<} = -\frac{\partial V^{>}}{\partial r}\bigg|_{r=R} + \frac{\partial V^{<}}{\partial r}\bigg|_{r=R} = \frac{\sigma(\theta)}{\varepsilon_0}$$

$$\sigma(\theta) = \varepsilon_0\left[\sum(\ell+1)\left(a_\ell\, R^{2\ell+1}\right) R^{-(\ell+2)} P_\ell(\cos\theta) + a_\ell\, R^{(\ell-1)}\, P_\ell(\cos\theta)\right]$$

$$= \varepsilon_0 \sum\left[2\,\ell+1\right] a_\ell\, R^{(\ell-1)}\, P_\ell(\cos\theta)$$

Inserting the charge density that was induced on the conducting sphere:

$$\sigma(\theta) = 3\,\varepsilon_0 E_0\; P_1(\cos\theta) = \varepsilon_0 \sum\left[2\,\ell+1\right] a_\ell\, R^{(\ell-1)}\, P_\ell(\cos\theta)$$

The coefficients of the Legendre polynomials must match term by term so $a_1 = E_o$ and $b_1 = a_1\, R^3 = E_o\, R^3$ with all other coefficients zero.

$$V^{<}(r,\theta) = E_0\; r\; P_1(\cos\theta) = E_0\; r\,\cos\theta\,;\; V^{>}(r,\theta) = E_0\, R^2\, r^{-2} P_\ell(\cos\theta) = E_0\, R^2\, r^{-2}\cos\theta$$

The surface charge contributes a uniform electric field $-E_0\hat{k}$ inside and the expected dipole field outside. In the interior ($r < R$) the addition of the external field and the field due to the surface charge makes the region field free as expected inside a conductor in electrostatic equilibrium. Always remember and never forget that the electrostatic field is due to the charges only. Conductors may allow charges to redistribute, but field is due to the charges in their final distribution. After the charge distribution is static, the conductor can be removed *leaving the charge distribution* and field unchanged.

Cheap Legendre Expansions: Another charged shell

The cheap Legendre method is presented in detail in the Tools of the Trade section below. The point is that, if the surface charge density can be expressed as a polynomial in cosθ, then one can find the resulting potential without heavy lifting. Suppose, for example, that $\sigma(\theta) = \sigma_0 \sin^2\theta = \sigma_0\left[1-\cos^2\theta\right]$. A polynomial in cosθ of order n can be expressed in terms of the first n Legendre polynomials by inspection. One just matches the nth power coefficient to set the same coefficient for $(\cos\theta)^n$ in a multiple of P_n. Then a multiple of P_{n-1} is used to match the remainder of the $(\cos\theta)^{n-1}$ dependence and so on.

The task:

Express $f(\theta)$ [$\sin^2\theta$ in this example] as a linear combination of powers of cosθ. Match highest power (call it m) by assigning a coefficient to $P_m(\cos\theta)$. Next look at the $[\cos\theta]^{m-1}$ term. Add enough $P_{m-1}(\cos\theta)$ to cover the $[\cos\theta]^m$ term plus any leftovers from the Legendre polynomials used previously to match the higher powers of cosθ.

$$\sin^2\theta = a + b\cos\theta + c\cos^2\theta + \ldots + d\,[\cos\theta]^m =$$
$$g\,P_m(\cos\theta) + h\,P_{m-1}(\cos\theta) + \ldots + k\,P_0(\cos\theta).$$

Hence:

$$1-\cos^2\theta = -\tfrac{2}{3}\,P_2(\cos\theta)+\tfrac{2}{3}\,P_0(\cos\theta)\,.$$

$$\sigma(\theta)=\sigma_0\sin^2\theta=\sigma_0\left[+\tfrac{2}{3}\,P_0(\cos\theta)-\tfrac{2}{3}\,P_2(\cos\theta)\right]=\varepsilon_0\sum\left[2\,\ell+1\right]a_\ell\,R^{(\ell-1)}\,P_\ell(\cos\theta)$$

Comparison yields: $a_0=\frac{2\sigma_0 R}{3\varepsilon_0},\ a_2=-\frac{2\sigma_0}{15\varepsilon_0 R}\ ;\ b_0=\frac{2\sigma_0 R^2}{3\varepsilon_0},\ b_2=-\frac{2\sigma_0 R^4}{15\varepsilon_0 R}$

Not So Cheap Legendre Expansions: Another charged shell

Suppose, for example, that $\sigma(\theta) = \sigma_o \sin\theta$. This form is not 'just a polynomial', and the full method must be employed – *projection of coefficients by invoking the orthogonality relation.*

$$\int_0^\pi P_m(\cos\theta)\left[\sigma_0(\theta)\right]\sin\theta\,d\theta=\int_0^\pi P_m(\cos\theta)\left[\sigma_0\sin\ \theta\right]\sin\theta\,d\theta$$

$$=\varepsilon_0\sum\left[2\,\ell+1\right]a_\ell\,R^{(\ell-1)}\int_0^\pi P_m(\cos\theta)\,P_\ell(\cos\theta)\sin\theta\,d\theta$$

$$=\varepsilon_0\sum_\ell\left[2\,\ell+1\right]a_\ell\,R^{(\ell-1)}\,\tfrac{2}{2\ell+1}\,\delta_{\ell m}=2\,\varepsilon_0\,R^{(m-1)}\,a_m$$

$$a_m=\frac{\sigma_0}{2\,\varepsilon_0\,R^{(m-1)}}\int_0^\pi P_m(\cos\theta)\,\sin\theta\ \sin\theta\,d\theta$$

An enormous amount of effort (not really) yields:

$$a_0=\frac{\sigma_0\pi R}{4\,\varepsilon_0}\ ;\ a_2=-\frac{\sigma_0\pi}{32\,\varepsilon_0\,R}\ ;\ a_4=-\frac{\sigma_0\,\pi}{256\,\varepsilon_0\,R^3}\ ;\ a_6=-\frac{5\,\sigma_0\,\pi}{4096\,\varepsilon_0\,R^5};a_8=-\frac{35\,\sigma_0\,\pi}{65536\,\varepsilon_0\,R^7};\ \ldots.$$

It should be noted that $\sin\theta$ is even about $\pi/2$ as are the even Legendre polynomials of $\cos\theta$. There is no odd Legendre polynomial of $\cos\theta$ character in $\sin\theta$.

Perturbation approach to problems that are nearly spherically symmetric:

This topic is non-standard. You may skip to the Tools of the Trade Section unless your instructor directs that you read it.

When boundary conditions are specified on a spherical surface, matching is relatively straightforward. In cases in which the boundary is slightly distorted, a perturbation approach can be attempted. No thought is to be given to establishing convergence. If the situation is one in which your good sense suggests that the small distortion should lead to small changes, attempt the process. The distortion should be smooth and small as compared to the radius of the surface.

The problem statement: The potential $V(\theta)$ is given on a surface defined by $r(\theta) = R\,[1 + \varepsilon f(\theta)]$ where ε is a *small* parameter. The spherical representation for potential fields with azimuthal symmetry is the starting point.

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell\ r^\ell + b_\ell\ r^{-(\ell+1)}\right] P_\ell(\cos\theta)$$
$$= \sum_{\ell=0}^{\infty}\ \sum_{m=0}^{\infty}\left[\varepsilon^m a_{\ell m}\ r^\ell + \varepsilon^m b_{\ell m}\ r^{-(\ell+1)}\right] P_\ell(\cos\theta)$$

Procedure: Beginning with ε^0, match the solution to the boundary conditions order by order noting that the coefficients of each $P_m(\cos\theta)$ must be matched in separately in each order.

Approximate symmetry sample calculation. Given: The potential is constant ($V(\theta) = V_o$) on the surface $r = R\,[1 + \varepsilon\ P_2(\cos\theta)]$. The regions inside of and outside of this surface are charge free. To low order:

$$V_<(r,\theta) \approx \left(a_{00} + \varepsilon\, a_{01} + \ldots\right) r^0 P_0(\cos\theta) + \left(a_{10} + \varepsilon\, a_{11} + \ldots\right) r\ P_1(\cos\theta)$$
$$+ \left(a_{20} + \varepsilon\, a_{21} + \ldots\right) r^2 P_2(\cos\theta) + \ldots..$$

$$V_>(r,\theta) \approx \left(b_{00} + \varepsilon\, b_{01} + \ldots\right) r^{-1} P_0(\cos\theta) + \left(b_{10} + \varepsilon\, b_{11} + \ldots\right) r^{-2} P_1(\cos\theta)$$
$$+ \left(b_{20} + \varepsilon\, b_{21} + \ldots\right) r^{-3} P_2(\cos\theta) + \ldots..$$

Surface: $r \approx R[1 + \varepsilon\, f(\theta)] \qquad r^n \approx R^n[1 + n\varepsilon\, f(\theta) + n(n-1)\varepsilon^2 f(\theta)^2/2 + \ldots]$

Matching the inside solution:

$$V_0 = V_<(r,\theta) \approx \left(a_{00} + \varepsilon\, a_{01}\right) P_0(\cos\theta) + \left(a_{10} + \varepsilon\, a_{11}\right) R[1 + \varepsilon\, P_2(\cos\theta)] P_1(\cos\theta)$$
$$+ \left(a_{20} + \varepsilon\, a_{21}\right) R^2 [1 + 2\varepsilon\, P_2(\cos\theta)] P_2(\cos\theta) + ...$$

To zero order in ε:

$$V_0 = V_0 P_0(\cos\theta) = a_{00}\ P_0(\cos\theta) + a_{10}\, R\, P_1(\cos\theta) + a_{20}\ R^2\, P_2(\cos\theta) + ...$$

Note that the constant V_o is written as $V_o\, P_o(\cos\theta)$ to make the coefficient matching for the Legendre polynomials explicit. It follows that $a_{oo} = V_o$ and that $a_{om} = 0$ for $m > 0$. The matching equation is reduced to:

$$V_0 = V_<(r,\theta) \approx \left(V_0 + \varepsilon\, a_{01}\right) P_0(\cos\theta) + \varepsilon\, a_{11}\, R[1 + \varepsilon\, P_2(\cos\theta)] P_1(\cos\theta)$$
$$+ \varepsilon\, a_{21}\ R^2 [1 + 2\varepsilon\, P_2(\cos\theta)] P_2(\cos\theta) + ...$$

The terms that are first order in ε:

$$0 = \varepsilon\, a_{01}\ R\ P_0(\cos\theta) + \varepsilon\, a_{11}\, R\, P_1(\cos\theta) + \varepsilon\, a_{21}\ R^2\, P_2(\cos\theta) + ...$$

All the coefficients of each $P_\ell(\cos\theta)$ are zero on the left-hand side so $a_{\ell 1} = 0$ in every case. To first order in ε, the potential is constant inside. (Think!)

Matching the outside solution:

$$V_0\, P_0(\cos\theta) \approx \left(b_{00} + \varepsilon\, b_{01}\right) R^{-1} [1 - \varepsilon\, P_2(\cos\theta)] P_0(\cos\theta)$$
$$+ \left(b_{10} + \varepsilon\, b_{11}\right) R^{-2} [1 - 2\varepsilon\, P_2(\cos\theta)] P_1(\cos\theta) + \left(b_{20} + \varepsilon\, b_{21}\right)\ R^{-3} [1 - 3\varepsilon\, P_2(\cos\theta)] P_2(\cos\theta) + ...$$

To zero order in ε:

$$V_0\, P_0(\cos\theta) = b_{00}\, R^{-1} P_0(\cos\theta) + b_{10}\, R^{-2} P_1(\cos\theta) + b_{20}\ R^{-3} P_2(\cos\theta) + ...$$

Matching the coefficients of reach Legendre polynomial, it follows that $b_{oo} = V_o\, R$ and that all other $b_{mo} = 0$.

To first order in ε:

$$0 = V_0\, \varepsilon \left\{ P_2(\cos\theta) P_0(\cos\theta) \right\} + \varepsilon\, b_{01}\, R^{-1} P_0(\cos\theta)$$
$$+ \varepsilon\, b_{11}\, R^{-2} P_1(\cos\theta) + \varepsilon\, b_{21}\ R^{-3} P_2(\cos\theta) + ...$$

In a general situation, the product $P_2(\cos\theta)\, P_0(\cos\theta)$ *must be* expressed as a linear combination of the Legendre polynomials perhaps using the cheap Legendre method. In this case the answer is simple: $P_2(\cos\theta)\, P_0(\cos\theta) \equiv P_2(\cos\theta)$.

$$0 = V_0\, \varepsilon\, P_2(\cos\theta) + \varepsilon\, b_{01}\, R^{-1} P_0(\cos\theta) + \varepsilon\, b_{11}\, R^{-2} P_1(\cos\theta) + \varepsilon\, b_{21}\; R^{-3} P_2(\cos\theta) + ...$$

By inspection, $b_{21} = V_o\, R^3$, and all other $b_{\ell 1} = 0$.

$$V_>(r,\theta) \approx V_0\, (R/r) + \varepsilon\, V_0\, (R/r)^3\; P_2(\cos\theta)$$

The small distortion of the equipotential surface (a shell of conducting material) leaves the potential equal V_o everywhere in the interior, but it contributes a first order perturbation to the potential in the exterior region.

EXERCISE: Why must the potential on the interior remain constant?

Perturbation schemes requiring that the discontinuity of the normal derivative be matched to a surface charge density are significantly more involved as the normal itself is perturbed so the derivative must by calculated consistently to the relevant order of ε as $\vec{\nabla} V \cdot \hat{n}$. No problems of this type are to be considered.

Tools of the Trade

A comment on weight functions: The Legendre polynomials $P_\ell(x)$ described above satisfy the orthogonality relation.

$$\int_{-1}^{+1} P_m(x)\, P_\ell(x)\, dx = \left(\frac{2}{2\ell+1}\right) \delta_{\ell m}$$

In many standard applications, x represents cosθ and so after a change of variable, the orthogonality relation is:

$$\int_0^{\pi} P_m(\cos\theta)\, P_\ell(\cos\theta) \sin\theta\, d\theta = \left(\frac{2}{2\ell+1}\right) \delta_{\ell m}$$

in which case the factor sinθ can be interpreted as a weight function. The cosθ

identification is made for problems in spherical coordinates in which the volume element is $r^2 \sin\theta \, dr \, d\theta \, d\phi$ demonstrating the angular range $d\theta$ is to be weighted by $\sin\theta$ as the area on the surface of a sphere in an angular range $d\theta$ is $2\pi \, r^2 \sin\theta \, d\theta$. There is more land area in a $1°$ band of latitude at the equator than in a $1°$ band through Greenland. Thus $\sin\theta$ is an appropriate weight function.

Cheap expansions in terms of Legendre polynomials: The first few Legendre polynomials are: (This exercise may be more helpful if you replace x by $\cos\theta$.)

$$P_0(x) = 1, \quad P_1(x) = x, \; P_2(x) = \tfrac{3}{2} x^2 - \tfrac{1}{2}, \; P_3(x) = \tfrac{5}{2} x^3 - \tfrac{3}{2} x, \quad P_4(x) = \tfrac{35}{8} x^4 - \tfrac{30}{8} x^2 + \tfrac{3}{8}$$

The task is to represent x^3 as a sum of the Legendre polynomials. The standard method to extract the coefficients is to multiply by one of the polynomials, integrate from –1 to +1 and to invoke the orthogonality relation. This process is brutal and time consuming.

In practice x^n can be represented as a sum of the polynomials or order n or less. Even- or odd-ness is a distinguishing character. So x^3, as 3 is odd, requires only odd polynomials of order less than or equal 3.

$$x^3 = a\left(P_3(x)\right) + b\left(P_1(x)\right) = a\left(\tfrac{5}{2} x^3 - \tfrac{3}{2} x\right) + b\,(x)$$

Match the highest power first: $1\,x^3 = a\left(\tfrac{5}{2} x^3\right)$ or $a = \tfrac{2}{5}$. This choice leaves an excess of $\tfrac{2}{5}\left(\tfrac{-3}{2}\right) x = \tfrac{-3}{5} x = \tfrac{-3}{5} P_1(x)$ which is easily remedied by adding $+\tfrac{3}{5} P_1(x)$. It follows that $x^3 = \tfrac{2}{5}\left(P_3(x)\right) + \tfrac{3}{5}\left(P_3(x)\right)$.

In general, after the highest power n is matched, a multiple of the n-2 polynomial is added to kill off the x^{n-2} part. Onward to n-4 and so on until the 0 or 1 is reached, and the task is accomplished.

In many applications, x represents $\cos\theta$, and it is functions $f(\theta)$ that are to be expanded. In special cases, the function can be re-expressed as a function of $\cos\theta$ and

the cheap expansion technique can be employed. Thus one finds that $f(\theta) = (\sin\theta)^2 \cos\theta$ equals $\cos\theta - (\cos\theta)^3$ and $P_3(\cos\theta) = {}^5/_2\,(\cos\theta)^3 - {}^3/_2.\cos\theta$. So, $f(\theta) = -{}^2/_5\, P_3(\cos\theta) + {}^2/_5 \cos\theta = -{}^2/_5\, P_3(\cos\theta) + {}^2/_5\, P_1(\cos\theta)$.

Conducting Spherical Shell: Charge induced on shell; no other charge nearby. That is: There is an external applied field due to charges located far away that are inducing a surface charge distribution on the shell. This sample calculation assumes that the net charge of the shell is zero. Be aware of this as you consider the potential of the shell.

If $r = R$ is the surface of an conducting spherical shell, then for the field just inside $E_r^< = 0$. The external applied field is taken to be:

$$V_{applied} = \sum_{\ell=0}^{\infty} d_\ell\ r^\ell\ P_\ell(\cos\theta)$$

The total potential is the applied potential plus that due to the induced charge density on the conducting shell.

$$V_>(r,\theta) = \sum_{\ell=0}^{\infty} d_\ell\ r^\ell\ P_\ell(\cos\theta) + \sum_{\ell=0}^{\infty} b_\ell\ r^{-(\ell+1)}\ P_\ell(\cos\theta)$$

Matching at $r = R$,

$$V_>(R,\theta) = \sum_{\ell=0}^{\infty} d_\ell\ R^\ell\ P_\ell(\cos\theta) + \sum_{\ell=0}^{\infty} b_\ell\ R^{-(\ell+1)}\ P_\ell(\cos\theta) = 0$$

The constant potential of the shell was chosen to be zero with little loss of generality. Matching the coefficient of the corresponding Legendre polynomials, $b_\ell = -\,d_\ell\, R^{(2\ell+1)}$.

$$V_>(r,\theta) = \sum_{\ell=0}^{\infty} d_\ell \left(r^\ell - R^{(2\ell+1)}\, r^{-(\ell+1)} \right) P_\ell(\cos\theta)$$

and just outside $E_r^> = -\frac{\partial V^>}{\partial r}\Big|_{r=R} = \frac{\sigma(\theta)}{\varepsilon_0} = -\sum_{\ell=0}^{\infty} d_\ell\,(2\ell+1)R^{(\ell-1)}\ P_\ell(\cos\theta)$ which is computed by differentiating the series for the potential term by term.

For a conducting sphere in a uniform external field E_o in the z direction, $d_1 = -E_o$. Substituting, we find $\sigma(\theta) = -\varepsilon_o(-E_o)(3)P_1(\cos\theta) = 3\,\varepsilon_o E_o P_1(\cos\theta)$. This result agrees with our previous findings.

Zeros of the Bessel functions: Eigenvalues in polar coordinates

The drumhead example used to introduce finding and ordering eigenvalues associated with Bessel functions **is not a Laplace equation example**, but rather it is a Helmholtz equation example.

$$\nabla^2\psi(r,\phi) = k^2\psi(r,\phi)\ ; \quad \Psi(r,\phi,t) = \psi(r,\phi)\cos[kvt+\delta]$$

The drumhead satisfies a wave equation which leads to a Helmholtz equation for the spatial variation after the temporal dependence has been separated.

The identification of the separation constants for Bessel's functions is more involved than was the case for trig functions. We only consider homogeneous boundary conditions. Consider a vibrating circular drum head clamped along the line $r = R$ such that the displacement is zero along that line. One must include all the $J_n(k_{nm}R)$ that satisfy $J_n(k_{nm}R) = 0$. [SL.35]

Table: Zeros of the Bessel Functions

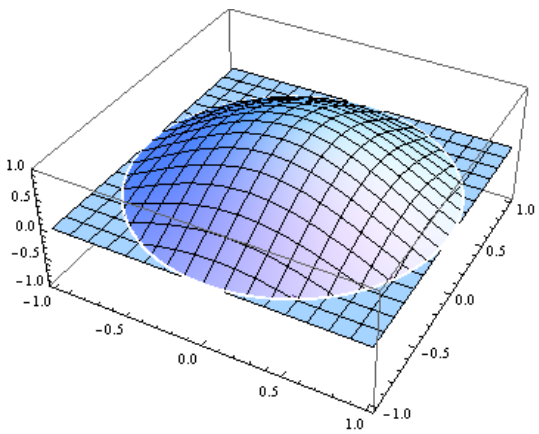
0#	$J_0(x) = 0$	$J_1(x) = 0$	$J_2(x) = 0$	$J_3(x) = 0$	$J_4(x) = 0$
1	2.40483	3.83171	5.13562	6.38016	7.58834
2	5.52008	7.01559	8.41724	9.76102	11.0647
3	8.65373	10.1735	11.6198	13.0152	14.3725
4	11.7915	13.3237	14.7960	16.2235	17.6160
5	14.9309	16.4706	17.9598	19.4094	20.8269

```
N[BesselJZero[0,3]] = 8.65373
```

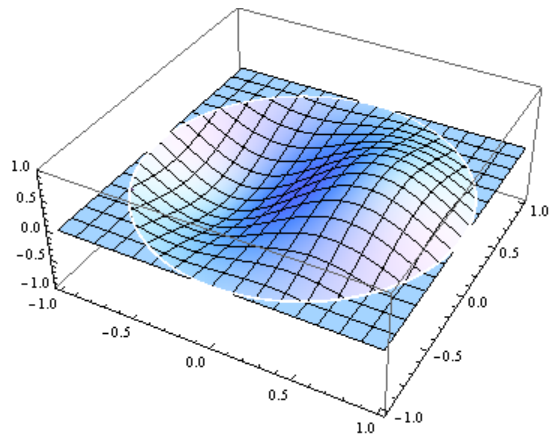
Examining the table, the lowest allowed values of $k_{nm}R$ follow as 2.4048, 3.8317, 5.1356, 5.5201, 6.3802, 7.0156, 7.5883, ... The corresponding vibration patterns of the drumhead are: http://www.youtube.com/watch?v=v4ELxKKT5Rw

Six lowest frequency vibration modes of a circular drumhead

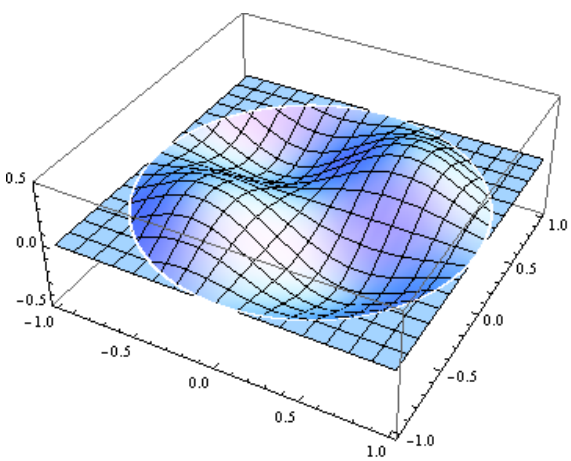
Plot $J_n(k_{nm}\, r)\, \sin(n\phi)$; $m \to m^{\text{th}}$ zero of $J_n(k\, R)$; arbitrary vertical scale



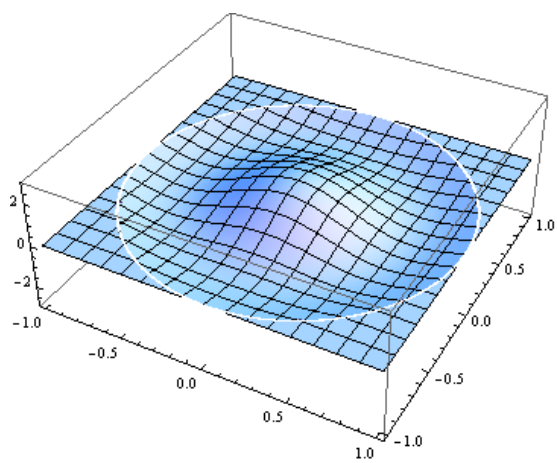
$J_0(2.4048\, r)$



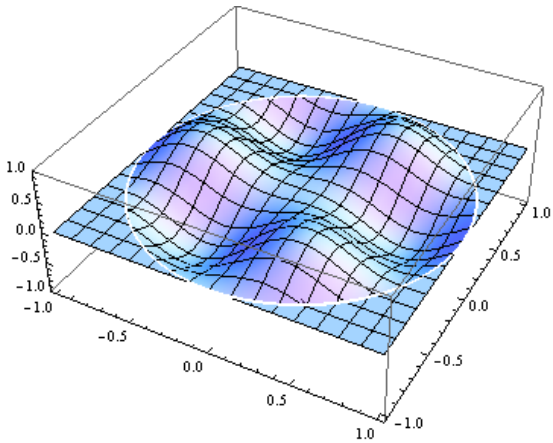
$J_1(3.8317\, r)\, \sin\phi$



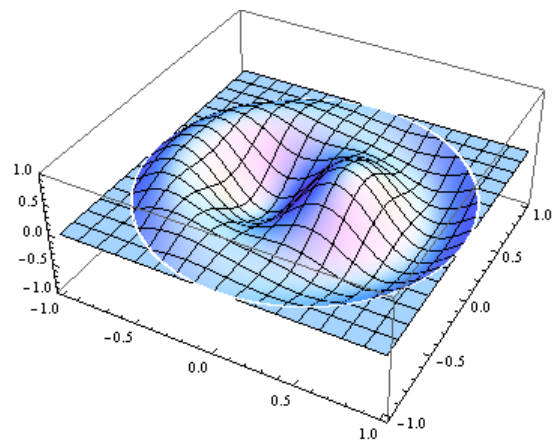
$J_2(5.1356\, r)\, \sin(2\phi)$



$J_0(5.5201\, r)$



$J_3(6.3802\, r)\, \sin(3\phi)$

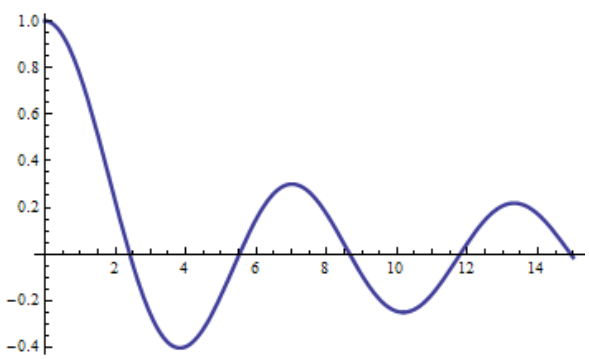


$J_1(7.0156\, r)\, \sin\phi$

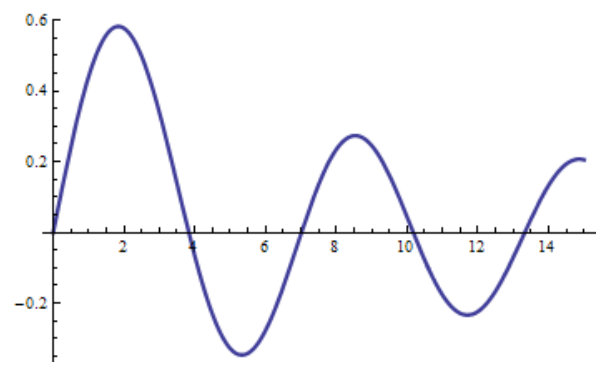
n = 0; m = 0; zm = 5.5201;Plot3D[(UnitStep[1 - Sqrt[x^2 +y^2]])BesselJ[n,zm Sqrt[x^2 +y^2]] ,{x,-1,1},{y,-1,1}, PlotRange → All]

Use: zm from table; sinϕ = x/Sqrt[x^2 +y^2]; sin2ϕ = 2xy/(x^2 +y^2)

Sample plots to find the Bessel functions zeros graphically.



Plot[BesselJ[0,r],{r,0,15}, PlotStyle → Thickness[0.007]]



Plot[BesselJ[1,r],{r,0,15}, PlotStyle → Thickness[0.007]]

Bessel Fourier Series **(**http://en.wikipedia.org/wiki/Fourier%E2%80%93Bessel_series**)**

Because Bessel functions are orthogonal with respect to the ***weight function*** x on the interval $[0, b]$, functions can be expanded in a Fourier–Bessel series defined by:

$$f(x) \sim \sum_{n=0}^{\infty} c_n J_\alpha(\lambda_n x / b)$$

where λ_n is the nth zero of $J_\alpha(x)$. From the orthogonality relationship:

$$\int_0^1 J_\alpha(x\lambda_m) J_\alpha(x\lambda_n)\, x\, dx = \frac{\delta_{mn}}{2}[J_{\alpha+1}(\lambda_n)]^2$$

the coefficients are given by

$$c_n = \frac{\int_0^b J_\alpha(\lambda_n x/b) f(x)\, x\, dx}{\int_0^b x J_\alpha^2(\lambda_n x/b) dx} = \frac{\langle f \mid J_\alpha(\lambda_n x/b)\rangle}{\langle J_\alpha(\lambda_n x/b) \mid J_\alpha(\lambda_n x/b)\rangle}$$

The lower integral may be evaluated, yielding:

$$c_n = \frac{\int_0^b J_\alpha(\lambda_n x/b) f(x)\, x\, dx}{b^2 J_{\alpha\pm1}^2(\lambda_n)/2}$$

where the plus and minus signs is equally valid.

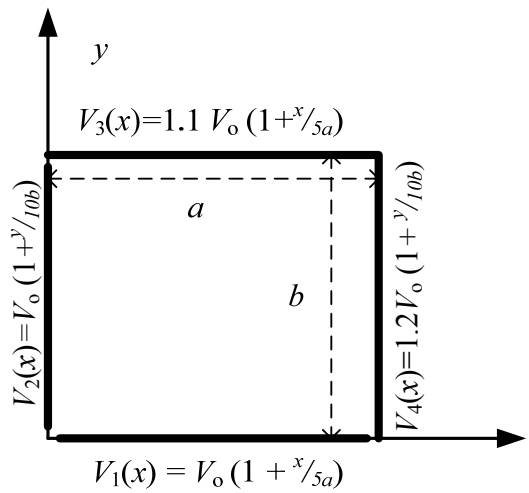
Warm Up Problems

WUP1. a.) Show that $A \sin[kx]\, e^{-cy}$ is a solution of the 2D Laplace (Cartesian) equation provided that a relationship between c and k is satisfied. Identify the required relationship.

b.) Show that $A \cos[kx] \sin[kx]\, e^{-2ky}$ satisfies the 2D Laplace equation. . Brute force works, but attempt a solution based on part a.

WUP2. The general spherical coordinate solution for azimuthal symmetry is $G(r,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell\, r^\ell + b_\ell\, r^{-(\ell+1)}\right] P_\ell(\cos\theta)$. Each $r^\ell P_\ell(\cos\theta)$ and $r^{-(\ell+1)} P_\ell(\cos\theta)$ is a solution of the Laplace equation. Show that $r\cos\theta$, $r^{-2}\cos\theta$, r^{-1} and $r^{-3}\,[\,{}^3/_2 \cos^2\theta - {}^1/_2]$ satisfy the 3D Laplace equation in spherical coordinates by direct substitution.

WUP3. The problem is a (2D) rectangular box with boundary points held at specified potentials. The sides are insulators that are charged to have the specified potentials as functions of position.



The channel has an *x-y* cross section of *a* by *b*, and the potentials along each edge are given on the diagram.

a.) Find the potential at all interior points.

b.) Find the *x* and *y* components of the electric field.

c.) Find the *x* and *y* components of the electric field along the lines $x = a$ and $y = b$.

d.) What type of boundary conditions do we have? Choose for the types discussed as being adequate to ensure uniqueness.

e.) Use $\varepsilon_o \vec{\nabla} \cdot \vec{E} = \rho$ to find the charge density at all points inside the *box*.

f.) Show that there are no relative maxima of minima of $V(x,y)$ in the interior of the channel.

Do not forget to try a zero frequency solution.

Problems

Problems requiring the solution of the Laplace equation are presented in most intermediate physics texts. The problems chosen here are to complement those problems rather than to exhaustively replace or duplicate them.

1.) Use the 2D relaxation model equation to find an approximate solution for the square box with one side held at V_o and the other sides held at 0.

Submit: the spreadsheet of final values, a surface plot of the final values, and lines plots along the centerline row and along the centerline column.

Setup a 21 x 21 array of cells in a spreadsheet (consider the values as 0-20). Set edges to zero except for the left edge row which is to set to values of one except for the end positions which are to be set to one half. Every interior cell is to compute the average value of its four nearest neighbor cells (up, down, left, right).

Set cell D5 to "=(C5+E5+D4+D6)/4"

The spreadsheet must iterate/recalculate. The number format should be set to show at least two paces after the decimal (0.00). Print the spreadsheet in the landscape orientation. What is the value at the mid point? Plot line cuts for the horizontal and vertical cuts through the mid point. Pressing the F9 key executes an iteration of the

calculations in an Excel spreadsheet with circular references. It is instructive to step through the iterations and to observe the values as they relax to the solution set. Go to the Tools Menu to Options to Calculation. Set Manual and Maximum Iterations = 1. Hit F9. Repeat, repeat, ….. Increase Maximum Iterations. Continue.

Excel Instructions for iterative solution of the Laplace equation. The procedure includes using circular references. Go to the Tools Menu to Options to the Calculation tab. Set Manual and Maximum Iterations = 1. Turn OFF recalculate on save. Build the spread sheet with zeros in all the boundary position. Paste zero at every interior point. Enter the average of neighbors procedure at an interior point (say for B6 ⇒ = (A6 + C6 + B5 + B7)/4). Paste the formula at all the remaining interior points. Insert Menu … Chart …. Surface ….. As a new sheet. Save the worksheet and set its properties to read only. Open the sheet. Change the boundary conditions to be those desired. Each corner point should be set to the average value to the two sides meeting at that corner. Save the starting point with a new name. To command excel to execute an iteration set, hit F9. Repeat, repeat, ….. . Watch the values develop in the spreadsheet and on the chart. Increase Maximum Iterations. Continue.

2.) Use the 2D relaxation model equation to approximate and to **plot** the solution for the slot with the short side at V_o and the top/bottom held at 0. Setup a 21 x 300 array of cells in a spreadsheet. Set edges to zero except for the 21 element left edge row which is to set to values of one except for the end positions which are to be set to one half.. Every interior cell is to compute the average value of its four nearest neighbor cells (up, down, left, right). The spreadsheet must iterate/recalculate. Compare the results with the Griffiths result $V(x,y) = \frac{2V_0}{\pi} \tan^{-1}\left[\frac{\sin(\pi y/b)}{\sinh(\pi x/b)}\right]$. Consider the index values as 0-20 with b = 20. Pressing the F9 key executes an iteration of the calculations in an Excel spreadsheet with circular references. It is instructive to step through the iterations and to observe the values as they relax to the solution set. Go to the Tools Menu to Options to Calculation. Set Manual and Maximum Iterations = 1. Hit F9. Repeat, repeat, ….. Increase Maximum Iterations. Continue.

3.) a.) Establish the relaxation approximation in three dimensions:

$$V(x,y,z) \cong \frac{V(x+\Delta,y,z)+V(x-\Delta,y,z)+V(x,y+\Delta,z)+V(x,y-\Delta,z)+V(x,y,z+\Delta)+V(x,y,z-\Delta)}{6}$$

b.) Derive the *relaxation approximation equation* in 2D that is correct for points distributed on a rectangular grid. That is: for points spaced by a in the x-direction and spaced by b in the y-direction. Generalize your result to the form for a 3D problem on a rectangular a-b-c grid rather than s *square* grid with the points evenly spaced in the three dimensions. $(x_k, y_m, z_n\} = \{ x_o + k\,a,\ y_o + m\,b,\ z_o + n\,c \}$

Start by finding an expression for the second derivative with respect to x using values on a grid of point with spacing a. For a 2D, Δ-spaced grid, the forms are:

$$\left.\frac{\partial V}{\partial x}\right|_{(x+\Delta/2,y)} \cong \frac{V(x+\Delta,y)-V(x,y)}{\Delta} \qquad \left.\frac{\partial V}{\partial x}\right|_{(x-\Delta/2,y)} \cong \frac{V(x,y)-V(x-\Delta,y)}{\Delta}$$

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{(x,y)} \cong \frac{\left.\frac{\partial V}{\partial x}\right|_{(x+\Delta/2,y)} - \left.\frac{\partial V}{\partial x}\right|_{(x-\Delta/2,y)}}{\Delta} = \frac{V(x+\Delta,y)-2V(x,y)+V(x-\Delta,y)}{\Delta^2}$$

Answer:

$$V(x,y,z) \cong \frac{b^2c^2[V(x+a,y,z)+V(x-a,y,z)]}{2[a^2b^2+b^2c^2+a^2c^2]} + \frac{a^2c^2[V(x,y+b,z)+V(x,y-b,z)]}{2[a^2b^2+b^2c^2+a^2c^2]} + \frac{a^2b^2[V(x,y,z+c)+V(x,y,z-c)]}{2[a^2b^2+b^2c^2+a^2c^2]}$$

4.) For the slot problem, the expansion terms in the solution have the form: $V_m(x,y) \to A_m \sin(m\pi y/b)\, e^{-m\pi x/b}$. The boundary condition at $x = 0$ could be set to $V(0,y) \to V_0 \sin(m\pi y/b)$ leading to the solution: $V(x,y) = V_0 \sin(m\pi y/b)\, e^{-m\pi x/b}$. Use the *average property* and the *relaxation equation* to explain why a higher frequency variation along the left boundary should lead to a more rapid decay with respect to increasing x.

Discovery Exercise for SL4: Draw the slot with two full cycles of a sine wave across the left end. Prepare a second sketch using the same slot width with 5 full cycles across the left end. Draw a line in the x direction than begins at one of the maxima of the sine wave near the centerline of the slot. Place points along that line at intervals of 0.1 of the slot height. Begin at the left end and work to the right. Sketch a circle around each point that is just tangent to the left end of the slot. Imagine the average of the potential values around the circumference of that circle. At what distance from the left end does the average begin to contain positive and negative values; almost as much negative as positive. Discuss the falloff in terms of the averaging property.

5.) Consider the solution to the 2D Laplace equation in polar coordinates for an annular region with inner radius a and outer radius b. The potential is specified on the inner and outer boundaries of the region.

$$V(a,\phi) = f(\phi) \qquad\qquad V(b,\phi) = g(\phi)$$

Give the sum of terms that represents the general form of a physically allowed solution in the annular region. Describe a strategy for matching the boundary conditions to evaluate the undetermined constants in the general solution. HINT: Review the comments on strategies for solving Cartesian box problems.

6.) In the absence of ϕ dependence and requiring solutions to be finite at $\theta = 0$ and π, the general form of a solution to the Laplace equation in spherical coordinates is: $G(r,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell\ r^{\ell} + b_\ell\ r^{-(\ell+1)}\right] P_\ell(\cos\theta)$. a.) Give the physically allowed form of the solution for the region inside a sphere of radius R. b.) A point charge has a potential that varies as r^{-1} so perhaps a term like $b_o\, r^{-1}$ should be allowed *inside*. Given that $V(r, \theta)$ is to satisfy the Laplace equation in the region $r < R$, explain why a point charge at the origin is not to be allowed.

7.) In the absence of ϕ dependence and requiring solutions to be finite at $\theta = 0$ and π, the form of a solution to the Laplace equation in spherical coordinates is:

$G(r,\theta)=\sum_{\ell=0}^{\infty}\left[a_\ell\ r^{\ell}+b_\ell\ r^{-(\ell+1)}\right]P_\ell(\cos\theta)$. Give the physically allowed form of the solution for the region outside a sphere of radius R.

8.) In the absence of ϕ dependence and restricting to solutions that are finite for $\theta = 0$ and π, the form of a solution to the Laplace equation in spherical coordinates is:

$G(r,\theta)=\sum_{\ell=0}^{\infty}\left[a_\ell\ r^{\ell}+b_\ell\ r^{-(\ell+1)}\right]P_\ell(\cos\theta)$. Give the physically allowed form of the solution for the region between spherical surfaces concentric with the origin and with radii of a and b.

SL9.) a.) Prove that if $\sum_{\ell=0}^{\infty} A_\ell\ P_\ell(\cos\theta)=\sum_{k=0}^{\infty} B_k\ P_k(\cos\theta)\ for\ 0<\theta<\pi$ then $A_m = B_m\ \forall\, m$.

Use the orthogonality relation for the Legendre polynomials to project out A_j and B_j.

Use the projection procedure, not the vector space result that the expansion coefficients in terms of a linearly independent set are unique. (A mutually orthogonal set of vectors are linearly independent.)

$$\int_0^{\pi} P_m(\cos\theta)\ P_\ell(\cos\theta)\sin\theta\ d\theta=\left(\frac{2}{2\ell+1}\right)\delta_{\ell m}$$

b.) The Legendre polynomials of $\cos\theta$ can be used as an expansion set for functions of θ.

$$f(\theta)=\sum_{\ell=0}^{\infty} A_\ell\ P_\ell(\cos\theta)\ for\ 0<\theta<\pi$$

Use the orthogonality relation to generate an expression for the coefficient A_p. The procedure is analogous to that used to develop the equation to compute the Fourier coefficient a_p.

c.) Express $\sin^2\theta$ as a Legendre-Fourier series. $[\sin(\theta)]^2=\sum_{\ell=0}^{\infty} A_\ell\ P_\ell(\cos\theta)$

10.) A Cartesian Laplace equation problem sets the infinite plane $x = 0$ at potential 0 and the infinite plane $x = a$ at potential V_0. As there is no dependence on either y or z,

the corresponding separation constants are zero. Set the constant in

$\frac{d^2X(x)}{dx^2} = C_x \; X(x)$ to the value appropriate for this problem. Find the general form of the solution and match the boundary conditions to evaluate the undetermined constants. Make a drawing to display the $^1/_3\, V_0$ and $^2/_3\, V_o$ equipotentials and a few field lines.

11.) The interior of a spherical shell of radius R is charge free, and the potential on its surface is described by $V = V_o \cos^2\theta$.

(a) Determine the functional description of $V(r,\theta)$ for $r < R$, and write down the function for $r < R$, $V(r, \theta) =$

_____

(b) Determine the functional description of $V(r,\theta)$ for $r > R$, and write down the function for $r > R$, $V(r, \theta) =$

_____

(c) Determine the **surface charge density** $\sigma(R,\theta)$ on the spherical shell at $r = R$.

11.ext) In the problem above, it results that $\frac{\sigma_{P_2}}{\sigma_{P_0}} = \{2(2)+1\}\frac{V_{P_2}}{V_{P_0}}\bigg|_{r=R}$. Repeat problem 11 for a potential $V_o \cos^3\theta$ to find out if $\frac{\sigma_{P_3}}{\sigma_{P_1}} = \left\{\frac{3(3)+1}{1(1)+1}\right\}\frac{V_{P_3}}{V_{P_1}}\bigg|_{r=R}$. If you work this problem, report the result to tank.

12.) A standard slot problem with no z dependence has a solution which is a sum of terms of the form: $[\, A_n e^{k_n x} + B_n e^{-k_n x}]\, [\, C_n \sin(k_n y) + D_n \cos(k_n y)\,]$. A solution inside the slot valid for $\{0<x<\infty;\ 0<y<a;\ \text{all } z\}$ is sought. The boundary conditions restrict the possible values for the symbols A_n, B_n, C_n, D_n and k_n.

(a) The condition that the solution is to be physically well-behaved as $x \to \infty$ requires that the constant _____ be restricted to the value(s) _____.

(b) The condition that $V = 0$ for $y=0$ requires that the constant _____ be restricted to the value(s) _____.

(c) The condition that for $y = a$, $V = 0$ requires that the constant _____ be restricted to the value(s) _____.

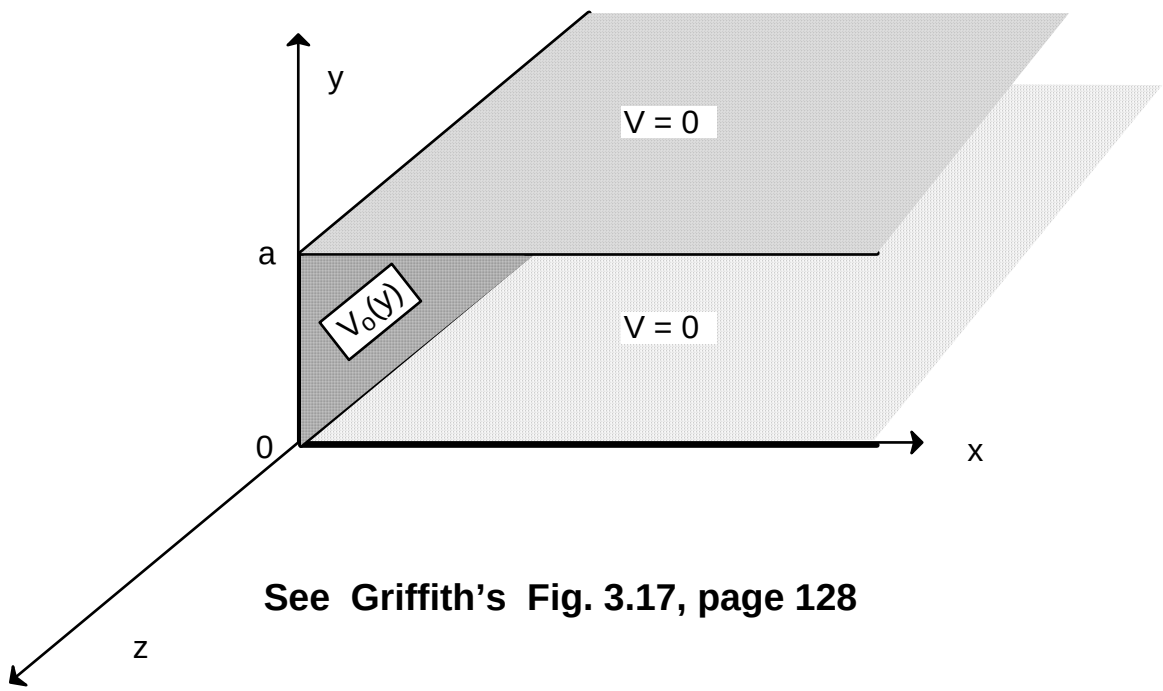
(d) Given that $V(x,y) = \Sigma\, F_n\, e^{-(n\,\pi\, x\,/a)}\, sin[n\,\pi\, y/_a]$ and that $V(0,y) = V_0(y)$ $= \mathcal{V}_0 \sin[\pi y/_a]$, find the F_n for all n and hence, for points in the slot.

$V(x,y) =$ _____

NOTE: The Griffiths slot has a constant potential along the left edge. This problem has a figure similar to Griffiths but differs in the fine details of $V_o(y)$.

(e) Find the surface charge density $\sigma(x)$ on the conducting plane at $y = 0$.

In electrostatics problems, the electric field just outside a conductor has magnitude to the local surface charge density on the conductor divided by the permittivity of free space and is directed normally away (for a positive charge density). $\vec{E}_{just\ outside} = {}^{\sigma_{local}}\!/_{\varepsilon_0}\, \hat{n}_{away.outward}$ *hence* $\sigma_{local} = \varepsilon_0\, E_n = -\varepsilon_0 \frac{\partial V}{\partial n} \rightarrow -\varepsilon_0 \frac{\partial V}{\partial y}$ *on the lower surface of the slot (at y = 0).*



See Griffith's Fig. 3.17, page 128

13.) The potential due to a point charge on the +z-axis at $z = d$ was expanded in terms of the solutions to the Laplace equation in a *Sample Calculation* for the case $r > d$.

a.) Reproduce the expansion for the $r > d$ case.

b.) Find the corresponding expansion for $r < d$.

c.) Identify $d\,\hat{k}$ as $\vec{r}'$ and note that the function that has been expanded is:

$\left(\frac{q}{4\pi\varepsilon_0}\right)\frac{1}{|\vec{r}-\vec{r}'|}$ and that $|\vec{r}-\vec{r}'|^{-1}$ satisfies the Laplace equation if $r > r'$ or if $r < r'$.

There is no loss of generality as the direction of the polar axis can be chosen as desired. The cosine of the angle between the primed and unprimed position vectors is $\hat{r}\cdot\hat{r}'$ .As the solution to the Laplace equation is unique if the values match on the boundary, conclude that:

$$\frac{1}{|\vec{r}-\vec{r}'|}=\sum_{\ell=0}^{\infty}\frac{(r')^{\ell}}{r^{(\ell+1)}}\,P_\ell(\hat{r}\cdot\hat{r}')\ \textit{for}\ r'<r \text{ and } \frac{1}{|\vec{r}-\vec{r}'|}=\sum_{\ell=0}^{\infty}\frac{(r)^{\ell}}{r'^{(\ell+1)}}\,P_\ell(\hat{r}\cdot\hat{r}')\ \textit{for}\ r'>r$$

These results may be summarized as:

$$\frac{1}{|\vec{r}-\vec{r}'|}=\sum_{\ell=0}^{\infty}\frac{(r_<)^{\ell}}{r_>^{(\ell+1)}}\,P_\ell(\hat{r}\cdot\hat{r}')$$ where $r_>$ ($r_<$) is the greater (lesser) of r and r'.

14.) The potential due to a uniform ring of charge of radius a concentric with the origin and in the x-y plane was expanded in terms of the solutions to the Laplace equation in a *Sample Calculation* for the case $r > a$. Find the corresponding expansion for $r < a$.

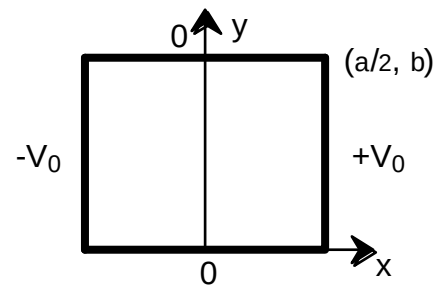
15.) Consider the prospects for an $r < a$ expansion of the potential due to a circular disk with uniform surface charge density σ and radius a concentric with the origin and in the x-y plane in terms of the solutions of the Laplace equation as requested in the problems just above this one. Discuss these prospects.

$$V(z)=\frac{\sigma}{2\varepsilon_0}\left[\sqrt{a^2+z^2}-|z|\right]$$

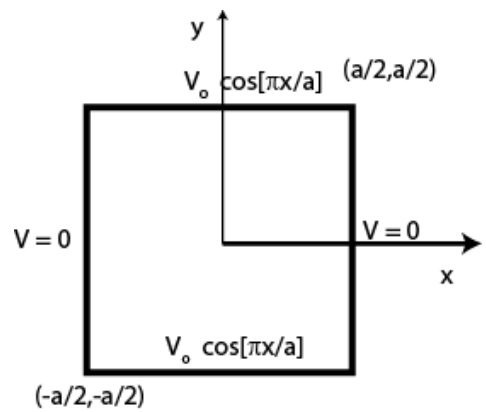
See problem 34 for the full calculation.

16.) Find the potential at points in the charge free interior of the 2D box illustrated. This problem requires fitting a function of y on the boundaries with $x = \pm ½\, a$, so the sinusoids are to be used for the y-

dependence. The problem is anti-symmetric in x. The box is held at 0 potential on the top and bottom and $-V_0$ at $x = -\frac{1}{2}a$ and $+V_0$ at $x = +\frac{1}{2}a$. Express $V(x,y)$ as a series expansion for points in the box.



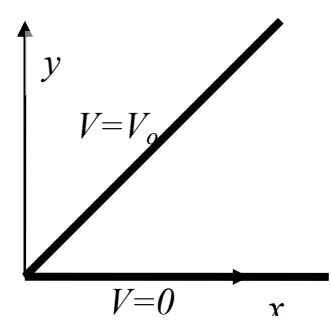
16.+1) Pollock & Stump 5.3. A long channel with square cross section is held at $V = V_o \cos[{}^{\pi x}/_{a}]$ long the sides at $y = \pm{}^{a}/_{2}$ and the conducting sides at $x = \pm{}^{a}/_{2}$ are held at V = 0. Find the potential at points inside the channel. Compute the electric field inside the channel. Why must the field be perpendicular to the sides at $x = \pm{}^{a}/_{2}$?



The electric field is not perpendicular to the sides at $y = \pm{}^{a}/_{2}$. Explain that behavior. Use the relation $\sigma = \varepsilon_o E_n$ to find the charge density on the sides at $x = \pm{}^{a}/_{2}$. Sketch field lines. Note that the sides at $y = \pm{}^{a}/_{2}$ are not conductors*. A *long channel* means that there is no z dependence.

* What angle does an electric field line make with the surface of a conductor in electrostatic equilibrium? Conductors are, of course, equipotentials.

17.) A problem in cylindrical coordinates has no z-dependence. The plane $\phi = 0$ is held at 0 potential and the plane $\phi = {}^{\pi}/_{4}$ at potential V_0. Find $V(r,\phi)$ for all r and $0 < \phi < {}^{\pi}/_{4}$. Sketch equipotentials for ${}^{1}/_{3}V_0$ and for ${}^{2}/_{3}V_0$ plus a few field lines. Compute: $\vec{E}(r,\phi) = -\vec{\nabla}V(r,\phi)$.



Details: In polar coordinates consider the Laplace equation problem with the constant coordinate half-plane $\phi = 0$ held at potential 0 and the constant coordinate half-plane $\phi = \phi_0$ held at potential V_o. Begin with the general solution form:

$$V(r,\phi) = \left[A_0 + B_0 \ln(r)\right]\left[C_0 + D_0 \ \phi\right] + \sum_{m=1}^{\infty}\left[A_m r^m + B_m r^{-m}\right]\left[C_m \sin(m\phi) + D_m \cos(m\phi)\right]$$

Note that the $D_0 \ \phi$ form is allowed as long as $\phi_o < 2\pi$. Exclude terms that diverge as $r \to 0$ or as $r \to \infty$. Boundary-value-match to set the remaining constants. [The problem would almost make sense with $\phi_o = 2\pi$ if that plane were occupied by a lipid membrane such as those in the walls of living cells. These lipid layers maintain a potential difference of about 70 mV across their 'microscopic' thickness.]

18.) Consider the Cartesian slot problem (see # 12). Suppose that the potential along the left boundary is $V_o(y) = V_o \sin({}^{m\,\pi y}/_{a})$. Find $V(x, y)$ for points in the slot. Find the electric field at points in the slot. The surface charge density on a conductor is the permittivity of free space times the normal component of the electric field just outside the conductor. a.) Assume that the potential at the left edge is set on the surface of a huge number of conducting strips running parallel to the z axis. **Set** $m = 1$. Find the effective surface charge density $\sigma(0, y)$ on the left edge on the slot. Find the charge density $\sigma(x, 0)$ of the conductor at the bottom of the slot ($x = 0$). Compute the net charge per unit length in the z direction on the surfaces. $\{ Q_{left\,in\,length\,L} = L\int_0^a \sigma(0,y)\ dy$; $Q_{lower\,in\,L} = L\int_0^\infty \sigma(x,0)\ dx\ \}$ Discuss the results.

b.) The model for the structure of the $x = 0$ edge may be too artificial. Compute the net flux of the electric field out of the strip at $x = 0$ from a length L. Relate that to the charge per length L on the top and bottom surfaces ($y = 0$ and $y = a$).

19. **Thin Insulating Charged Shell**: No externally applied field. Given $r = R$ is an thin insulating spherical shell with an azimuthally symmetric surface charge density $\sigma(\theta)$, show that the coefficients in the $r < R$ region can be computed as:

$$a_m = -\frac{1}{2\,\varepsilon_0\,R^{(m-1)}}\int_0^{\pi}\sigma(\theta)\;P_m(\cos\theta)\,\sin\theta\;d\theta$$

20. **Dielectric Sphere**: No externally applied field. Given that the region $r < R$ is filled with a uniform linear dielectric with constant ε_{r1} and the region $r > R$ is filled with a uniform linear dielectric with constant ε_{r2}, show that the coefficients in the $r < R$ region can be computed as:

$$a_m = -\frac{2m+1}{2\left[(\varepsilon_{r1}+\varepsilon_{r2})m+\varepsilon_{r2}\right]\varepsilon_0\,R^{(m-1)}}\int_0^{\pi}\sigma_{free}(\theta)\;P_m(\cos\theta)\,\sin\theta\;d\theta$$

This result has not been verified.

21. **Conducting Shell**: (In a uniform applied field) If $r = R$ is the outer surface of a conducting sphere in a uniform applied electric field $E_0\,\hat{k}$, then with $V(R) = 0$.

$$V^{>}(r,\theta) = E_0\left[\frac{R^3}{r^2} - r\right]\cos\theta = E_0\left[\frac{R^3}{r^2} - r\right]P_1(\cos\theta)$$

This is just the external potential plus that due to the electric dipole moment induced on the sphere. Find the surface charge density $\sigma(\theta)$ on the conducting sphere. Use the equation, $\frac{\sigma(\theta)}{\varepsilon_0} = E_r^{>} - E_r^{<} = -\frac{\partial V^{>}}{\partial r}\Big|_{r=R} + -\frac{\partial V^{<}}{\partial r}\Big|_{r=R}$. Discuss its application. Use the equation, $\vec{p} = \int \rho(\vec{r})\;\vec{r}\;dV \rightarrow \int \sigma(\theta)\;\vec{r}(\theta,\phi)\;R^2\sin\theta\,d\theta\,d\phi$ where $\vec{r}(\theta,\phi) = R\left[\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\;\hat{k}\right]$ to compute the electric dipole moment induced on the sphere. What would be the potential due to that moment at points $r > R$?

22. **Dielectric Sphere**: (In a uniform applied field)

The region $r < R$ is filled with a uniform linear dielectric with constant $\varepsilon_{r1} = \varepsilon_r$ and the region $r > R$ is filled with a uniform linear dielectric with constant $\varepsilon_{r2} = 1$ and if σ_{free}, the free surface charge density on the insulating dielectric sphere, is zero, find $V^{>}(r,\theta)$ and $V^{<}(r,\theta)$.

$$V^{>}(r,\theta) = -E_0\, r\cos\theta + \sum_{\ell=0}^{\infty} b_\ell\ r^{-(\ell+1)}\ P_\ell(\cos\theta)$$

Find $\vec{E}^{<}(r,\theta)$. Sketch it. The field $\vec{E}^{>}(r,\theta)$ is the superposition of the applied electric field and the field due to the dipole induced on the dielectric sphere. Given that σ_{free} =0, use $\frac{\sigma(\theta)}{\varepsilon_0} = E_r^{>} - E_r^{<} = -\frac{\partial V^{>}}{\partial r}\Big|_{r=R} + -\frac{\partial V^{<}}{\partial r}\Big|_{r=R}$ to find the polarization charge density on the surface of the dielectric sphere. Use the equation,

$$\vec{p} = \int \rho(\vec{r})\ \vec{r}\ dV \rightarrow \int \sigma(\theta)\ \vec{r}(\theta,\phi)\ R^2 \sin\theta\, d\theta\, d\phi$$

where $\vec{r}(\theta,\phi) = R\left[\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\right]$ to compute the electric dipole moment induced on the sphere. What would be the potential due to that moment at points $r > R$?

23. Review the two-dimensional slot problem. Griffiths presents a closed form representation of the solution in his E&M text.

$$V(x,y) \ = \frac{2V_0}{\pi}\ \tan^{-1}\left[\frac{\sin(\pi y/b)}{\sinh(\pi x/b)}\right]$$

Show that this function satisfies the boundary conditions. Next, show that

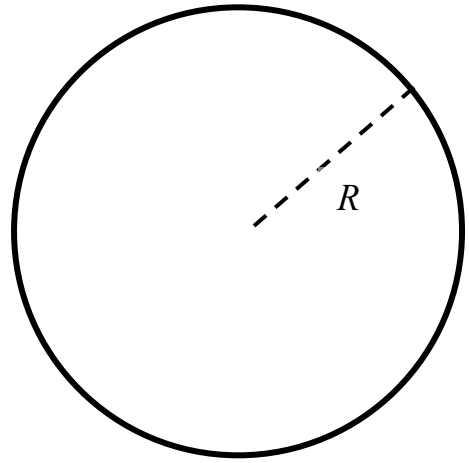
$$G(x,y) \ = \ \tan^{-1}\left[\frac{\sin(y)}{\sinh(x)}\right]$$

is a solution of the 2D Laplace equation. [Maple, Mathematica or some other symbolic tool is recommended for this part.] If none is available, adopt the common denominator: $\left[1+\left(\operatorname{csch}(x)\ \sin(y)\right)^2\right]^2$. It then follows from uniqueness that:

$$V(x,y) \;=\frac{2V_0}{\pi}\;\tan^{-1}\left[\frac{\sin(\pi y/b)}{\sinh(\pi x/b)}\right] \;=\sum_{m=0}^{\infty}\frac{4V_0}{[2m+1]\pi}\;e^{-([2m+1]\pi x/b)}\;\sin([2m+1]\pi y/b)$$

Note that, with some effort, the series can be summed to establish the equality.

24. A spherical shell of radius R has surface charge while the regions $r<R$ and $r>R$ are charge free. The problem has azimuthal symmetry ($\Rightarrow$ no dependence on ϕ). The surface charge on the sphere sets its potential to be $V(\mathrm{R}, \theta) = \mathcal{V}_0 \sin^2\theta$. Assume that the potential vanishes as $r \to \infty$.



a.) Name the equation that $V(\theta)$ satisfies for $r < R$ and for $r > R$.

b.) Give the sum that represents the allowed form of the inside solution (for $r<R$).

c.) Give the sum that represents the form of the outside solution (for $r>R$).

d.) Express $\sin^2\theta$ in terms of $\cos\theta$.

e.) Express $\sin^2\theta$ in terms of the $P_\ell(\cos\theta)$.

f.) Find $V(r, \theta)$ for $r < R$.

g.) Find the form of E_r for $r < R$.

h.) Find the surface charge density $\sigma(\theta)$ on the shell. See sample calculation SC3.

$$E_r^{>}(R,\theta) - E_r^{<}(R,\theta) = \frac{\sigma(\theta)}{\varepsilon_0} \quad \text{or} \quad \left.\frac{\partial V^{<}}{\partial r}\right|_{r=R} - \left.\frac{\partial V^{>}}{\partial r}\right|_{r=R} = \frac{\sigma(\theta)}{\varepsilon_0}$$

25. The potential is constant ($V(\theta) = V_o$) on the surface $r = R\,[1 + \varepsilon\,\cos\theta]$. The regions inside of and outside of this surface are charge free. Use the perturbation approach to find the potential inside and outside the surface through all terms up to

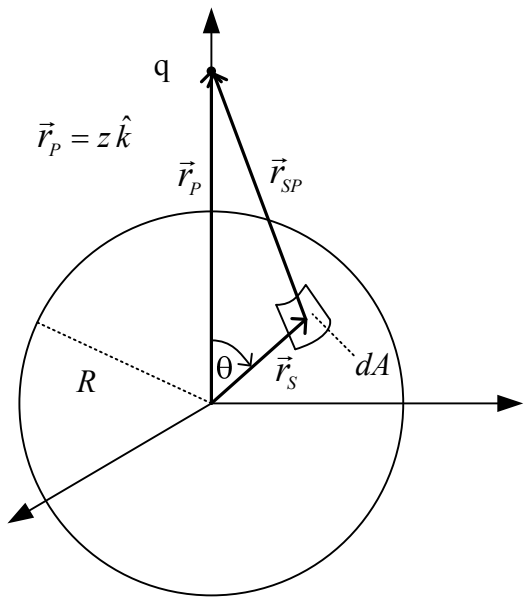
first order in ε. Begin by expressing $\cos\theta$ in terms of the Legendre polynomials of $\cos\theta$.

26. The potential is constant ($V(\theta) = V_o$) on the surface $r = R\,[1 + \varepsilon\ \cos\theta]$. The regions inside of and outside of this surface are charge free. Use the perturbation approach to find the potential inside and outside the surface through all terms up to first order in ε. Begin by expressing $\cos\theta$ in terms of the Legendre polynomials of $\cos\theta$. Repeat for $V(\theta) = V_o \cos\theta$.

27. The potential is constant ($V(\theta) = V_o$) on the surface $r = R\,[1 + \varepsilon\ P_2(\cos\theta)]$. The regions inside of and outside of this surface are charge free. Use the perturbation approach to find the potential inside and outside the surface through all terms up to first order in ε.

28. The potential is $V(\theta) = V_o \cos\theta$ on the surface $r = R\,[1 + \varepsilon\ P_2(\cos\theta)]$. The regions inside of and outside of this surface are charge free. Use the perturbation approach to find the potential inside and outside the surface through all terms up to first order in ε.

29. The potential due to a point charge is to be averaged over a spherical surface of radius R.



A point charge is located a distance z up the polar axis. The average value of the potential due to that charge is to be computed over the surface of a sphere of radius R centered on the origin. The distance from the charge to a patch on the surface of the sphere is, by the law of cosines, $\sqrt{R^2 + z^2 - 2\,z\,R\cos\theta}$. The potential on the surface is: $\left[\, q / \left(4\,\pi\,\varepsilon_0 \sqrt{R^2 + z^2 - 2\,z\,R\cos\theta}\right)\right]$.

Compute V_{ave} for $z > R$ and for $z < R$. In which case does V satisfy the Laplace equation inside the shell. What is the potential at the origin due to the single point charge on the z axis? The average value of the potential over the sphere is computed as:

$$V_{ave} = \frac{1}{4\,\pi\,R^2}\int_0^{\pi}\int_0^{2\pi} \frac{q R^2 \sin\theta\; d\theta\; d\phi}{4\,\pi\,\varepsilon_0 \sqrt{R^2 + z^2 - 2\,z\,R\cos\theta}} = \frac{1}{2}\int_0^{\pi} \frac{q\;\sin\theta\;d\theta}{4\,\pi\,\varepsilon_0 \sqrt{R^2 + z^2 - 2\,z\,R\cos\theta}}$$

29½.) Consider a insulating spherical shell of radius R with charges distributed on and exterior to the shell to set its potential to $V(R,\theta)$. The interior is charge-free so the potential satisfies the Laplace equation in the interior.

a.) What is the general form of the expansion for the potential in the interior expressed in its spherical coordinate form for no ϕ dependence?

b.) Specialize the expansion to $r = 0$ to find the potential at $r = 0$ in terms of the expansion.

c.) Specialize the general solution to $r = R$ and set it equal to $V(R,\theta)$. Project out the coefficient of $P_o(\cos\theta)$ using the orthogonality relation for the Legendre polynomials of $\cos\theta$.

d.) The average property states that the value of a solution to the Laplace equation at a point is equal to the average of the values of that solution at a full set of equally distant points. Do your results support the proposition that the potential at the center of the sphere is the average of the value of the solution as averaged over all the points on a concentric spherical shell?

$$V(r,\theta)\big|_{r=0} = \frac{1}{4\pi R^2}\int_0^{\pi}\int_0^{2\pi} V(R,\theta)\, R^2 \sin\theta\, d\theta\, d\phi$$

30.) Generate an $r > a$ expansion of the potential due to a uniform circular disk of charge of radius a concentric with the origin and in the x-y plane in terms of the solutions of the Laplace equation in spherical coordinated with no ϕ dependence.

31.) Use the first four terms in the expansion representing the potential due to a point charge q at $a\ \hat{k}$ to compute the electric field at $10\ a\ \hat{j} + 11\ a\ \hat{k}$. Compare your result to the exact result computed using Coulomb's Law.

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\right)\frac{a^{\ell}}{r^{(\ell+1)}}\ P_{\ell}(\cos\theta)$$

32.) For a general far-field expansion representing the potential in terms of the solutions of the Laplace equation in spherical coordinated with no ϕ dependence, compute the term-by-term form of the electric field. $V(r,\theta) = \sum_{\ell=0}^{\infty} A_{\ell}\ r^{-(\ell+1)}\ P_{\ell}(\cos\theta)$

Use: $$\frac{dP_n(\cos\theta)}{d\theta} = (\sin\theta)\ \frac{\left[n\ \cos(\theta)\ P_n(\cos\theta) - n\ \ P_{n-1}(\cos\theta)\right]}{1-(\cos\theta)^2}$$

Do not despair just because the result is ugly.

33.) Compute $\frac{dP_2(\cos\theta)}{d\theta}$ and $\frac{dP_3(\cos\theta)}{d\theta}$. Compare with the results found using the identity: $\frac{dP_n(\cos\theta)}{d\theta} = (\sin\theta)\ \frac{\left[n\ \cos(\theta)\ P_n(\cos\theta) - n\ \ P_{n-1}(\cos\theta)\right]}{1-(\cos\theta)^2}$

33.) Charge is glued on the surface of a spherical shell of radius r in such a fashion that the potential on the shell is $V_o \cos(2\theta)$.

a.) Express $V_o \cos(2\theta)$ in the form $\sum_{\ell=0}^{\infty} a_\ell\ \ P_\ell(\cos\theta)$.

b.) Find the series representations for the potential in the regions $r < R$ and $r > R$. Assume that space is charge-free except for that glued on the shell.

c.) Find the charge density $\sigma(\theta)$ on the surface of the shell. Use the equation:

$$\frac{\sigma(\theta)}{\varepsilon_0} = E_r^> - E_r^< = -\frac{\partial V^>}{\partial r}\bigg|_{r=R} + \left(-\frac{\partial V^<}{\partial r}\bigg|_{r=R}\right).$$

Answer: $\sigma(R,\theta) = (\varepsilon_o V_o/3R)\ [20\ P_2(\cos\theta) - 1\ P_0(\cos\theta)]$

34.) a.) Compute the potential of the z axis due to a uniformly charged circular disk of radius a and charge density σ centered in the x-y plane.

$$V(\vec{r}) = \int \frac{\sigma\, dA}{4\pi\,\varepsilon_0\,|\,\vec{r}_p - \vec{r}_s\,|}.$$

b.) Use the binomial theorem to generate ($z>a$) and ($z<a$) expansions for the potential on the z axis.

c.) Match these expansions with the forms $V(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell\ r^{-(\ell+1)}\ P_\ell(\cos\theta)$ and $V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell\ r^{\ell}\ P_\ell(\cos\theta)$ to generate series representations of the potential in both regions.

$$\textit{Answer}:\ V\left(z\,\hat{k}\right)=\frac{\sigma}{2\,\varepsilon_0}\left(\sqrt{a^2+z^2}-|z|\right)$$

$$V(r,\theta)=\frac{\sigma a^2}{4\,\varepsilon_0}\left[r^{-1}\,P_0(\cos\theta)+\sum_{m=1}^{\infty}\frac{(-1)^m(2\,m-1)!!\,a^{2m}}{2^m\,(m+1)!\,r^{(2m+1)}}\;P_{2m}(\cos\theta)\right]$$

for $r>a$

$$=\frac{\sigma a}{2\,\varepsilon_0}P_0(\cos\theta)-\frac{\sigma r}{2\,\varepsilon_0}P_1(\cos\theta)+\frac{\sigma r^2}{4\,\varepsilon_0\,a}P_2(\cos\theta)+\frac{\sigma}{4\,\varepsilon_0}\sum_{m=1}^{\infty}\frac{(-1)^m(2\,m-1)!!\,r^{(2m+2)}}{2^{(m+1)}\,(m+1)!\,a^{(2m+1)}}P_{2m+2}(\cos\theta)$$

for $r<a$ and $\theta<\pi/2$

$$=\frac{\sigma a}{2\,\varepsilon_0}P_0(\cos\theta)+\frac{\sigma r}{2\,\varepsilon_0}P_1(\cos\theta)+\frac{\sigma r^2}{4\,\varepsilon_0\,a}P_2(\cos\theta)+\frac{\sigma}{4\,\varepsilon_0}\sum_{m=1}^{\infty}\frac{(-1)^m(2\,m-1)!!\,r^{(2m+2)}}{2^{(m+1)}\,(m+1)!\,a^{(2m+1)}}P_{2m+2}(\cos\theta)$$

for $r<a$ and $\theta>\pi/2$

35.) Begin with $\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+\left[C_{\ell m}-\frac{m^2}{\sin^2\theta}\right]\Theta(\theta)=0$ specialized to $m=0$, the case of no ϕ dependence. $\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+C_\ell\ \Theta(\theta)=0$. Make the magical assumption that $\Theta(\theta)=P(\cos\theta)$. a.) Compute: $\frac{d\Theta(\theta)}{d\theta}$ as expressed using $P(\cos\theta)$.

b.) Compute $\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)$ as expressed using $P(\cos\theta)$.

c.) Write $\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+C_\ell\ \Theta(\theta)=0$ as expressed using $P(\cos\theta)$.

d.) Replace $\cos\theta$ by x everywhere in the equation. Compare the equation with the Legendre equation in the Differential Equations handout. Comment.

e.) Solve the problem using the power series method.

f.) The solution does not converge as an infinite series for $x=+1$ so values of C_ℓ must be chosen that case the series to terminate as an ℓth order polynomial. Give the of C_ℓ that terminates the series as an ℓth order polynomial.

36.) Compare the far-field expansion for problems with azimuthal symmetry:

$$V(r,\theta) = \sum_{\ell=0}^{\infty} b_\ell \; r^{-(\ell+1)} \, P_\ell(\cos\theta)$$

with the multipole version of the expansion for the potential:

$$V(\vec{r}) = \frac{q_{total}}{4\pi\varepsilon_o r} + \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_o r^3} + \frac{1}{4\pi\varepsilon_0}\sum \frac{x_i\, x_j}{2\; r^5}\, Q_{ij} + \ldots .$$

Azimuthal symmetry (no ϕ dependence) means that there is no x or y dependence, but rather, only z and r. Show that: $q_{total} = 4\pi\varepsilon_o\, b_o$; $p_x = 0$; $p_y = 0$; $p_z = 4\pi\varepsilon_o\, b_1$ and Q_{xx}, $Q_{yy} = -\,½\, Q_{zz} = -\,4\pi\varepsilon_o\, b_2$.

37.) The average value of the potential over a sphere of radius R concentric with the origin in spherical coordinates is: $\langle V \rangle = \frac{1}{4\pi R^2}\int_0^{\pi}\int_0^{2\pi} V(R,\theta) R^2 \sin\theta \, d\theta \, d\phi$. Use the inside and outside expansions for the potential to show that, if the interior is charge free, then the potential at the center is $\langle V \rangle$. Show that if there are charges located inside the shell ($r < R$), but not on or outside the shell, then $\langle V \rangle = \frac{q_{net\ inside}}{4\pi\varepsilon_0 R}$.

38.) The potential due to a point charge q on the z axis at $a\ \hat{k}$ has been expressed using the Legendre polynomials as:

$$V(r,\theta) = \begin{cases} \sum_{\ell=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\right)\frac{r^\ell}{a^{(\ell+1)}}\; P_\ell(\cos\theta) & \text{for } r < a \\ \sum_{\ell=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\right)\frac{a^\ell}{r^{(\ell+1)}}\; P_\ell(\cos\theta) & \text{for } r > a \end{cases}$$

Use these expressions to compute the average value of the potential due to that charge over a sphere of radius R concentric with the origin in the cases that $R < a$ and $R > a$.

39.) Use the expressions included in the statement of the previous problem for the potential due to a point charge q on the z axis at $a\ \hat{k}$ to develop an expression for the

potential field when a conducting sphere of radius R concentric with the origin is added. The potential in the region exterior to the spherical conductor must have the form:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\right)\frac{r^{\ell}}{a^{(\ell+1)}}\ P_\ell(\cos\theta) \quad + \sum_{\ell=0}^{\infty} b_\ell\ r^{-(\ell+1)}\ P_\ell(\cos\theta)$$

That is: the potential due to the original point charge plus an additional contribution that vanishes as $r \to \infty$. Begin by matching the potential at $r = R$ where $V(R,\theta) = V_o$. Find the forms of the b_ℓ for $\ell \geq 1$. Compare them with the $r > b$ expansion for a charge q' at $b\ \hat{k}$. [$\sum_{\ell=0}^{\infty}\left(\frac{q'}{4\pi\varepsilon_0}\right)\frac{b^{\ell}}{r^{(\ell+1)}}\ P_\ell(\cos\theta)$]. Identify q' and b. Compute the net charge of the conductor in terms of q' and V_o. Note that the total expression

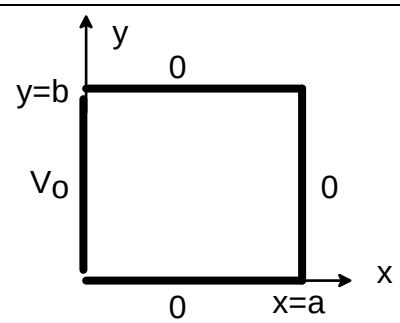
$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\right)\frac{r^{\ell}}{a^{(\ell+1)}}\ P_\ell(\cos\theta) \quad + \sum_{\ell=0}^{\infty} b_\ell\ r^{-(\ell+1)}\ P_\ell(\cos\theta)$$

corresponds to a charges: q at $a\ \hat{k}$; q' at $b\ \hat{k}$ and $4\pi\varepsilon_o R V_o$ at $0\ \hat{k}$.

Answers: $q' = -\,q\,({}^{R}/_{a})$ and $b = {}^{R^2}/_{a}$.

40.) Consider a 2D square box of side a with one side held at V_o and the other three sides at ground ($V = 0$.). The potential is found to be: [SL.12]

$$V(x,y) = \sum_{m=0}^{\infty}\frac{4V_0}{[2m+1]\pi}\ \frac{\sinh\left[(2m+1)\pi\left[a-x\right]/b\right]}{\sinh\left[(2m+1)\pi a/b\right]}\sin([2m+1]\pi y/b)$$



$$V(x,y) = \sum_{m=0}^{\infty}\frac{4V_0}{[2m+1]\pi}\ \frac{\sinh\left[(2m+1)\pi\left[a-x\right]/b\right]}{\sinh\left[(2m+1)\pi a/b\right]}\sin([2m+1]\pi y/b)$$ [SL.12]

with $a = b$.

$$V(x,y) \quad = \sum_{m=0}^{\infty} \frac{4V_0}{[2m+1]\pi} \; \frac{\sinh\left[(2m+1)\pi\left[1-{}^{x}\!/_{a}\right]\right]}{\sinh\left[(2m+1)\pi\right]} \sin({}^{[2m+1]\pi y}\!/_{a})$$

a.) What should the potential be at (½ a, ½ a)?

b.) Show that $\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{[2m+1]} \; \frac{\sinh\left[(m+\tfrac{1}{2})\pi\right]}{\sinh\left[(2m+1)\pi\right]} = \frac{\pi}{16}$.

41.) Following Example 3.4 on page 133 of the third edition of Griffiths, a rectangular channel running from $-b$ to b in x and from 0 to a in y has the $y = 0$ and $y = a$ sides held 0 and the $x = -b$ and $x = b$ sides held at V_o.

$$V(x,y) \quad = \sum_{m=0}^{\infty} \frac{4V_0}{[2m+1]\pi} \; \frac{\cosh\left[(2m+1)\pi\left[{}^{x}\!/_{a}\right]\right]}{\cosh\left[(2m+1)\pi\left[{}^{b}\!/_{a}\right]\right]} \sin({}^{[2m+1]\pi y}\!/_{a})$$

a.) Set b = ½ a. What should the potential be at (0, ½ a)?

b.) Show that $\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{[2m+1]} \; \frac{1}{\cosh\left[(m+\tfrac{1}{2})\pi\right]} = \frac{\pi}{8}$.

Data for the next three problems:

Problems based on the expansion $\displaystyle\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{m=0}^{\infty} \frac{r_<^m}{r_>^{m+1}} P_m(\hat{r}\cdot\hat{r}')$

$r_>$ is the larger of r and r' and $r_>$ is the smaller of r and r'

Choose to have the field point on the z axis so that $\hat{r}\cdot\hat{r}' = \cos\theta$.

Use the orthogonality relation $\displaystyle\int_0^{\pi} P_m(\cos\theta)\, P_n(\cos\theta) \sin\theta \, d\theta = \frac{2}{2\ell+1}\delta_{mn}$

Note that: $\displaystyle\int_0^{\pi} P_m(\cos\theta)\, \sin\theta \, d\theta = \int_0^{\pi} P_m(\cos\theta)\, P_0(\cos\theta) \sin\theta \, d\theta$; $\quad P_m(1) = 1$.

42.) Find the potential due to a uniform spherical shell of charge of radius R and surface charge density σ_o at a point a distance z from the center of the shell. Begin

with the equation: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{all\ sources} \frac{dq'}{|\vec{r}-\vec{r}'|}$. Use $\sum_{m=0}^{\infty} \frac{r_<^m}{r_>^{m+1}} P_m(\hat{r}\cdot\hat{r}')$ to replace $\frac{1}{|\vec{r}-\vec{r}'|}$. Be alert to the necessity to change the form of the integrand for field points inside the shell ($z > R$) as compared to outside the shell ($z < R$).

43.) Find the potential due to a uniform spherical ball of charge of radius R and surface volume density ρ_o at a point a distance z from the center of the shell. Begin with the equation: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{all\ sources} \frac{dq'}{|\vec{r}-\vec{r}'|}$. Use $\sum_{m=0}^{\infty} \frac{r_<^m}{r_>^{m+1}} P_m(\hat{r}\cdot\hat{r}')$ to replace $\frac{1}{|\vec{r}-\vec{r}'|}$. Be alert to the necessity to change the form of the integrand for ($z > r'$) as compared to outside the shell ($z < r'$).

Answer: $V(r) = (\frac{1}{2}\rho_o/\varepsilon_o)[R^2 - \frac{1}{3}z^2)$

44.) Find the potential due to a spherical shell of charge of radius R and surface charge density $\sigma_o \cos\theta$ at a point on the z axis a distance z from the center of the shell. Begin with the equation: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{all\ sources} \frac{dq'}{|\vec{r}-\vec{r}'|}$. Use $\sum_{m=0}^{\infty} \frac{r_<^m}{r_>^{m+1}} P_m(\hat{r}\cdot\hat{r}')$ to replace $\frac{1}{|\vec{r}-\vec{r}'|}$. Be alert to the necessity to change the form of the integrand for field points inside the shell ($z > R$) as compared to outside the shell ($z < R$).

45.) Find the potential due to a spherical shell of charge of radius R and surface charge density $\sigma_o (\cos\theta)^2$ at a point on the z axis a distance z from the center of the shell. Begin with the equation: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{all\ sources} \frac{dq'}{|\vec{r}-\vec{r}'|}$. Use $\sum_{m=0}^{\infty} \frac{r_<^m}{r_>^{m+1}} P_m(\hat{r}\cdot\hat{r}')$ to replace $\frac{1}{|\vec{r}-\vec{r}'|}$. Note that $(\cos)^2 = \frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_o(\cos\theta)$. Be alert to the necessity to change

the form of the integrand for field points inside the shell ($z > R$) as compared to outside the shell ($z < R$).

46.) The values D and L are known radii of concentric surfaces at which azimuthally symmetric solutions are matched. a.) Suppose that $D = L$, solve for a_ℓ in terms of b_ℓ.

$$\begin{aligned} a_\ell\ D^\ell &= c_\ell\ D^\ell + d_\ell\ D^{-(\ell+1)} \\ c_\ell\ L^\ell + d_\ell\ L^{-(\ell+1)} &= b_\ell\ L^{-(\ell+1)} \end{aligned} \qquad \text{[SL.34]}$$

b.) For $D \neq L$, attempt to solve for a_ℓ in terms of b_ℓ. Discuss the outcome in terms of the boundary condition specification required to ensure that a function that satisfies the Laplace equation except possibly on the surfaces $r = D$ and $r = L$ is uniquely determined.

c.) The matching equations above are appropriate for the potential being continuous at the surfaces. Add the conditions that the normal derivative (equivalently the radial component of the electric field) is continuous at the surfaces. Find the relation between a_ℓ and b_ℓ with this additional set of conditions.

47.)

51.) **The potential due to a ring of charge on and off the z-axis**

The potential of a uniform ring of charge of radius a concentric with the origin in the x-y plane at points along the z-axis is:

$$V(0,0,z) = \frac{Q}{4\pi\varepsilon_0\sqrt{a^2+z^2}} \rightarrow \frac{Q}{4\pi\varepsilon_0 z}\left[1+(a/z)^2\right]^{-1/2} \text{ for } z > a$$

$$V(0,0,z)=\frac{Q}{4\pi\varepsilon_0\, z}\left[1+\left(a/z\right)^2\right]^{-1/2}\approx\frac{Q}{4\pi\varepsilon_0\, z}\left[1+\left(\frac{-1/2}{1!}\right)\left(a/z\right)^2+\left(\frac{\left(-1/2\right)\left(-3/2\right)}{2!}\right)\left(a/z\right)^{2*2}+\ldots+\left(\frac{(-1)(-3)\ldots(2n-1)}{2^n\, n!}\right)\left(a/z\right)^{2*n}+\ldots\right]$$

Use our matching technique to find the potential as a function of r and θ given that $r > a$. Find an independent form the works for $r < a$.

Something has been gained. The original expression is only valid on axis, but the final expressions are convergent and correct in the entire regions $r > a$ and $r < a$. The results can be made more compact by using the double factorial notation which is defined as: $n!! = (n)(n\text{-}2)\,(n\text{-}4)$ …. ending with 2 or 1.Explain why the expansion for the potential of the ring only includes Legendre polynomials of even index. What is $P_o(\cos\theta)$?

52.) Following separation in Cartesian coordinates, there is a very special case solution in which $C_x = C_y = C_z = 0$. In this case show that $X(x)$ has the form $m\,x + b$ where m and b are constants. Give the general form of $V(x, y, z)$ in this special case. Give the form of the electric field described by this $V(x, y, z)$. Find the specific form for an electrostatic potential that corresponds to a uniform electric field: $\vec{E} = E_0\,\hat{j}$.

Recall: $\vec{E} = -\vec{\nabla} V$

53.) **The potential due to a point charge on the z-axis at points on and off the z-axis. Prepare a sketch.** The potential due to a point charge q at $d\,\hat{k}$ at points on the z axis with $z > d$ is: concentric with the origin in the x-y plane at points along the z-axis is:

$$V(0,0,z)=\frac{Q}{4\pi\varepsilon_0\,(z-d)}\rightarrow\frac{Q}{4\pi\varepsilon_0\, z}\left[1-d/z\right]^{-1}\ \text{for } z > d$$

a.) Use the binomial theorem to generate an expansion valid for $z > d$.

b.) Give an expression of the potential appropriate for an expansion to be used for $0 < z < d$. Use the binomial theorem to generate an expansion valid in that region.

c.) Match the form $V_>(r,\theta) = \sum_{\ell=0}^{\infty} b_\ell \; r^{-(\ell+1)} \; P_\ell(\cos\theta)$ for points on the z axis where each $P_\ell(\cos\theta) = 1$ and $z > d$. Use the fact that each power of r is an independent function (vector) in the sense of VS17. Give the form of $V_>(r,\theta)$for all θ.

d.) Match the form $V_<(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell \; r^{\ell} \; P_\ell(\cos\theta)$ for points on the z axis where each $P_\ell(\cos\theta) = 1$ and $z < d$. Use the fact that each power of r is an independent function (vector) in the sense of VS17. Give the form of $V_<(r,\theta)$for all θ.

e.) Replace cosθ by $\hat{r}\cdot\hat{r}'$ a restate the forms of $V_<(r,\theta)$ and $V_>(r,\theta)$. Compare with Griffiths (3.94) and the equation for cosθ′ a few lines above (3.98).

The Point: The function $|\vec{r}-\vec{r}'|^{-1}$ is a solution of the Laplace equation (except at $\vec{r}=\vec{r}'$), which remain finite at θ = 0 and π, so it has a representation as $\sum_{\ell=0}^{\infty}\left[a_\ell \; r^{\ell} + b_\ell \; r^{-(\ell+1)}\right] P_\ell(\cos\theta)$. In this problem $\vec{r}' = d\,\hat{k}$ and θ is the angle that the field point position vector makes with respect to the polar axis. Replacing cosθ by $\hat{r}\cdot\hat{r}'$ makes the result coordinate system independent. The portable result is:

$$\frac{1}{\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty}\left(\frac{r_<^{\ell}}{r_>^{\ell+1}}\right) P_\ell(\hat{r}\cdot\hat{r}')$$

where $r_<(r_>)$ is the smaller (larger) of the magnitudes of $\vec{r}$ and $\vec{r}'$.

54.) Consider two concentric conducting shells. An inner shell R_1 held at potential V_1 and an outer shell of radius R_2 held at potential V_2 in otherwise charge-free space. Give the general form of a solution to the Laplace equation in each of the three regions of r. Use a projection argument to identify the P_l terms that will contribute in each region. Find the potential for all r. Suppose that the inner shall has a radius of 4

cm and a potential of -12 *V*, and the outer shell has a radius of 12 *cm* and a potential of $+12$ *V*. Get answers with numbers!!!

$$\textit{Partial Answer}: V(r) = \frac{V_2 R_2 - V_1 R_1}{R_2 - R_1} + \frac{(V_1 - V_2) R_1 R_2}{R_2 - R_1} r^{-1} \qquad (\text{for } R_1 < r < R_2)$$

55.) Consider the charge distribution that is a single point charge q located at $d\,\hat{k}$.

a.) The potential of a point charge at $d\,\hat{k}$ is shown to be:

$$V(x,y,z) = \frac{q}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z-d)^2}} = V(r,\theta) = \sum_{\ell=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\right)\frac{d^\ell}{r^{(\ell+1)}}\ P_\ell(\cos\theta)\,.$$

Write out the first three terms in the expansion on the right.

$$P_0(z) = 1; \qquad P_1(z) = z; \qquad P_2(z) = {}^3/_2\, z^2 - {}^1/_2$$

b.) The charge distribution of a point charge at $d\,\hat{k}$ has the following multipole moments. $q_{\text{net}} = q$; $\vec{p} = q\,d\,\hat{k}$, $Q_{zz} = 2\,q\text{a}^2$; $Q_{xx} = Q_{yy} = -\,q\text{a}^2$ and $Q_{ij} = 0$ otherwise. Expand the multipole far-field estimate of the potential:

$$\begin{aligned} V(\vec{r}) &= V_0(\vec{r}) + V_1(\vec{r}) + V_2(\vec{r}) + \ldots \\ &= \frac{q_{total}}{4\pi\varepsilon_o r} + \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_o r^3} + \frac{1}{4\pi\varepsilon_0}\sum\frac{x_i\,x_j}{2\,r^5}\,Q_{ij} + \ldots \end{aligned}$$

Compare with the first three terms of the expansion of part a. Use the standard relation $z = r\cos\theta$.

56.) The region inside a spherical of radius R centered on the origin is charge-free. What equation does the potential satisfy in the region? The form of a spherical shell coordinate expansion was found in problem SL6.

$$G(r,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell\ r^\ell\right] P_\ell(\cos\theta)$$

Based on the series expansion above, what is $G(r = 0,\theta)$?

Use the form above and compute the average value of the potential on the sphere of radius R. Use the orthogonality relation [SL.28] after multiplying the relation by $P_o(\cos\theta) = 1$.

$$G_{ave} = \frac{1}{4\pi R^2}\int_0^{\pi}\int_0^{2\pi} G(R,\theta)\ R^2 \sin\theta\ d\theta\ d\phi$$

Relate these results to the average property of solutions to the Laplace equation.

57.) A theorem for the representation of the product of two Legendre polynomials is proposed.

$$P_n(\cos\theta)\,P_m(\cos\theta) = \sum_{\substack{\ell=|m-n| \\ \Delta\ell=2}}^{\ell=m+n} c_\ell\ P_\ell(\cos\theta)$$

That is: The product of the mth and n^{th} Legendre polynomials can be expressed as a linear combination of the Legendre polynomials of order $|m - n|$ to order $m + n$. All of the polynomials represented will be even or odd to match the evenness or oddness of $m + n$.

a.) Represent the product of the 1st and 3rd Legendre polynomials as a linear combination of the Legendre polynomials of order 2 to order 4.

b.) Represent the product of the 2nd and 2nd Legendre polynomials as a linear combination of the Legendre polynomials of order 0 to order 4.

Partial Answer: $P_1(x)\ P_3(x) = {}^4/_7\ P_4(x) - {}^3/_7\ P_2(x)$

The result that: The product of the mth and n^{th} Legendre polynomials can be expressed as a linear combination of the Legendre polynomials of order $|m - n|$ to order $m + n$ is often compared to the triangle inequality and a vector model of the mixing in products of Legendre polynomials.

Mathematica syntax: $P_n(x) \Rightarrow$ LegendreP[n,x]; also use Expand[…]

{1/5, 2/7,18/35}

58.) Charge is glued on the surface of a spherical shell of radius r in such a fashion that the charge density on the shell is $\sigma_o \cos(\theta) = \sigma_o P_1(\cos(\theta))$.

a.) Give the allowed form of the potential, $V^<(r, \theta)$ for the region $r < R$ and $V^>(r, \theta)$ for the region $r > R$. Assume that space is charge-free except for that glued on the shell.

b.) Prepare a sketch of the electric field line pattern that you expect for this charge distribution.

c.) Find the relation between the coefficients in the two expansions by requiring that:

$$V^<(R, \theta) = V^>(R, \theta)$$

d.) Use the relation below to generate the final relation needed to identify all the expansion coefficients.

$$\frac{\sigma(\theta)}{\varepsilon_0} = E_r^> - E_r^< = -\frac{\partial V^>}{\partial r}\bigg|_{r=R} + \left(-\frac{\partial V^<}{\partial r}\bigg|_{r=R}\right).$$

e.) Compute the forms of the potential for $r < R$ and for $r > R$.

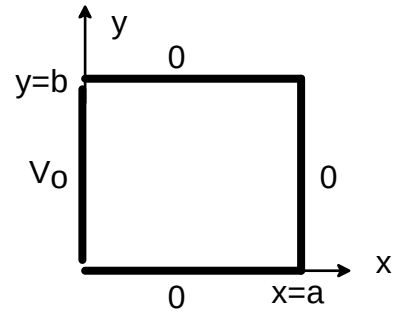
f.) Compute the full electric field for $r < R$.

g.) Express $\hat{k}$ in terms of $\hat{r}$ and $\hat{\theta}$. Use the result as you describe the electric field in the region $r < R$.

h.) Set $\sigma_o = 3\ \varepsilon_o E_o$ and then compare the field with $E_o\ \hat{k}$. Use this result to discuss the field interior to the charge conducting sphere, the example discussed in class.

i.) Discuss the relation between this problem and SC3.

59.) The general solution to the 2D Laplace equation has the form:



$$G(x,y) = [A + B\,x][C + D\,y]$$
$$+\sum_{m=1}^{\infty}\left[E_m \sin(k_m x) + F_m \cos(k_m x)\right]\left[G_m e^{k_m y} + H_m e^{-k_m y}\right]$$
$$+\sum_{n=1}^{\infty}\left[S_n e^{k_n y} + T_n e^{-k_n y}\right]\left[U_n \sin(k_n y) + V_n \cos(k_n y)\right]$$

The solution chosen omitted $[A + B\,x]\,[C + D\,y]$ part. Test the linear terms to see if they could fit the boundary conditions. (See page 14.)

60.) Compute the electrostatic potential due to a uniform line of source charge of linear density λ running along the z axis from $-a$ to $+a$ at points $z > a$. Match this result to a solution of the Laplace equation in spherical coordinates with azimuthal symmetry to find a series representation for $V(r, \theta)$ valid for $r > a$.

$$V(\vec{r}_p) = \frac{1}{4\pi\varepsilon_0}\int_{\substack{all\\sources}} \frac{dq_s}{\left|\vec{r}_p - \vec{r}_s\right|}$$ Answer: $V(r,\theta) \to \dfrac{Q}{4\pi\varepsilon_0 r}\sum_{m=0}^{\infty}\dfrac{a^{2m}}{(2m+1)\,r^{2m}}P_{2m}(\cos\theta)$; $r>a$, $Q = 2\lambda a$

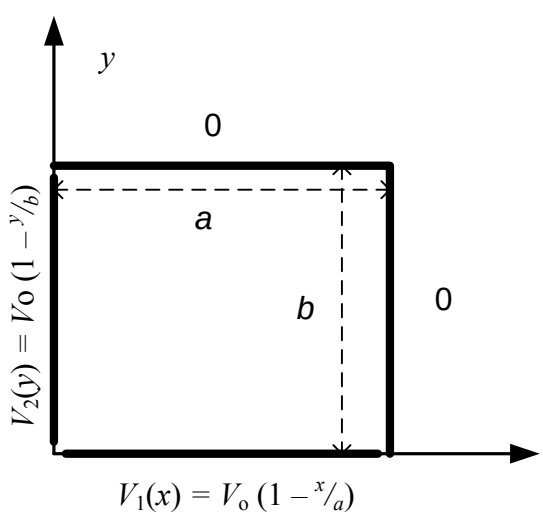
61.) This problem was assigned as an E&M integration problem. Repeat it using $\dfrac{1}{\left|\vec{r} - \vec{r}'\right|} \to \sum_{m=0}^{\infty}\dfrac{r^m}{z^{m+1}}P_m(\cos\theta)$ for the case of the problem. Use the orthogonality of the Legendre polynomials. Consider a solid sphere of radius R and charge density $\rho_0\,(r/R)\cos\theta$. Compute the electric field due to this distribution at points on the polar axis with $z > R$. It is recommended that the electrostatic potential be computed for points on the z axis and that the field be found as a derivative of the potential. Considerable patience is required! Possible answer: $E_z = \dfrac{2\rho_0 R^4 z}{15\varepsilon_0 z^4}$ for $|z| > R$.

62.) $\dfrac{1}{\left|\vec{r}\ - d\hat{k}\right|} = \dfrac{1}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} = \sum_{\ell=0}^{\infty} \dfrac{d^\ell}{r^{(\ell+1)}}\ P_\ell(\cos\theta)$ for $r > d$

Make a binomial expansion of $\dfrac{1}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} = \dfrac{1}{r}\left[1 + (d^2/r^2) - 2(d/r)\cos\theta\right]^{-1/2}$.

Make the identifications n = - ½ and $x = (d^2/r^2) - 2(d/r)\cos\theta$. Expand to order x^3. Group the terms in the form: $\dfrac{1}{r}(\cdots) + \dfrac{d}{r^2}(\cdots) + \dfrac{d^2}{r^3}(\cdots) + \dfrac{d^3}{r^4}(\cdots) + \cdots$. Identify the *stuff* that fills each of the sets of parentheses up uo the d^3/r^4 set. There are some pieces of higher order such as d^4/r^5, …, and d^6/r^7. Why do we not attempt to identify the factors that multiply d^4/r^5, …, and d^6/r^7 at this stage?

SL63. The problem is a (2D) rectangular box* with two conducting sides held at zero potential. The third and fourth sides are insulators that are charged to have the specified potentials as functions of position. (* The corresponding 3D problem is a infinitely long conducting channel of rectangular cross section with no z dependence.)



The channel has an x-y cross section of a by b, and the potential along the lower edge is $V_1(x) = V_o\,(1 - x/a)$ while the potential along the left edge is $V_2(y) = V_o\,(1 - y/b)$.

a.) Find the potential at all interior points.

b.) Find the x and y components of the electric field in the interior.

c.) Find the x and y components of the electric field along the lines $x = a$ and $y = b$.

d.) What type of boundary conditions do we have? Choose for the types discussed as

being adequate to ensure uniqueness.

e.) Find the charge densities on the two sides held at zero potential.

f.) Show that there are no relative maxima of minima of $V(x,y)$ in the interior of the channel.

64.) The interior of a spherical shell of radius R is charge free, and the potential on its surface is described by $V = V_0 \cos^3\theta$.

(a) Determine the functional description of $V(r,\theta)$ for $r < R$, and write down the function for $r < R$, $V(r, \theta) =$

(b) Determine the functional description of $V(r,\theta)$ for $r > R$, and write down the function for $r > R$, $V(r, \theta) =$

(c) Determine the **surface charge density** $\sigma(R,\theta)$ on the spherical shell at $r = R$.

$P_0(\cos\theta) = 1$; $P_1(\cos\theta) = \cos\theta$; $P_2(\cos\theta) = {}^3/_2 \cos^2\theta - {}^1/_2$; $P_3(\cos\theta) = {}^5/_2 \cos^2\theta - {}^3/_2\cos\theta$

Do you results support the proposition that: $\dfrac{\sigma_{P_3}}{\sigma_{P_1}} = \left\{\dfrac{3(3)+1}{1(1)+1}\right\} \dfrac{V_{P_3}}{V_{P_1}}\Bigg|_{r=R}$

65.) The interior and exterior around a charged spherical shell of radius R is charge free, and the surface charge density is $\sigma(R,\theta) = \sigma_0 \cos^3\theta$.

Express $\cos^3\theta$ as a linear combination of the Legendre polynomials below.

(a) Match the forms of $V_<(r, \theta)$ and $V_>(r, \theta)$ at $r = R$ to find the b_ℓ in terms of the a_ℓ.

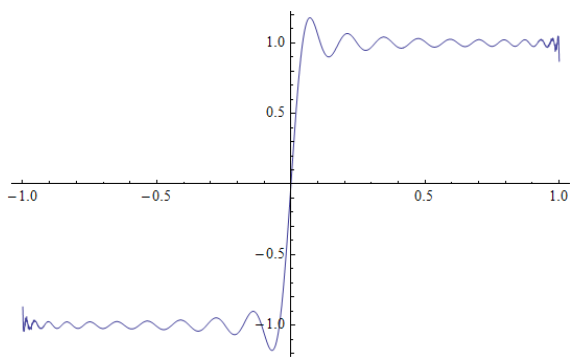
(b) Find the charge density due to these general forms. Match it to $\sigma_0 \cos^3\theta$. Identify the values of all the a_ℓ.

(c) Find $V(r,\theta)$ inside and outside the shell.

66.) Consider a sphere of radius R divided into two conducting caps. The $0 < \theta < \frac{1}{2}\pi$ cap is at potential V_o and the $\frac{1}{2}\pi < \theta < \pi$ cap is held at $-V_o$. Find the potential everywhere keeping the $\ell = 0, 1, 2$ and 3 contributions. Search for a reference claiming to contain the complete solution. It follows from the result that:

$$f(x)=\begin{cases} 1 \text{ if } 0<x<1 \\ -1 \text{ if } -1<x<0 \end{cases} = \sum_{m=0}^{\infty} \frac{(-1)^m (4m+3)\,(2m)!}{2^{2m+1}\, m!\;\;(m+1)!} P_{2m+1}(x)$$

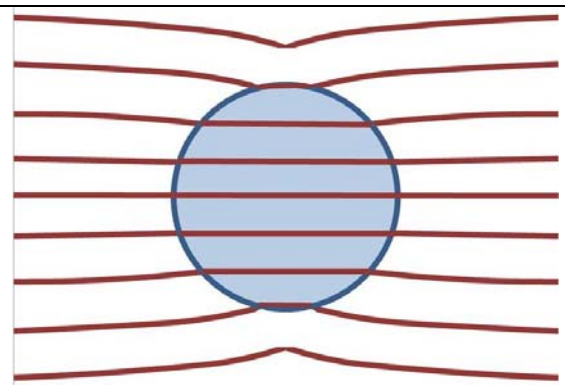
Does this expansion support the values that you found for the first four coefficients? Note that the expression above is derived by replacing P_k by its Rodriques formula representation and integrating. (If one integrates by parts, an interesting question arises. Discuss the issue that arises for extra credit. Use $P_k(1) = 1$ for all k, and apply the ratio test to the expansion.)



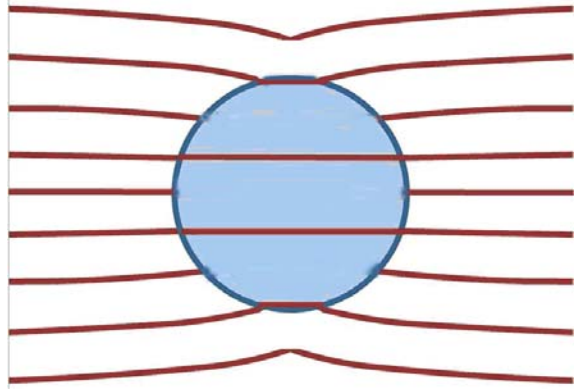

```
f[x_] = Sum[((-1)^m (4 m + 3)  Factorial[2
m]LegendreP[2 m + 1, x])/( 2^(2 m+1) Factorial[m]
Factorial[m +1]), {m, 0, 21}]; Plot[f[x],{x,-1,1}]
```

The plot supports the validity of the expansion above. It also displays a Gibbs overshoot, and poor convergence in the neighborhoods of $x = \pm 1$. (Repeat the plot summing up to $m = 40$ and compare.)

SL68.) A uniform field is established in a large region of otherwise empty space that is 'centered' on the origin. This external potential is due to source charges located at great distances from the origin. a.) Give a potential V_{ext} that corresponds to $\vec{E} = E_o \hat{k}$. A dielectric sphere with a relative susceptibility $\varepsilon_r = 4$ and radius R is moved in from a great distance and placed centered on the origin. b.) Find expressions for the potential and electric field inside and outside of the sphere. c.) Find the surface charge density on the sphere. d.) Find the charge density inside the sphere. e.)



Illustrates D, the electric displacement. Lines of D are continuous. Below, the electric field. E_{internal} is weaker than the applied field for $\varepsilon_r > 1$. Lines of E begin and end on the bound surface charge.

Compute the multipole moments of the dielectric sphere. f.) Compute the electric field at $r = 0$. g.) Find diamagnetism on Wikipedia.	

By analogy, develop a definition for dielectricism[1]. Based on the picture of an applied field polarizing the atoms of a dielectric, explain the dielectricism of dielectrics.

Hints: Match $V_<$ and $V_>$ at $r = R$; $V_> \to V_{ext}$ as $r \to \infty$; $\varepsilon_r \varepsilon_o E_r^< = \varepsilon_o E_r^>$ at $r = R$.

References:

1. Mary L. Boas, ***Mathematical Methods in the Physical Sciences***, 2nd Edition, chapter 3, John Wiley & Sons (1983).
2. Bruce Kusse and Erik Westwig, ***Mathematical Physics***, John Wiley & Sons (1998).
3. David J. Griffiths, ***Introduction to Electrodynamics***, 3rd Edition, Prentice Hall (1999).
4. K. F. Riley, M. P. Hobson and S. J. Bence, ***Mathematical Methods for Physics and Engineering***, 3rd Ed., Cambridge, Cambridge UK (2006).
5. The Wolfram web site: mathworld.wolfram.com/

Sample Problems:

1.) **ASL51. The potential due to a ring of charge on and off the z-axis !!!**

Prepare a sketch.

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[1] Dielectricism is a new term to denote the action of dielectric materials. Dielectricity might be preferred. Compare with diamagnetism, paramagnetism and ferromagnetism. Find paraelectricity and ferroelectric on Wikipedia.

The potential of a uniform ring of charge of radius a concentric with the origin in the x-y plane at points along the z-axis is:

$$V(0,0,z) = \frac{Q}{4\pi\varepsilon_0\sqrt{a^2+z^2}} \to \frac{Q}{4\pi\varepsilon_0 z}\left[1+\left(\tfrac{a}{z}\right)^2\right]^{-1/2} \text{ for } z > a$$

a.) Use the binomial theorem to generate the next three non-zero terms in the expansion below. $[1+x]^{-1/2} = 1 + \frac{(-1/2)}{1!}\,x + \frac{(-1/2)(-3/2)}{2!}\,x^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}\,x^3 + \ldots$

$$V(0,0,z) = \frac{Q}{4\pi\varepsilon_0\, z}\left[1+\left(\tfrac{a}{z}\right)^2\right]^{-1/2} \approx \frac{Q}{4\pi\varepsilon_0\, z}\left[1+ \qquad\qquad\qquad\qquad \right]$$

The binomial theorem gives $[1+x]^{-1/2} = 1 + \frac{(-1/2)}{1!}\,x + \frac{(-1/2)(-3/2)}{2!}\,x^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}\,x^3 + \ldots$

where $x = (^a/_z)^2$.

$$V(0,0,z) = \frac{Q}{4\pi\varepsilon_0\, z}\left[1+\left(\tfrac{a}{z}\right)^2\right]^{-1/2} \approx \frac{Q}{4\pi\varepsilon_0\, z}\left[1 - \frac{1}{2}\left(\tfrac{a}{z}\right)^2 + \frac{3}{8}\left(\tfrac{a}{z}\right)^4 - \frac{15}{48}\left(\tfrac{a}{z}\right)^6 + \;\ldots \right]$$

b.) Assume that on the z-axis, the form below is valid for $z > a$ and that the c_k are known values

$$V(0,0,z) = \frac{Q}{4\pi\varepsilon_0\, z}\left[1+\left(\tfrac{a}{z}\right)^2\right]^{-1/2} = \frac{Q}{4\pi\varepsilon_0}\left[z^{-1} + \sum_{k=1}^{\infty} c_k\, z^{-(2k+1)}\right]$$

Use our matching technique to find a representation of the potential as a function of r and θ in the entire region $r > a$. Your expression should involve the known constants, the c_k. Do not substitute your values from part a.

The form is to match $G(r,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell\, r^\ell + b_\ell\, r^{-(\ell+1)}\right] P_\ell(\cos\theta) \to \sum_{\ell=0}^{\infty} b_\ell\, r^{-(\ell+1)}\, P_\ell(\cos\theta)$ in a large region like $r > a$. On the z axis, $r = z$ and $P_\ell(\cos\theta) = 1$. For θ = 0 and hence cosθ = 1,

$$V(0,0,z) = \frac{Q}{4\pi\varepsilon_0}\left[z^{-1} + \sum_{k=1}^{\infty} c_k\, z^{-(2k+1)}\right] \to \frac{Q}{4\pi\varepsilon_0}\left[r^{-1}P_0(1) + \sum_{k=1}^{\infty} c_k\, r^{-(2k+1)}\, P_{2k+1}(1)\right]$$

$$= \sum_{\ell=0}^{\infty} b_\ell \; r^{-(\ell+1)} \; P_\ell(\cos\theta) \quad \text{for } \theta = 0.$$

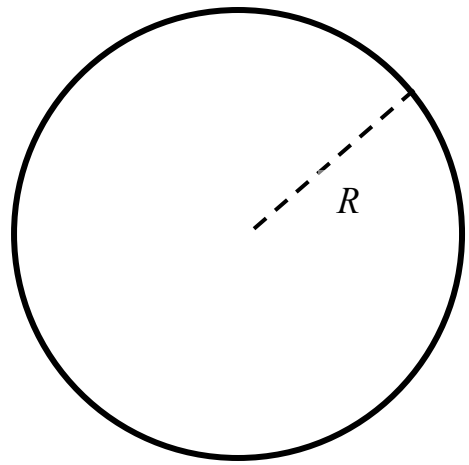
Matching power-by-power along z axis so that r varies with $\theta = 0$ and hence each $P_\ell = 1$. Then freeing θ to roam from 0 to π; that is put the matched coefficient in $\sum_{\ell=0}^{\infty} b_\ell \; r^{-(\ell+1)} \; P_\ell(\cos\theta)$.

See the statement of VS17b on the last page. Each power $r^{-(\ell+1)}$ is an independent function (vector).

$b_\ell = 0$ for ℓ odd; $b_o = Q/4\pi\varepsilon_o$; $b_\ell = Qc_k/4\pi\varepsilon_o$. for $\ell = 2\,k \Rightarrow$even

$$V(r,\theta) = \frac{Q}{4\pi\varepsilon_0}\left[r^{-1}P_0(\cos\theta) + \sum_{k=1}^{\infty} c_k \, r^{-(2k+1)} \, P_{2k+1}(\cos\theta) \right]$$

24. A spherical shell of radius R has surface charge while the regions $r<R$ and $r>R$ are charge free. The problem has azimuthal symmetry ($\Rightarrow$ no dependence on ϕ). The surface charge on the sphere sets its potential to be $V(R, \theta) = \mathcal{V}_0 \sin^2\theta$. Assume that the potential vanishes as $r \to \infty$.



a.) Name the equation that $V(\theta)$ satisfies for $r < R$ and for $r > R$. the Laplace equation

b.) Give the sum that represents the allowed form of the inside solution (for $r<R$).

$$V_<(r,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell\ r^\ell\right] P_\ell(\cos\theta)$$

c.) Give the sum that represents the form of the outside solution (for $r>R$).

$$V_>(r,\theta) = \sum_{\ell=0}^{\infty} b_\ell\ r^{-(\ell+1)}\ P_\ell(\cos\theta)$$

d.) Express $\sin^2\theta$ as a linear combination of powers of $\cos\theta$.

e.) Express $\sin^2\theta$ in terms of the $P_\ell(\cos\theta)$. Match the coefficient of the highest power first.

$$\sin^2\theta = a + b\cos\theta + c\cos^2\theta + \ldots + d\,[\cos\theta]^m =$$
$$f\ P_m(\cos\theta) + g\ P_{m-1}(\cos\theta) + \ldots + k\ P_0(\cos\theta).$$

f.) Find $V(r, \theta)$ for $r < R$. The solution inside must match the potential at $r = R$. On a constant r surface, the $P_\ell(\cos\theta)$ vary independently so the coefficients a_ℓ of each

$P_\ell(\cos\theta)$ must match across the equal sign. Once the coefficients are determined, they must be substituted in the full form $V_<(r,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell\ r^\ell\right] P_\ell(\cos\theta)$. See the statement of VS17b on the last page. Each $P_\ell(\cos\theta)$ is an independent function (vector).

Match the coefficients of independent behaviors term-by-term.

$$-\,{}^2/_3\,\mathcal{V}_0\ P_2(\cos\theta) + {}^2/_3\,\mathcal{V}_0\ P_0(\cos\theta) = V_<(R,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell\ R^\ell\right] P_\ell(\cos\theta)$$

We conclude that $a_o = {}^2/_3\,\mathcal{V}_0$ and that $a_2 = -\,{}^2/_3\,\mathcal{V}_0\,R^{-2}$. Substituting,

$$V_<(R,\theta) = {}^2/_3\,\mathcal{V}_0\ \ P_0(\cos\theta) - {}^2/_3\,\mathcal{V}_0\,({}^r/_R)^2\ P_2(\cos\theta)$$

g.) Find $E_r(r, \theta)$ for $r < R$. $\quad E_r(r, \theta) = -\dfrac{\partial V_<(r,\theta)}{\partial r} = +\,{}^4/_3\,\mathcal{V}_0\,R^{-2}\,r\ \ P_2(\cos\theta)$

12.) [***highly modified ⇒read carefully***] A long ***vertical*** slot runs for all $\boldsymbol{y > 0}$ **and** $\boldsymbol{0 < x < a}$. There is no z dependence. The potential along the $y = 0$ end is set to the function: $V(x, 0)$. Conducting planes parallel to the y-z plane and through $x = 0$ and $x = a$ are grounded and hence are a potential zero. Prepare a sketch before proceeding.

A standard slot problem with no z dependence has a solution which is a sum of terms of the form:

$$[\ A_n\ e^{k_n x} + B_n\ e^{-k_n x}]\ [\ C_n\ \sin(k_n\ y) + D_n\ \cos(k_n\ y)\ ]$$

or of the form: $[\ C_n\ \sin(k_n\ x) + D_n\ \cos(k_n\ x)\ ]\ [\ A_n\ e^{k_n y} + B_n\ e^{-k_n y}]$

(a) Choose one of the forms above for this problem? **COPY your choice onto the next line**:

$$V(x, y) \approx \boxed{[\ C_n\ \sin(k_n\ x) + D_n\ \cos(k_n\ x)\ ]\ [\ A_n\ e^{k_n y} + B_n\ e^{-k_n y}]}$$

Justify your choice: The solution must approach zero for large $y \Rightarrow e^{-ky}$ and we need to fit a function of x so we need cos(kx) and sin(kx) to fit $V(x, 0)$ with a Fourier series.

A solution inside the slot valid for $\{0 < x < a;\ 0 < y < \infty;\ \text{all } z\}$ is sought. The boundary conditions restrict the possible values for the symbols A_n, B_n, C_n, D_n and k_n.

(b) The condition that the solution is to be physically well-behaved as $y \to \infty$ requires that the constant A_n be restricted to the value zero.

(c) The condition that $V = 0$ for $x = 0$ requires that the constant D_n be restricted to the value zero.

(d) The condition that for $x = a$, $V = 0$ requires that the constant k_n be restricted to the value(s) $n\,({}^{\pi}/_{a})$ where $n = 1, 2, 3, 4, \ldots$.

(e) Given that $V(x, 0) = \mathcal{V}_0 \sin[{}^{5\pi x}/_{a}]$ and the constraints of the previous four parts, find the potential in the slot: $V(x,y) = \sum_{n=1}^{\infty} E_n \sin[n\frac{\pi}{a}x]\, e^{-[n\frac{\pi}{a}y]}$.

$$V(x,0) = \sum_{n=1}^{\infty} E_n \sin[n\tfrac{\pi}{a}x]\,(1) = \mathcal{V}_0 \sin[{}^{5\pi x}/_{a}] \quad \Rightarrow \quad E_5 = \mathcal{V}_0;\ E_n = 0 \text{ for } n \neq 5.$$

See the statement of VS17b on the last page. Each $\sin[{}^{n\pi x}/_{a}]$ is an independent function (vector).

Match the coefficients independent behavior by independent behavior. The equation immediately above is only valid for $y = 0$. Once the coefficients are set, they must be substituted into the general form $\sum_{n=1}^{\infty} E_n \sin[n\frac{\pi}{a}x]\, e^{-[n\frac{\pi}{a}y]}$ to provide $V(x,y)$ at all points in the slot.

$$V(x,y) = \mathcal{V}_0 \sin[{}^{5\pi x}/_{a}]\ e^{-[5(\frac{\pi}{a})y]} \qquad\qquad \vec{E} = -\vec{\nabla} V$$

VS17b.) The representation of a vector as a linear combination of vectors in a linearly independent set is unique. Prove this proposition. Begin by assuming that there are two distinct linear combinations of the vectors that equal the vector $|v\rangle$.

$$|v\rangle = a_1|1\rangle + a_2|2\rangle + \ldots + a_N|N\rangle \text{ and } |v\rangle = b_1|1\rangle + b_2|2\rangle + \ldots + b_N|N\rangle$$

Show that this assumption leads to a contradiction. It follows immediately that $a_1|1\rangle + a_2|2\rangle + \ldots + a_N|N\rangle = b_1|1\rangle + b_2|2\rangle + \ldots + b_N|N\rangle$ requires that $a_i = b_i$ for $1 \le i \le N$. <u>Compare this problem with FS3 and LS9.</u> Add the result from VS18 and compare again! Mutually orthogonal vectors are linearly independent.

In the case of an expansion in terms of a set that obeys an orthogonality relation and $|\text{ZERO}\rangle$ is not a member, an alternative proof uses ***projection***. The jth coefficient is projected out using the inner product.

$$\langle j|v\rangle = a_1\langle j|1\rangle + a_2\langle j|2\rangle + \ldots + a_N\langle j|N\rangle = \sum_{k=1}^{N} b_k\langle j|k\rangle$$

$$\langle j|v\rangle = a_j\langle j|j\rangle = \sum_{k=1}^{N} b_k\,\delta_{jk}\langle j|k\rangle = b_j\langle j|j\rangle \quad \Rightarrow a_j = b_j = \frac{\langle j|v\rangle}{\langle j|j\rangle}$$

Shows equality and generates a component calculator.