Introduction to Linear Vector Spaces

Concepts of primary interest:

- **Closure and axioms for addition and multiplication**
- **Linear independence**
- **Span, basis and dimension**
- **Inner Product**

Sample calculations:

- **Span of a set**
- **Certifying a vector space**
- **Certifying an inner product**
- **Gram-Schmidt basis construction**

Application examples:

- **Legendre polynomials**

Tools of the trade:

Desirable background:

- **Fourier Series***

***** Fourier series are used as examples. They should be skipped if you are not familiar with Fourier series.

Study Guide: Definitions and statements of fundamental rules (such as axioms) must be memorized as soon as one begins the study of a new topic. One must identify the definition of each concept. Do not blur the concept by using an example of the concept rather than its definition. You should be able to present prose statements of definitions and axioms using complete sentences. You statements should be complete enough to remove the need for any other defining material such as an equation. If you cannot do so, you do not know the definition. Memorizing definitions and statements of rules is the anchor step. Understanding cannot be achieved without it.

Mathspeak: Fundamental precise definition. Mathspeak definitions are absolutely required knowledge, but often difficult to interpret.

PhysicsSpeak: A statement in the physics context providing a useful application or interpretation that is in harmony with all physical law. PhysicsSpeak can be less formal than mathspeak

Fuzzyspeak: An interpretation definition or principle that may contribute to appreciating the importance of the concept. A fuzzyspeak *definition* is not a proper substitute for a formal mathspeak definition. It can be added after the formal definition. It cannot replace a formal definition.

Vector quantities play a central role in physics. At the lowest level, a vector is a quantity that has both magnitude and direction. A more complete and general description of vectors, vector spaces and their properties is to be developed in this section and in a later section on linear transformations.

The Displacement Model

The properties of the collection of all possible displacements $\Delta\vec{r}_i$ of a particle in our familiar model, a flat, infinite (Euclidean) three-dimensional universe, are presented to introduce vector concepts.

Rules for the Addition of Displacements

A1. The sum of two displacements is another allowed displacement.

$$\Delta\vec{r}_1 + \Delta\vec{r}_2 = \Delta\vec{r}_3$$

A2. Additions can be regrouped without changing the resultant displacement.

$$\Delta\vec{r}_1 + \left(\Delta\vec{r}_2 + \Delta\vec{r}_3\right) = \left(\Delta\vec{r}_1 + \Delta\vec{r}_2\right) + \Delta\vec{r}_3$$

A3. The order of additions can be changed without changing the sum.

$$\Delta\vec{r}_1 + \Delta\vec{r}_2 = \Delta\vec{r}_2 + \Delta\vec{r}_1$$

A4. There is a ***zero*** displacement or $\vec{0}$ that does not change the position of the particle.

$$\Delta\vec{r}_i + \vec{0} = \Delta\vec{r}_i \quad \text{for all } \Delta\vec{r}_i .$$

A5. For any displacement $\Delta\vec{r}_i$, there is another (***additive inverse***) $-\Delta\vec{r}_i$ that has the opposite action on a particle's position.

$$\text{for all } \Delta\vec{r}_i , \quad \Delta\vec{r}_i + \left(-\Delta\vec{r}_i\right) = \vec{0} .$$

Rules for the Multiplication of Displacements by Scalars

M1. If $\Delta\vec{r}_i$ is an allowed displacement, then any *real* scalar multiple of that displacement, $c\,\Delta\vec{r}_i$, is also an allowed displacement.

M2. The scalar multiple of a sum of displacements is the same as the sum of the same multiple of the individual displacements.

$$c\left(\Delta\vec{r}_1 + \Delta\vec{r}_2\right) = c\Delta\vec{r}_1 + c\,\Delta\vec{r}_2 \text{, an allowed displacement}$$

M3. A sum of scalars times a displacement is the sum of the individual scalars times that displacement. $\left(c + d\right)\Delta\vec{r}_i = c\,\Delta\vec{r}_i + d\,\Delta\vec{r}_i$, an allowed displacement

M4. The scalar multiplication of displacements is associative. $c\left(d\,\Delta\vec{r}_i\right) = \left(c\,d\right)\Delta\vec{r}_i$.

M5. One times any displacement is that same displacement.

$$1\,\Delta\vec{r}_i = \Delta\vec{r}_i \;\forall\; \Delta\vec{r}_i .$$

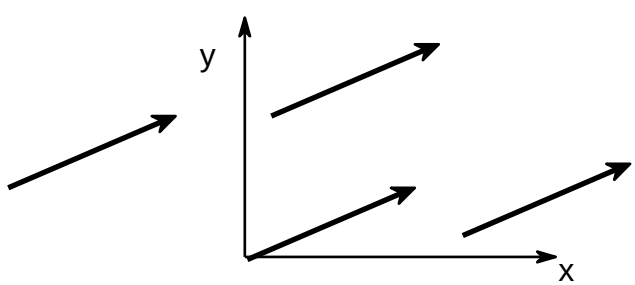
The collection of all displacements has other features such as an inner (dot) product that yields a scalar-valued measure of the component of one displacement along the direction of a second displacement times the magnitude of the second displacement. It gauges the degree to which the two displacements embody common behavior, displacing in the same direction.[1] The inner product of a displacement with itself is always positive. Thus the inner product can be used to assign lengths to all the

[1] This sentence is an example of fuzzyspeak. Find a formal definition of *inner product*.

displacements in the space (to set a *norm* for the space: $|\Delta\vec{r}_i| = \sqrt{\Delta\vec{r}_i \cdot \Delta\vec{r}_i}$ |). Further, we can find a set of three displacements $\left\{\vec{b}_1, \vec{b}_2, \vec{b}_3\right\}$ such that all the allowed displacements can be represented as a sum of multiples of the members of this set, a linear combination. That is: all $\Delta\vec{r}_i = c_1\ \vec{b}_1 + c_2\ \vec{b}_2 + c_3\ \vec{b}_3$ for some scalars: $\left\{c_1, c_2, c_3\right\}$. This set of displacements $\left\{\vec{b}_1, \vec{b}_2, \vec{b}_3\right\}$ is a spanning set for our three dimensional universe. The minimum number of vectors $\left\{\vec{b}_1, \vec{b}_2, ..., \vec{b}_n\right\}$ required such that every displacement can be represented as $\Delta\vec{r}_i = c_1\ \vec{b}_1 + c_2\ \vec{b}_2 + ... + c_n\ \vec{b}_n$ is the dimension of the space of displacements (three for our example). A spanning set with the minimum required number of elements is a basis set. A common choice for the basis set for displacements in three dimensions is $\{\hat{i}, \hat{j}, \hat{k}\}$.

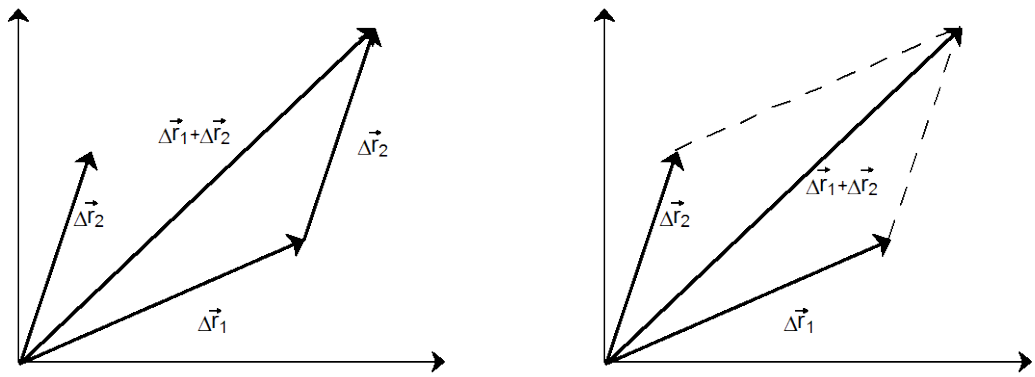
Graphical Representations

Dropping back to two dimensions to simplify the drawings, the figure below illustrates the concept of equality for displacements.



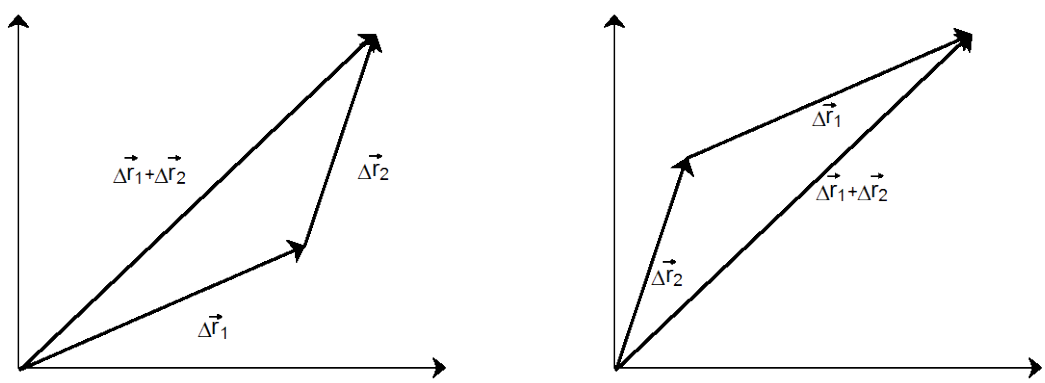
All the displacements illustrated have the same magnitude (length) and direction; they are all equal. The 'location' of the displacement is not considered, and, in fact, the location of the vector is not an intrinsic property of the displacement, but rather is an adjustable parameter that may be used to be facilitate the solution of the problem of interest. Move the displacement vector around without changing its magnitude or direction, and the action of the displacement is unchanged. (*Moving a vector without*

changing its direction is sometimes called parallel transport.) Vector addition illustrates the utility of the concept.



The sum of two displacements should be the net displacement when the two displacements are executed in sequence.

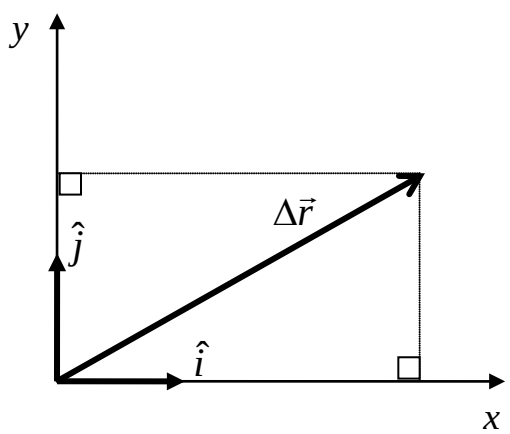
The left-hand figure illustrates transporting the tail of $\Delta\vec{r}_2$ to the tip of $\Delta\vec{r}_1$ to yield the sum of the displacements. The parallelogram rule diagram presents another graphical representation of vector addition.

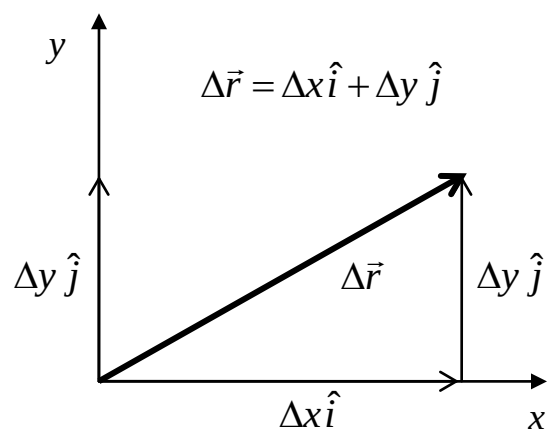


Commutative Addition: Diagrams illustrating the commutative property (A3) for the addition of displacements in two dimensions. $\Delta\vec{r}_1 + \Delta\vec{r}_2 = \Delta\vec{r}_2 + \Delta\vec{r}_1$

Analytic (Component) Methods

Graphical techniques are perhaps superior for thinking about problems, but, for the heavy lifting, analytic methods are needed. A representation is adopted; a coordinate system is chosen. The axes are chosen to be perpendicular (orthogonal). The values Δx and Δy are the projections of $\Delta\vec{r}$ onto the *x* and *y* axes. The unit (normalized basis) vectors $\hat{i}$ and $\hat{j}$ are chosen to have the directions of the *x* and *y* axes. The displacement $\Delta\vec{r}$ can then be represented in terms of its components as $\Delta x\,\hat{i} + \Delta y\,\hat{j}$. Once vector addition, scalar multiplication and the inner product operations are defined in terms of these components, all problems can be submitted to a computer for analysis.





The component representation of a vector is found by representing the vector as a sum of pieces with the number of pieces matching the dimension of the space and with each of the pieces running along one of the coordinate directions. In the figure above, one can drop perpendiculars to the *x* and *y* axes to fin the corresponding components. Notice that $\Delta\vec{r} = \Delta x\,\hat{i} + \Delta y\,\hat{j}$ is clearly a representation of $\Delta\vec{r}$ as the sum of a piece (displacement) in the *x* direction plus a piece in the *y* direction. Note that changing Δx does not change Δy. Displacements in the *x* and *y* directions are ***orthogonal*** to one another.

Component-based representation of:

Addition: $\Delta\vec{r}_1 + \Delta\vec{r}_2 = \left(\Delta x_1\,\hat{i} + \Delta y_1\,\hat{j}\right) + \left(\Delta x_2\,\hat{i} + \Delta y_2\,\hat{j}\right) = (\Delta x_1 + \Delta x_2)\,\hat{i} + (\Delta y_1 + \Delta y_2)\,\hat{j}$

Scalar Multiplication: $c\,\Delta\vec{r}_1 = c\left(\Delta x_1\hat{i} + \Delta y_1\hat{j}\right) = \left(c\,\Delta x_1\right)\hat{i} + \left(c\,\Delta y_1\right)\hat{j}$

Inner Product: $\Delta\vec{r}_1 \cdot \Delta\vec{r}_2 = \left(\Delta x_1\hat{i} + \Delta y_1\hat{j}\right)\cdot\left(\Delta x_2\hat{i} + \Delta y_2\hat{j}\right) = \left(\Delta x_1\,\Delta x_2\right) + \left(\Delta y_1\,\Delta y_2\right)$

The operations of addition, scalar multiplication and inner product are defined in terms of the scalar components of the vectors. The properties of the vector operations can now be based on the properties of operations on the scalar components. The term '*scalars*' means elements of a scalar field. Fields are discussed briefly below.

The *direction cosines* provide a particularly useful representation of the direction of a vector. A direction has components that are the cosine of the angle between that direction and the coordinate direction of interest.

$$\hat{e} = \left(\hat{i}\cdot\hat{e}\right)\hat{i} + \left(\hat{j}\cdot\hat{e}\right)\hat{j} + \left(\hat{k}\cdot\hat{e}\right)\hat{k} = \left(\cos\theta_{e,x}\right)\hat{i} + \left(\cos\theta_{e,y}\right)\hat{j} + \left(\cos\theta_{e,z}\right)\hat{k}$$

Each direction cosine gauges the degree to which the direction $\hat{e}$ points in a given coordinate direction.

The inner product is a gauge of common behavior. The inner product of two displacements gauges the degree to which the displacements are in the same direction and scales that result by the product of the magnitudes of the individual displacements.

This statement is in fuzzy-speak. It is not a math-worthy definition.

The Big Shift: A summary of the properties of displacements has been presented. Those properties are those that are characteristic of vectors as studied in an introductory physics course. They apply to a number of physical vectors. Mathematicians collect the common rules from these behaviors and remove the context to define *mathematical* vector spaces. The spaces will include examples with various dimensions from one to infinity. The vector elements of these spaces can be quite different from our model of displacements in three dimensions. A space may be the collection of all periodic functions with a period *T* or of all 7 x 7 matrices with complex elements.

Don't expect outcomes based on your prior experiences. Make your highest priority memorizing the definitions of terms and statements of rules (axioms) as they arise. Base your arguments and conclusions on the new definitions and rules rather than

on any prejudices based on your understanding of displacements.

We will return to the concept of a physical vector in another section. At that time, the properties required to qualify an entity as a physical vector will be discussed.

An Abstract Linear Vector Space: By definition, a vector space is any collection of physical or mathematical entities (elements) with defined ***binary addition*** and ***scalar multiplication*** operations that satisfy the axioms below. The scalar multipliers are chosen from a field. The real numbers and complex numbers are examples of scalar fields. Just to ensure that you stay abstract and that you can't identify the vectors discussed with any that you may have encountered in the past, vectors will be represented as kets. $\{|A\rangle, |B\rangle, |C\rangle, ...\}$. The name ket is chosen because the combination $\langle \ | \ \rangle$ is a bracket. It must follow that $\langle A|$ is a bra and that $|B\rangle$ is a ket. A bra and ket $\langle A|B\rangle$ together represents the inner product of the two vectors.

A collection of entities (elements) forms a vector space $\mathbb{V}$ if and only if the entities have defined operations for ***binary*** addition (for combining ***two*** vectors to yield another vector) and for multiplication of a vector by a scalar to yield another vector that are closed (that yield results which are also in $\mathbb{V} \Rightarrow$ A1 & M1 below) and that obey the rules: A2-A5 & M2-M5. The property blocks below first state the ***closure*** property and then list four more axioms satisfied by the addition and multiplication processes respectively.

TABLE I: Vector Space Axioms for Addition and Multiplication

Axioms for Vector Addition for elements of a vector space $\mathbb{V}$

A1. *Closure under vector addition*:

$$\text{For all } |A\rangle, |B\rangle \in \mathbb{V},\ |A\rangle + |B\rangle = |C\rangle \in \mathbb{V}$$

A2. The addition is *associative*.

$$|A\rangle + (|B\rangle + |C\rangle) = (|A\rangle + |B\rangle) + |C\rangle$$

A3. The addition is *commutative*.

$$|A\rangle + |B\rangle = |B\rangle + |A\rangle$$

A4. There is an *additive identity* (*zero*) *vector*.

$$|A\rangle + |0\rangle = |A\rangle \text{ for all } |A\rangle \in \mathbb{V}$$

A5. There is an a*dditive inverse* for each vector.

$$\text{For all } |A\rangle \in \mathbb{V} \quad \exists |-A\rangle \in \mathbb{V} \text{ such that } |A\rangle + |-A\rangle = |0\rangle$$

Axioms for Multiplication by a Scalar

M1. *Closure*: The product of a scalar *c* and any vector in $\mathbb{V}$ is also in $\mathbb{V}$.

$$\text{For all } |A\rangle \in \mathbb{V},\ c\,|A\rangle \in \mathbb{V}$$

M2. Scalar multiplication is *distributive across a sum of vectors*.

$$c\left(|A\rangle + |B\rangle\right) = c|B\rangle + c|A\rangle$$

M3. Scalar multiplication is *distributive across a scalar sum*. For scalars *c* and *d*,

$$(c+d)|A\rangle = c|A\rangle + d|A\rangle$$

M4. Scalar multiplication is *associative*.

$$(c\,k)|A\rangle = c\left(k|A\rangle\right)$$

M5. Action of the scalar *identity*

$$1|A\rangle = |A\rangle \text{ for all } |A\rangle \in \mathbb{V}$$

Learn to express each axiom in prose form!

If only real scalars $\mathbb{R}$ are allowed, the space is a **real linear vector space**. If complex scalars $\mathbb{C}$ are allowed, then the vector space is a **complex vector space**. For a general case, the notation is to use *a*, *b*, *c*, ... as elements of an unspecified scalar field $\mathbb{K}$. Only spaces over $\mathbb{R}$ and $\mathbb{C}$ are of immediate interest to us. We must admit that there are other fields to cover all the cases of interest to mathspeak persons.

Closure example: Consider the addition of displacements. Start with $\boldsymbol{S_1}$**,** set of all displacements between points in an infinite three dimensional space. The sum of two such displacements is also in that set. Hence the set is closed under addition. Contrast this with $\boldsymbol{S_2}$**,** set of all displacements of magnitude 5 meters or less. One can add two displacements of magnitude 4 meters (which are members of that set) to get a displacement with a magnitude greater than 5 meters (which is not a member of the set). The second example set $\boldsymbol{S_2}$ is not closed under addition and so the set $\boldsymbol{S_2}$ (with its operations) is not a vector space.

Exercise: Consider the collection of objects of the form $\vec{v} = a\,\hat{i} + b\,\hat{j}$ where *a, b, and all other scalars* used are any real numbers. The vector addition and scalar (*c*) multiplication operations using real numbers are defined as:

$$\vec{v}_1 + \vec{v}_2 = (a_1 + a_2)\,\hat{i} + (b_1 + b_2)\,\hat{j}$$

$$c\,\vec{v}_1 = c\left(a_1\hat{i} + b_1\hat{j}\right) = (c\,a_1)\hat{i} + (c\,b_1)\,\hat{j}$$

Show that the collection of all $\vec{v}$ as prescribed is a linear vector space. Assume that scalar addition and multiplication of the real numbers obey all the common rules.

Notation Alert: The use of the vector label notation $|\,0\rangle$ representing the additive identity is sometimes preempted by another convention. For example, in quantum mechanics, $|0\rangle$ may represent the *ground state* (lowest energy state) for a problem. In these cases the additive identity is to be represented as either $|\text{ZERO}\rangle$ or $|\text{NULL}\rangle$. Stay alert; this conflict will arise when you study the quantum oscillator.

What is a field? From: http://mathworld.wolfram.com/Field.html

A field is any set of elements that satisfies the field axioms for both addition and multiplication and is a commutative division algebra. An archaic name for a field is

rational domain. The French term for a field is corps and the German word is Körper, both meaning "body." A field with a finite number of members is known as a finite field or Galois field.

As the identity elements of a field must be different for addition and multiplication, every field must have at least two elements. Examples of fields include the complex numbers, rational numbers, and real numbers, but not the integers (as integers do not include multiplicative inverses for all integers).

TABLE II: The field axioms written in addition-multiplication pairs.

Property	Addition	Multiplication
Commutative	$a+b=b+a$	$a\,b=b\,a$
Associative	$a+(b+c)=(a+b)+c$	$a\,(b\,c)=(a\,b)\,c$
Distributive	$a\,(b+c)=a\,b+a\,c$	$(a+b)\,c=a\,c+b\,c$
Identity	$a+0=a$	$1a=a$
Inverse	$a+(-a)=0$	$a\,a^{-1}=1$ (*)

(*) *The additive identity (0) is an exception; it does not have a multiplicative inverse.*

Notably, vector spaces lack self-multiplication and hence multiplicative inverses.

Math Trivia: The vector space rules or axioms require that a space contain at least one element, the additive null $|0\rangle$. If it contains any other element $|A\rangle$, then it must include the additive inverse $|-A\rangle$ and all elements of the form $c\,|A\rangle$ where c is an element of the associated scalar field. All the fields *that we consider* have an infinite number of elements so our vector spaces are likewise infinite, although they may be (and often are) finite dimensional.

Linear dependence, spanning sets and basis sets:

Def$^{\underline{n}}$: Linearly dependent set: A set of vectors $\{...,|V_i\rangle,...\}$ is linearly dependent if there is a linear combination of those vectors (sum of scalars times vectors) that equals the zero vector. $c_1|V_1\rangle + c_2|V_2\rangle + ... + c_i|V_i\rangle + ... = |0\rangle = |\text{zero}\rangle$ and not all the scalars c_i are zero.

If $c_4 \neq 0$, then $|V_4\rangle$ can be represented as a sum of multiples of the other members of the set. That is $|V_4\rangle$ is linearly dependent on the other members of the set. If the zero vector $|0\rangle$ is a member of the set then the set is linearly dependent because the sum of one times $|0\rangle$ plus zero times the remaining members sums to the zero vector $|0\rangle$.

If a set of vectors is not linearly dependent, it is **linearly independent**.

The span of a set of vectors $\mathbb{S} = \{...,|V_i\rangle,...\}$ is all the vectors that can be represented as finite linear combinations of the members of the set ($\sum a_i |V_i\rangle$ where the a_i are arbitrary scalar coefficients). The set $\mathbb{S} = \{...,|V_i\rangle,...\}$ is said to ***span*** the space of elements $\sum a_i |V_i\rangle$. Span($\{...,|V_i\rangle,...\}$) $= \left\{....,\sum a_i |V_i\rangle,.....\right\}$ for any $\{\ldots a_i \ldots\} \in \mathbb{K}$.

The span of a set of vectors meets the requirements to be a vector space. The rules for forming linear combinations assure the axioms for binary addition and scalar multiplications are met.

A spanning set for a space: Any set of vectors $\mathbb{S} = \{...,|S_i\rangle,...\}$ complete enough that every vector in that space $\mathbb{V}$ is of the form $\sum a_i |S_i\rangle$ is a spanning set for that space. That is: If the span of the set $\mathbb{S}$ is the entire space $\mathbb{V}$, then $\mathbb{S}$ is a spanning set for $\mathbb{V}$.

A spanning set must include at least one vector embodying each distinct characteristic behavior found in members of the space. A reduced spanning set is the spanning set less one or more members which also spans the complete vector space. The requirement that the reduced set span the same space ensures that any member that can be removed is a linear combination of the members that remain.

Reducing a spanning set: Any members of the spanning set $\mathbb{S} = \{..., |V_i\rangle, ...\}$ that are linearly dependent on the other members of that spanning set can be removed from the set without removing any vectors from the span of the set = { …., $\sum a_i |V_i\rangle$, …. }, the collection of all possible linear combinations of the members of $\mathbb{S}$.

A basis for a space: Any spanning set for a space that cannot be reduced is a basis set for that space. A basis is a spanning set with the smallest possible number of elements. If a member is removed from a basis set, then the span of the remaining members no longer includes all the elements of the vector space. Equivalently, a basis set is a spanning set that is linearly independent.

If a vector is removed from a basis set, one characteristic behavior found in the original space is lost.

Dimension: The number of members in the smallest spanning set for a space is the **dimension** of that space. A set with the smallest number of members that still spans the space is a ***basis*** for the space. This number (or dimension) is the same as the largest number of linearly independent vectors in the space.

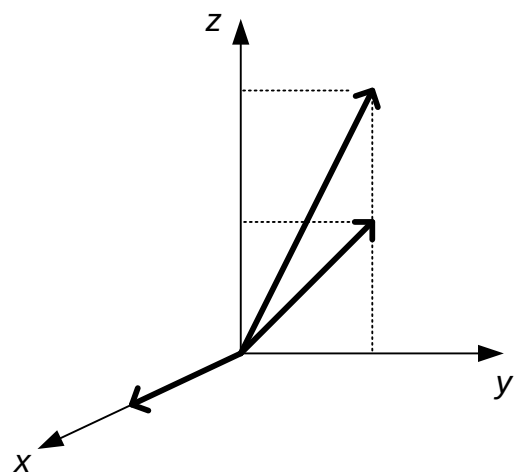
The dimension of our space of all three-space displacements is, of course, three.

A **subspace** is a vector space $\mathbb{W}$ all of whose elements are contained in another vector space $\mathbb{V}$. It follows that Dimension($\mathbb{W}$) $\leq$ Dimension($\mathbb{V}$). The span of any set is a vector space. If one vector is removed from a basis for $\mathbb{V}$ then the span of the remaining vectors in the set is a subspace of $\mathbb{V}$ with dimension = Dimension($\mathbb{V}$) -1.

The concepts just presented are for general *abstract* vector spaces. A set of concrete examples follows. Do not lose sight of the more general nature of the concepts.

Examples of the concepts restricted to 3D Cartesian displacement space.

Linearly independent: The set of vectors $\{\hat{i}, \hat{j}+\hat{k}, \hat{j}+2\hat{k}\}$ is linearly independent. It would be dependent if $a\hat{i}+b(\hat{j}+\hat{k})+c(\hat{j}+2\hat{k})=\vec{0}$ for some {a,b,c }, not all zero. That would require: a = 0; b + c = 0, and b + 2 c = 0. Those conditions are contradictory. It can't happen. The conditions require that a = b = c = 0. More directly, in order to be a linear combination of the other two vectors, the sum of multiples of the vectors must be a triangle, a closed figure. This would require that one of the vectors lies in the plane defined by the other two.



Span of a set: The span of $\{\hat{i}, \hat{j}+\hat{k}\}$ is all vectors of the form $a\hat{i}+b(\hat{j}+\hat{k})$ or all the vectors in a plane that includes the x axis and the line y = z. The span of $\{\hat{i}, \hat{j}+\hat{k}, \hat{j}+2\hat{k}\}$ is all vectors of the form $a\hat{i}+b(\hat{j}+\hat{k})+c(\hat{j}+2\hat{k})$ or all the vectors in the Cartesian three-dimensional space.

Spanning set: The set $\{\hat{i}, \hat{j}+\hat{k}\}$ is a spanning set for the space of all the vectors in a plane that includes the *x* axis and the line *y* = *z*. The set $\{\hat{i}, \hat{j}+\hat{k}, \hat{j}+2\hat{k}\}$ is a spanning set for the space of all the vectors in the Cartesian three-dimensional space. The set $\{\hat{i}, \hat{j}+\hat{k}, \hat{j}+2\hat{k}, \hat{k}\}$ is a spanning set for the space of all the vectors in the Cartesian three-dimensional space. Note that the last spanning set contains more members than the dimension of the space that it spans.

Basis sets: A spanning set with the fewest number of members or equivalently that is linearly independent. The set $\{\hat{i}, \hat{j}+\hat{k}\}$ is a basis set for the space of all the vectors in a plane that includes the *x* axis and the line *y* = *z*. The set $\{\hat{i}, \hat{j}+\hat{k}, \hat{j}+2\hat{k}\}$ is a basis set for the space of all the vectors in the Cartesian three-dimensional space. The set $\{\hat{i}, \hat{j}+\hat{k}, \hat{j}+2\hat{k}, \hat{k}\}$ is a spanning set for the space of all the vectors in the Cartesian three-dimensional space; it is not a basis set because the set is not linearly independent (or has more members than the minimum required). The minimum number of members is called the dimension of the space.

Exercise: A vector is represented as $\vec{v} = a\,\hat{i} + b(\hat{j}+\hat{k}) + c(\hat{j}+2\hat{k}) + d\,\hat{k}$, a linear combination of a spanning set. Represent the vector as a combination of the members of a basis. $\vec{v} = a'\,\hat{i} + b'(\hat{j}+\hat{k}) + c'(\hat{j}+2\hat{k})$. What are a', b' and c'?

The representation of a vector as a linear combination of members of a particular ***basis set*** is ***unique*** while there are many possible representations as linear combinations of the members of a ***spanning set*** with more members than are found in a basis.

Sample calculation of a span and identifying its dimension:

Consider the set of vectors $\mathbb{S} = \left\{ \vec{B}_1 = 2\,\hat{i} + 2\,\hat{j} - 2\,\hat{k},\ \vec{B}_2 = 4\,\hat{i} + 3\,\hat{j} - 3\,\hat{k},\ \vec{B}_3 = -\,\hat{j} + \hat{k} \right\}$ over the real numbers $\mathbb{R}$. The span is the collection of all linear combinations of the three

members of the set. $Span(\mathbb{S}) = a\left(2\,\hat{i} + 2\,\hat{j} - 2\,\hat{k}\right) + b\left(4\,\hat{i} + 3\,\hat{j} - 3\,\hat{k}\right) + c\left(-\,\hat{j} + \hat{k}\right) \quad \forall a,b,c \in \mathbb{R}$.

Defining $\alpha = a - 2\,c$ and $\beta = b + c$, $Span(\mathrm{S}) = \left(2\alpha + 4\beta\right)\hat{i} + \left(2\alpha + 3\beta\right)\hat{j} - \left(2\alpha + 3\beta\right)\hat{k} \quad \forall \alpha, \beta \in \mathbb{R}$.

The last form is a curious rewriting, but it demonstrates that the *Span(*$\mathbb{S}$*)* depends on only two free parameters and therefore must be suspected to be only two dimensional. Further $\vec{B}_3 = -2\,\vec{B}_1 + \vec{B}_2$. That is $\vec{B}_3$ is linear combination of $\vec{B}_1$ and $\vec{B}_2$ hence the set is linearly dependent.

$$Span(\mathbb{S}) = a\left(\vec{B}_1\right) + b\left(\vec{B}_2\right) + c\left(-\,2\,\vec{B}_1 + \vec{B}_2\right) = \alpha\left(\vec{B}_1\right) + \beta\left(\vec{B}_2\right) \quad \forall a,b,c,\alpha,\beta \in \mathbb{R}$$

The space is two-dimensional as two vectors form a *smallest* spanning set for it.

Exercise: Consider the set of vectors, $\left\{\hat{i} + \hat{j} + \hat{k}, \hat{i} + \hat{j}, \hat{k}\right\}$. Display the general form for the span of this set. Assume a space over the real numbers. The set is linearly dependent. Express $\hat{k}$ as a linear combination of the other two vectors. Display a reduced set that spans the same space as does the original set. Display a basis set for the space spanned by $\left\{\hat{i} + \hat{j} + \hat{k}, \hat{i} + \hat{j}, \hat{k}\right\}$. What is the dimension of that space? Describe the locus of points described if the vectors are considered to be a set of displacements from the origin.

The locus of points designated by the vectors spanned satisfies the equation $y - x = 0$, a plane perpendicular to the x-y plane (as z is unrestricted) and intersecting with it on the line $y = x$. The normal to the plane lies along the cross product of any two (non-parallel) vectors in the plane. Here $\left(\hat{i} + \hat{j} + \hat{k}\right) \times \left(\hat{i} + \hat{j}\right) = \left(\hat{j} - \hat{i}\right)$, so one normal to the plane is $\hat{n} = \dfrac{\left(\hat{j} - \hat{i}\right)}{\sqrt{2}}$.

A Few Examples of Vector Spaces:

***N*-tuple space**. For a field $\mathbb{K}$ with members *a*, *b*, *c*, ... consider the set of ordered collections of scalar values $|A\rangle = (a_1, a_2, ..., a_{N-1}, a_N)$ where addition is defined as $|A\rangle + |B\rangle = (a_1 + b_1, a_2 + b_2, ..., a_{N-1} + b_{N-1}, a_N + b_N)$ and multiplication is defined as $c\,|A\rangle = (c\,a_1, c\,a_2, ..., c\,a_{N-1}, c\,a_N)$. A basis for the space is {(1,0, ….,0); (0,1,0,…,0), … , (0,0, … ,1)}. No application has been discovered for this space; it is native to mathematics texts. (Dimension = *N*.)

3-D displacement space. For the field of the real numbers $\mathbb{R}$ with members, *a*, *b*, *c*, *d*, ... consider the $|\Delta\,\vec{r}_i\rangle = a_i\,\hat{i} + b_i\,\hat{j} + c_i\,\hat{k}$ where addition is defined as $|\Delta\,\vec{r}_i\rangle + |\Delta\,\vec{r}_j\rangle = (a_i + a_j)\,\hat{i} + (b_i + b_j)\,\hat{j} + (c_i + c_j)\,\hat{k}$ and multiplication is defined as $d\,|\Delta\,\vec{r}_i\rangle = d\,a_i\,\hat{i} + d\,b_i\,\hat{j} + d\,c_i\,\hat{k}$. A basis for the space is { $\hat{i}, \hat{j}, \hat{k}$ }. Examples of this space are common in physics texts.

Polynomial (*n*th order) space. For the field of the real numbers $\mathbb{R}$ or complex $\mathbb{C}$ numbers with members, *a*, *b*, *c*, *d*, ... consider the $|P_{An}(x)\rangle = a_o + a_1\,x + ... + a_n\,x^n$ where addition is defined as $|P_{An}(x)\rangle + |P_{Bn}(x)\rangle = (a_o + b_o) + (a_1 + b_1)\,x + ... + (a_n + b_n)\,x^n$ and multiplication is defined as $d\,|P_{An}(x)\rangle = d\,a_o + d\,a_1\,x + ... + d\,a_n\,x^n$. (Dimension = *n* + 1.) A basis for the space is {x^0, x^1, …, x^n}. One could also use the first *n* + 1 Hermite polynomials. There are many possibilities. A strange, but legal choice, is {1 (= x^0), 1 + *x*, 1 + *x* + x^2,…, 1 + *x* + x^2 + … + x^n}.

Taylor's series space. For the field of the real numbers $\mathbb{R}$ or complex $\mathbb{C}$ numbers with members, d_{An} , d_{Bn}, d_{Cn}, d_{Dn}, ... consider the $|T_A(x)\rangle = \sum_{n=0}^{\infty} d_{An}\,x^n$ where addition is defined as $|T_A(x)\rangle + |T_B(x)\rangle = \sum_{n=0}^{\infty} (d_{An} + d_{Bn})\,x^n$ and multiplication is defined as $c\,|T_A(x)\rangle = \sum_{n=0}^{\infty} (c\,d_{An})\,x^n$.

The space is to be restricted to series that converge. (Dimension = infinite.) A basis for the space is $\{x^0, x^1, \ldots, x^n, \ldots, \ldots\}$.

Oscillator space. The space of all functions specifying the position as a function of time for a simple harmonic oscillator: For the field of the real numbers $\mathbb{R}$ with members, *a*, *b*, *c*, *d*, ... consider the $\left|x_i(t)\right\rangle = a_i \cos(\omega t) + b_i \sin(\omega t)$ where addition is defined as $\left|x_i(t)\right\rangle + \left|x_j(t)\right\rangle = (a_i + a_j)\cos(\omega t) + (b_i + b_j)\sin(\omega t)$ and multiplication is defined as $d\left|x_i(t)\right\rangle = d\, a_i \cos(\omega t) + d\, b_i \sin(\omega t)$. (Dimension = homework.)
A basis for the space is $\{\cos(\omega t), \sin(\omega t)\}$.

Fourier series space. For the field of the real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$ with members, *a*, *b*, *c*, *d*, ... consider the Fourier space of elements
$f(t) = c_{f0}(1) + \sum_{n=1}^{\infty} a_{fn} \cos[n\omega_0 t] + \sum_{m=1}^{\infty} b_{fm} \sin[m\omega_0 t]$ where addition is defined by summing the corresponding Fourier coefficients and multiplication by *d* is defined by multiplying each Fourier coefficient by *d*. (Dimension = infinite.) A basis for the space is $\{1, \{\ldots, \cos(n\omega_o t), \ldots\}, \{\ldots, \sin(m\omega_o t), \ldots\}\}$ where *m* and *n* are strictly positive integers.

$$f(t) + g(t) = c_{f0}\left(c_{f0} + c_{g0}\right) + \sum_{n=1}^{\infty}(a_{fn} + a_{gn})\cos\left[n\omega_0 t\right] + \sum_{m=1}^{\infty}(b_{fm} + b_{Gm})\sin\left[m\omega_0 t\right]$$

$$d\, f(t) = \left(d\, c_{f0}\right) + \sum_{n=1}^{\infty}\left(d\, a_{fn}\right)\cos\left[n\omega_0 t\right] + \sum_{m=1}^{\infty}\left(d\, b_{fm}\right)\sin\left[m\omega_0 t\right]$$

Legendre Polynomial of cosθ Space for *f*(θ) defined for 0 ≤ θ ≤ π.

For the field of the real numbers $\mathbb{R}$ with members, a_ℓ consider the Legendre space of well-behaved (*finite-valued*) elements for $0 \le \theta \le \pi$: $\left|LS_f\right\rangle = f(\theta) = \sum_{\ell=0}^{\infty} a_{f\ell} P_\ell(c\cos\theta)$

where addition is defined by summing the corresponding expansion coefficients and multiplication by *d* is defined by multiplying each expansion coefficient by *d*. (Dimension = infinite.) The basis is the set of $P_\ell(\cos\theta)$ for ℓ a positive integer.

General Fourier series space. For the field of the complex $\mathbb{C}$ numbers with members *a, b, c, d, ...* consider a Fourier space of elements $|\Psi_i\rangle = \sum_{m=1}^{\infty} a_{im}\, \phi_m(x)$ where the $\{...,\phi_m(x),.....\}$ are a set of linearly independent (preferably orthogonal) functions. For example, $\Psi_i(x)$ might be the wave function for an electron in hydrogen. The $\{...,\phi_m,.....\}$ would be the set of distinct energy eigenfunctions for the electron in hydrogen. The state ϕ_1 could be the 1s ground state with energy -13.6 *eV* and spin up. Suggest definitions for addition and scalar multiplication for this space. The basis is the set $\{...,\phi_m(x),.....\}$ of linearly independent (preferably orthogonal) functions.

The Space of Null: A space containing only one element $|0\rangle$, the zero or additive identity element. Note: space of null is the "dull" space. The basis set for the space is the single element $|0\rangle$.

Matrix Space: The collection of all *m* x *n* matrices with complex scalar elements can be considered to be a vector space with complex numbers as the field.

Exercise: For matrix space, propose a definition or procedure for matrix addition and for scalar multiplication. What is the dimension of the space of all *m* x *n* matrices? Propose a set of basis vectors for the space. (Each matrix in the basis set is a *vector* in the sense of abstract vector spaces.

Certifying a vector space:

A vector space is a set of elements that has as defined operations a binary addition and a scalar multiplication that satisfy the 10 axioms required of vector space operation.

Step 1: Identify the members of the set that is to be investigated. That identification should include what is essentially a rule to test for membership in the set.

Step 2: The set must support two operations on it members, binary addition (the combining of two members to yield a third) and scalar multiplication of each member to yield a member. The operations must be defined. The scalar field involved should be specified in this step.

Step 3: Verify that each of the defined operations satisfies each of its five axioms. I suggest beginning with closure for each operation. If the vectors have a component representation, the operations are usually defined in terms of operations on the scalar components which satisfy the field axioms. There is a field axiom that is parallel to each vector space axiom. Use it to establish that the vector space axiom is satisfied.

The certification process should be exhaustive and exhausting. If it fails to be so, the certification is probably incomplete.

SAMPLE CALCULATION: Certifying a set with its field and operations as a vector space**:**

Show that the set of ordered 3-tuples of real numbers satisfies the requirements to be a vector space with respect to the scalar field, the real numbers.

The game plan is to note that the vector space operations are defined in terms of parameters that are members of a scalar field. The operations of addition and multiplication are defined in terms of the operations for that scalar field. Scalar fields satisfy axioms similar to those that are to be established in order that the set of objects be certified as a vector space. In fact, a scalar field is itself a one-dimensional vector space with one or two extra features. For the case of a field, the multiplication is an

operation on two members of the same set instead of one from the vector set and one from a separate scalar set, and a multiplicative inverse exists as a member of the set for all elements in the set except for the additive identity (zero).

Set the stage by reviewing the contestants and the rules:

Members: all ordered triplets $|V_i\rangle = (a_i, b_i, c_i)$ where $a_i, b_i, c_i \in \mathbb{R}$

Addition: $|V_i\rangle + |V_j\rangle = (a_i + a_j,\ b_i + b_j,\ c_i + c_j)$

Multiplication: $d\,|V_i\rangle = (d\ a_i, d\ b_i, d\ c_i)$

A1: *Closure under vector addition*: The real numbers are a field and hence have closure for addition. For a_i and a_j real numbers, $a_i + a_j$ is also a real number. Thus $(a_i + a_j,\ b_i + b_j,\ c_i + c_j)$ is a triplet of real numbers and therefore a member of the set of all ordered triplets of real numbers.

A2. The addition is *associative*. The game plan is to use the properties of the field in each case to establish the analogous property for the set of elements be examined for certification as a vector space. Let's not waste time. The addition of the triplets is defined component-wise (position by position in the triplet). The components are real numbers and their addition is associative. Therefore, the addition of the ordered triplets of real numbers is also associative.

To receive full credit, a student should include the detail of the next few lines plus a few lines of prose justifying each step based on the properties of the scalar field.

$$\begin{aligned}
\left(|I\rangle + |J\rangle\right) + |K\rangle = |I + J\rangle + |K\rangle &= \left((a_i + a_j) + a_k, (b_i + b_j) + b_k, (c_i + c_j) + c_k\right) \\
&= \left(a_i + a_j + a_k, b_i + b_j + b_k, c_i + c_j + c_k\right) = \left(a_i + (a_j + a_k), b_i + (b_j + b_k), c_i + (c_j + c_k)\right) \\
&= |I\rangle + \left(|J + K\rangle\right) = |I\rangle + \left(|J\rangle + |K\rangle\right)
\end{aligned}$$

A3. The addition is *commutative*. The addition of the triplets is defined component-wise (position by position in the triplet). The components (coefficients in the linear

combinations) are real and their addition is commutative. Therefore, the addition of the ordered triplets of real numbers is also commutative.

$$|I\rangle + |J\rangle = (a_i + a_j, b_i + b_j, c_i + c_j) = (a_j + a_i, b_j + b_i, c_j + c_i) = |J\rangle + |I\rangle$$

A4. There is an *additive identity* (*zero vector*). The number 0 is the unique additive identity (zero or null) for the real numbers. The zero vector must be $|0\rangle = (0,0,0)$. As 0 is the unique scalar additive identity (zero), the zero vector is unique.

$$|I\rangle + |0\rangle = (a_i + 0, b_i + 0, c_i + 0) = (a_i, b_i, c_i) = |I\rangle \; \forall \, |I\rangle \in \mathbb{V}$$

A5. There is an a*dditive inverse* for each vector. If a_i is a real number, then its additive inverse $-a_i$ is also a real number. Hence for every (a_i, b_i, c_i) in the set, the ordered triplet $(-a_i, -b_i, -c_i)$ is also in the set. It is the additive inverse of (a_i, b_i, c_i).

$$|I\rangle + |-I\rangle = (a_i - a_i, b_i - b_i, c_i - c_i) = (0, 0, 0) = |0\rangle$$

M1. *Closure*: under multiplication by a real. If $a_i \in \mathbb{R}$ then $d\, a_i \in \mathbb{R}$. Thus $d\,|V_i\rangle = (d\, a_i, d\, b_i, d\, c_i)$ is also a triplet of real numbers.

SLOW DOWN: $d\, a_i \in \mathbb{R}$ is true if $d\, a_i \in \mathbb{R}$. The field $\mathbb{K}$ must be the reals. Ordered triplets of real numbers are a vectors space ***over*** the real numbers. You should make the source fields for the components or coefficients match the field for the vector space multiplication. You might be able to make a space of triplets of complex numbers over the real numbers, but it works better if the members of the multiplet are members of the same field as the scalar field used in the definition of scalar multiplication for the space.

M2. Scalar multiplication is *distributive across a sum of vectors*. The multiplication is defined component-wise and the multiplication of real numbers is distributive. Therefore multiplication is distributive across a sum of triplets of real numbers.

$$d\left(|I\rangle+|J\rangle\right)=\left(d\left[a_i+a_j\right],d\left[b_i+b_j\right],d\left[c_i+c_j\right]\right)=\left(d\,a_j+d\,a_i,d\,b_j+d\,b_i,d\,c_j+d\,c_i\right)=d\,|I\rangle+d\,|J\rangle$$

M3. Scalar multiplication is *distributive across a scalar sum*. The multiplication is defined component-wise and the multiplication of reals is distributive. Therefore the multiplication by a sum is distributive.

$$(d+e)\,|I\rangle=\left(\left[d+e\right]a_i,\left[d+e\right]b_i,\left[d+e\right]c_i\right)=\left(d\,a_i+e\,a_i,d\,b_i+e\,b_i,d\,c_i+e\,c_i\right)=d\,|I\rangle+e\,|I\rangle$$

M4. Scalar multiplication is *associative*. The multiplication is defined component-wise and the multiplication of real numbers is associative. Therefore the multiplication of the triplets is associative.

$$(d\,e)\,|I\rangle=\left(\left[d\,e\right]a_i,\left[d\,e\right]b_i,\left[d\,e\right]c_i\right)=\left(d\left[e\,a_i\right],d\left[e\,b_i\right],d\left[e\,c_i\right]\right)=d\left(e\,|I\rangle\right)$$

M5. Multiplicative *identity*. The real numbers have a scalar identity 1. As $1\,a_i=a_i$ and the multiplication is defined component-wise, $1\left(a_i,b_i,c_i\right)\equiv\left(a_i,b_i,c_i\right)$, and thus 1 is the scalar multiplicative identity for the vector space of ordered triplets of real numbers.

$$(1)\,|I\rangle=\left([1]\,a_i,[1]\,b_i,[1]\,c_i\right)=\left(\,a_i,b_i,c_i\right)=|I\rangle$$

A possible basis set for the triplets is: {(1,0,0); (0,1,0), (0,0,1)} :

$$(a_i,\,b_i,\,c_i)\equiv a_i\,(1,0,0)+b_i\,(0,1,0)+c_i\,(0,0,1)$$

Summary: Proving that a set of elements is a vector space given its associated rule for addition, rule for multiplication and scalar field is *tedious*. It is accomplished by assuming the validity of the axioms for the associated field $\mathbb{K}$ and invoking them repeatedly to establish the properties to qualify the set of elements with its operations as a vector space. The set *n*-tuples and infinite-tuples of real or complex numbers are mathematical twins of (isomorphic to) the spaces (sets of interest) to be studied. Given that the *n*-tuples and infinite-tuples of real numbers or complex numbers are vector spaces, the spaces of interest are also fully qualified vector spaces.

The space of triplets was certified to be a vector space in this handout. The proof was component-wise without reference to the number of components so the arguments are adequate to certify that *n*-tuple space is a vector space. The space of a set of *n* dimensional vectors has addition and multiplication defined analogously to that for *n*-tuple space so once the coefficient of each vector in the spanning set is identified with the scalar number at a position in the *n*-tuple. Hence the span of any finite set of vectors is a vector space.

******** PROOFS related to Basis, Dimension and Linear Independence ********

<u>Review of **linear dependence**</u>: A set of vectors $\{...,|B_i\rangle,...\}$, where the label *I* takes on several values, is linearly dependent if there exists a set of scalars $\{...,c_i,...\}$ such that not all the c_i are zero and $\sum c_i|B_i\rangle = 0$. Suppose that $c_4 \neq 0$. Then $|4\rangle$ can be represented as a sum of multiples of the other members of the set (*or* $|4\rangle = |0\rangle$ *which is a sum of zero multiples of the other members of the set*). That is $|4\rangle$ is linearly dependent on the other members of the set. If a set of vectors is not linearly dependent, it is **linearly independent**.

Beware the ZERO vector: Consider a set of *N* vectors $\{...,|V_i\rangle,...\}$ in which the mth member $|V_m\rangle$ is the zero vector $|0\rangle$. The following sum is zero, and it has one constant that is non-zero. Hence the set is linearly dependent.

$$0|V_1\rangle + 0|V_2\rangle + \ldots + 0|V_{m-1}\rangle + 1|V_m\rangle + 0|V_{m+1}\rangle + \ldots + 0|V_N\rangle = |0\rangle$$

$$\Uparrow$$

$|V_m\rangle$ is the zero vector $|0\rangle$.

Trivial (but important) Result: Any set of vectors that includes the zero vector is linearly dependent. Begin each linear independence problem by addressing $|0\rangle$.

The Replacement Lemma: Suppose that $\{|v_1\rangle, |v_2\rangle, ..., |v_m\rangle\}$ spans a vector space $\mathbb{V}$. Given any non-zero $|w\rangle \in \mathbb{V}$, the vector $|w\rangle$ can be substituted into the spanning set for one of the $|v\rangle$'s (say $|v_i\rangle$) and the resulting set $\{|w\rangle, |v_1\rangle, |v_2\rangle, \ldots, |v_{i-1}\rangle, |v_{i+1}\rangle, \ldots, |v_m\rangle\}$ still spans the space $\mathbb{V}$.

As $|w\rangle \in \mathbb{V}$, it is in the span of $\{|v_1\rangle, |v_2\rangle, ..., |v_m\rangle\}$ which means that

$$|w\rangle = c_1|v_1\rangle + c_2|v_2\rangle + ... + c_m|v_m\rangle$$

and that at least one of the c_j is not zero as $|w\rangle$ is not the zero vector. Suppose that c_i is not zero. Then

$$|v_i\rangle = (1/c_i)|w\rangle - (c_1/c_i)|v_1\rangle - (c_2/c_i)|v_2\rangle - ... - (c_{i-1}/c_i)|v_{i-1}\rangle - c_{i+1}/c_i)|v_{i+1}\rangle - (c_m/c_i)|v_m\rangle$$

The vector $|w\rangle$ can now replace $|v_i\rangle$ in the set without reducing the span. All the vectors in $\mathbb{V}$ are in the span of the original basis and have representations of the form $|u\rangle = d_1|v_1\rangle + d_2|v_2\rangle + ... + d_i|v_i\rangle + \ldots + d_m|v_m\rangle$. That same vector can be represented in terms of the once-replaced set as:

$$\begin{aligned}|u\rangle = {} & d_1|v_1\rangle + d_2|v_2\rangle + ... + \\ & d_i\left[(1/c_i)|w\rangle - (c_1/c_i)|v_1\rangle - (c_2/c_i)|v_2\rangle - ... - (c_{i-1}/c_i)|v_{i-1}\rangle - c_{i+1}/c_i)|v_{i+1}\rangle - (c_m/c_i)\right] \\ & + \ldots + d_m|v_m\rangle\end{aligned}$$

Clearly every vector that is contained in the span of the original set is also in the span of the once-replaced set.

NOTE: If the original set is a basis set, each vector in the set has some character or behavior that is not in any other member of the basis. If the vector $|w\rangle$ is to replace a vector $|v_m\rangle$, it must possess that particular character or behavior of the vector $|v_m\rangle$ (in

addition to other characters embodied by other members of the basis) that is not present in the other members of the basis. (For a basis set, the coefficient c_i in the expansion of $|w\rangle$ will be non-zero only if $|w\rangle$ possesses the character of $|v_i\rangle$.) A spanning set may have multiple copies of a given behavior as it may have more members than the minimum present in a basis set.

The Replacement Lemma is important as it assures us that, if we are a little careful, we can replace a basis set with N members by another set of N vectors that can also be used as a basis. For 3D, we may switch from $\{\hat{i}, \hat{j}, \hat{k}\}$ to $\{\hat{i}, \hat{j}+\hat{k}, \hat{j}-\hat{k}\}$, or just rotate your axes. In quantum mechanics, one could use the chemist's $\{s, p_x, p_y, p_z\}$ set or the sp^3 hybridized set as a basis for the (angular) behavior of the electrons in a valence shell. The set $\{s, p_x, p_y, p_z\}$ has one non-directional state and the $\{p_x, p_y, p_z\}$ that are appropriate for bonds along the Cartesian axis directions. The sp^3 hybridized set is better for the four tetrahedral bonding directions of a molecule like methane.

Basis sets: Earlier, a basis set was defined to be a spanning set from which no member could be removed without reducing the span. An alternative definition is to say that a basis set is a spanning set consisting of linearly independent vectors. The basis set is $\{|B_1\rangle, |B_2\rangle, \ldots\ldots, |B_n\rangle\}$. **Assume** that at least *one vector can be removed from the set without reducing the span of the set*. Remove $|B_\ell\rangle$ for definiteness. The vector $|B_\ell\rangle$ is a member of the space and so can be represented as a sum over the elements in a spanning set for the space. Hence:

$$|B_\ell\rangle = d_1|B_\ell\rangle + \ldots + d_{\ell-1}|B_{\ell-1}\rangle + d_{\ell+1}|B_{\ell+1}\rangle + \ldots + d_n\,|B_n\rangle \quad \text{or}$$

$$|0\rangle = d_1|B_\ell\rangle + \ldots + d_{\ell-1}|B_{\ell-1}\rangle - |B_\ell\rangle + d_{\ell+1}|B_{\ell+1}\rangle + \ldots + d_n\,|B_n\rangle$$

The final line directly contradicts the assertion that the original set is linearly independent.

The assumption lead to a contradiction $\Rightarrow$ the assumption is false.

A basis set is a linearly independent spanning set for a space, or equivalently a set from which no member can be removed without reducing the span. A basis contains the smallest number of vectors necessary to span the space.

The Span of a Set is a Vector Space: Given any set of vectors $\{|I_1\rangle, |I_2\rangle, \ldots, |I_N\rangle\}$, the collection of vectors of the form $|u_i\rangle = k_{1i}\,|I_1\rangle + k_{2i}|I_2\rangle + \ldots + k_{Ni}\,|I_N\rangle$ *for* $k_i \in \mathbb{K}$ is a vector space with a binary addition defined as:

$$|u_i\rangle + |u_j\rangle = \left(k_{1i} + k_{1j}\right)|I_1\rangle + \left(k_{2i} + k_{2j}\right)|I_2\rangle + \ldots + \left(k_{Ni} + k_{Nj}\right)|I_N\rangle$$

and scalar multiplication defined as:

$$c|u_i\rangle = c\,k_{1i}\,|I_1\rangle + c\,k_{2i}|I_2\rangle + \ldots + c\,k_{Ni}\,|I_N\rangle$$

where c and the k_i are elements of the associated scalar field $\mathbb{K}$. The proof of this claim is tedious and long - a perfect homework problem!

Exercise: For the field of the real numbers $\mathbb{R}$ with members, *a, b, c, d, ...* consider the set of displacements of the form $|\Delta\,\vec{r}_i\rangle = a_i\,\hat{i} + b_i\,\hat{j} + c_i\,\hat{k}$ where addition is defined as $|\Delta\,\vec{r}_i\rangle + |\Delta\,\vec{r}_j\rangle = \left(a_i + a_j\right)\hat{i} + \left(b_i + b_j\right)\hat{j} + \left(c_i + c_j\right)\hat{k}$ and multiplication is defined as $d\,|\Delta\,\vec{r}_i\rangle = d\,a_i\,\hat{i} + d\,b_i\,\hat{j} + d\,c_i\,\hat{k}$. It is a three dimensional vector space. Identify the dimension of the vector space if the set of vectors is restricted such that (a.) the third component $c_n = 0$ for all n, (b.) the third component $c_n = 7$ for all n and (c.) all the components are equal $a_n = b_n = c_n$ for all n.

The Component Representation

Components of Vectors: Given that a basis $\{|B_1\rangle, |B_2\rangle, \ldots\ldots, |B_n\rangle\}$ has been chosen for a vector space $\mathbb{V}$, every vector in than space can be represented as a linear combination of those basis vectors: $|V_i\rangle = a_{i1}\,|B_1\rangle + a_{i2}\,|B_2\rangle + \ldots\ldots + a_{in}\,|B_n\rangle$. As a basis set is linearly

independent (and is a *smallest* spanning set), **the coefficients or *components*** a_{im} are uniquely determined for each vector $|V_i\rangle$.

Unique Expansion in terms of a Linearly Independent Set:

Assume that *there are two distinct linear combinations of the vectors in a basis set that represent the vector* $|V_i\rangle$. Say $|V_i\rangle = a_{i1}|B_1\rangle + a_{i2}|B_2\rangle + ... + a_{in}|B_n\rangle$ and $= d_{i1}|B_1\rangle + d_{i2}|B_2\rangle + ... + d_{in}|B_n\rangle$. Then,

$$|0\rangle = |V_i\rangle - |V_i\rangle = (a_{i1} - d_{i1})|B_1\rangle + (a_{i2} - d_{i2})|B_2\rangle + ... + (a_{in} - d_{in})|B_n\rangle$$

As the linear combinations are to be distinct, not all the factors in parentheses are zero. But, if not all the those factors are zero, then there is a linear combination of the basis vector in which not all coefficients are zero that sums to $|0\rangle$, a clear contradiction of linear independence.

The assumption lead to a contradiction $\Rightarrow$ the assumption is false.

Therefore each vector $|V_i\rangle$ has a unique set of expansion coefficients (*components*) given the basis $\{|B_1\rangle, |B_2\rangle, ..., |B_n\rangle\}$. It is expected that these coefficients would be different if a different basis were chosen. For a ***fixed basis set***, each vector has a unique set of components.

The representation of a vector as a linear combination of linearly independent vectors is ***unique***. This result is incredibly important. In a problem, you will establish that a set of mutually orthogonal vectors (that does not contain the zero vector) is necessarily linearly independent. What do you conclude about representations of vectors as linearly combinations of the vectors in a mutually orthogonal set?

How are the components of a vector computed? It's not clear! Fortunately science and engineering applications primarily deal with inner product spaces. The inner

product provides a tool to ***project out*** the values of the components for a given basis. A trivial example is the identification of the *x*-component of a displacement.

$$\left(\Delta\vec{r}\right)_x = \hat{i}\cdot\Delta\vec{r} = \hat{i}\cdot\left(\Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}\right) = \Delta x$$

The *x*-component is extracted by computing the inner product of the *x*-basis vector with the displacement $\Delta\vec{r}$. The inner product gauges the degree to which two vectors share characteristic behaviors. The procedure above gauges *x*-pointing displacement behavior in $\Delta\vec{r}$. There are a few complications that can be removed by constructing a basis set with members that have vanishing inner products with any other members of that basis. A vector pair with zero inner product is said to be ***orthogonal***. Our mantra is: always choose an orthogonal basis set. *Let's move onward and upward to the inner product.*

Inner Product Spaces: A space is an inner product space if there is a procedure that operates on any two vectors in the space to yield a scalar and if that procedure obeys the rules below. The symbol $\langle I|J\rangle$ represents the inner product of $|I\rangle$ and $|J\rangle$. $\langle I|J\rangle = c_{ij}$, the resulting scalar value. **(The Bra-Ket combination represents the inner product.)**

> *The inner products that appear in science and engineering applications gauge the degree to which the two vectors share characteristic behaviors. Do not dwell on such touchy-feely concepts as you master the mathematics of the abstract inner product. Neither should you think in terms of the dot product of vectors as illustrated above. Just lean back, relax and let your mind drift away from all practical reality.*

Formal Definition: INNER PRODUCT: An inner product is a procedure that inputs two vectors and outputs a scalar value and that satisfies the four axioms below. When a process is proposed to be an inner product, it must be certified by identifying the process that inputs two vectors and yields a scalar, and by showing that the process satisfies the requirements of the axioms I1 through I4.

Inner product: An operation on two vectors that yields a scalar: $\langle I \vert M \rangle = c_{im}$	
Properties of an inner product:	
I1. $\langle I \vert (\vert M \rangle + \vert N \rangle) = \langle I \vert M \rangle + \langle I \vert N \rangle = c_{im} + c_{in}$	*Distributive*
I2. $\langle I \vert (d \vert M \rangle) = d \langle I \vert M \rangle$ where d is any scalar.	*Scaling*
I3. $\langle I \vert M \rangle = \langle M \vert I \rangle^*$ * means complex conjugate	*'Hermitian'*
I4. $\langle I \vert I \rangle \geq 0$ and $\langle I \vert I \rangle = 0$ if and only if $\vert I \rangle = 0$.	*Can be a Norm*

Learn to express each axiom in prose form!

***FUZZY-SPEAK* DEFINITION of INNER PRODUCT:** A process that acts on two entities in a set (vectors) that gauges the degree to which those entities share characteristic behavior. This fuzzy-speak form is not valid when prompted for the definition of an inner product. One must respond with the formal mathspeak. However, one should consider the action of each inner product encountered *to determine the manner in which the inner product gauges the degree to which two entities share characteristic behavior*.

NORM: The inner product can be used to define the **norm** ('length') of a vector $\| I \rangle \| = |I| = \sqrt{\langle I \vert I \rangle}$.

ORTHOGONAL: Two vectors are **orthogonal** if their inner product is zero. For our space of displacements in 3D, the inner product is just the familiar dot product, the product of the magnitudes of the vectors times the cosine of the angle between their directions.

$$\langle \Delta\vec{r}_1 \vert \Delta\vec{r}_2 \rangle = \Delta\vec{r}_1 \cdot \Delta\vec{r}_2 = \Delta r_1 \, \Delta r_2 \cos\theta_{12}$$

or equivalently

$$\begin{aligned} \langle \Delta\vec{r}_1 \vert \Delta\vec{r}_2 \rangle = \Delta\vec{r}_1 \cdot \Delta\vec{r}_2 &= (\Delta x_1 \hat{i} + \Delta y_1 \hat{j} + \Delta z_1 \hat{k}) \cdot (\Delta x_2 \hat{i} + \Delta y_2 \hat{j} + \Delta z_2 \hat{k}) \\ &= \Delta x_1 \Delta x_2 + \Delta y_1 \Delta y_2 + \Delta z_1 \Delta z_2 \end{aligned}$$

For our purposes, $\left\langle \Delta\vec{r}_1 \middle| \Delta\vec{r}_2 \right\rangle = \Delta\vec{r}_1 \cdot \Delta\vec{r}_2 = \Delta r_1 \, \Delta r_2 \cos\theta_{12}$ *defines the inner product of two displacements, and the equality of* $\Delta r_1 \, \Delta r_2 \cos\theta_{12}$ *and* $\Delta x_1 \Delta x_2 + \Delta y_1 \Delta y_2 + \Delta z_1 \Delta z_2 =$ *is an example of a* ***Parseval relation****, a relation that computes the value of the* ***defined*** *inner product as the sum of the products of the corresponding expansion coefficients using an orthogonal basis set.* Here that orthogonal set is: $\{\hat{i}, \hat{j}, \hat{k}\}$.

The basis set for the 3D displacement example has been chosen to be orthogonal and unity normalized. $\{\hat{i}, \hat{j}, \hat{k}\} \Leftrightarrow \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$

$$\left\langle \hat{e}_i \middle| \hat{e}_j \right\rangle = \delta_{ij} \qquad \left\langle \hat{e}_1 \middle| \hat{e}_1 \right\rangle = \hat{i} \cdot \hat{i} = 1; \;\; \left\langle \hat{e}_1 \middle| \hat{e}_2 \right\rangle = \hat{i} \cdot \hat{j} = 0; \ldots ; \ldots..$$

Look for an equivalent component based procedure to evaluate the inner product for each inner product space that you study.

The Dagger$^{\mathbf{t}}$ Operation:

1.) The dagger operation converts a KET vector to its corresponding (conjugate) BRA vector. That is: $|M\rangle^{\dagger} = \langle M|$.

2.) The dagger operation also converts a BRA vector to its corresponding (conjugate) KET vector.

3.) If the dagger operation is performed twice, the original vector is recovered.

$\left[|M\rangle^{\dagger} \right]^{\dagger} = \langle M|^{\dagger} = |M\rangle$. $\qquad \left[\langle M|^{\dagger} \right]^{\dagger} = \left[|M\rangle \right]^{\dagger} = \langle M|$

4.) The dagger on a BRA-KET follows as $(\langle I|M\rangle)^{\dagger} = \langle M|I\rangle$. The inner product of two vectors is a scalar and $\langle I|M\rangle = \langle M|I\rangle^{*}$ so the dagger operation acting on a complex (or real) scalar returns the complex conjugate of that scalar; $(d\,)^{\dagger} = d^*$.

5.) Dagger operation on multiplying constants: $(d\,|M\rangle\,)^{\dagger} = |d\,M\rangle^{\dagger} = \langle d\,M| = d^*\,\langle M|$.

NOTE: Dagger should appear as † but font wars are leading to † and even to a superscript t (^{t}). Computers are simplifying our lives! $\Rightarrow$ Read † and † as †.

The details follow from the requirement that the BRA-KET combination follows the rules for the chosen inner product. Properties (I2) and (I3) of the inner product provide additional information about the action of the dagger operation.

I2. $\langle I|(d|M\rangle) = d\,\langle I|M\rangle$ where d is any scalar. *Scaling*

I3. $\langle I|M\rangle = \langle M|I\rangle^*$ asterisk means complex conjugate *'Hermitian'*

$$\langle I|d\,M\rangle^* = \left(d\,\langle I|M\rangle\right)^* = d^*\,\langle I|M\rangle^* = d^*\,\langle M|I\rangle$$

and $\langle I|d\,M\rangle^* = \langle dM|I\rangle = \langle I|d\,M\rangle^\dagger = d^*\,\langle M|I\rangle$

$$\boxed{\text{The conclusion is that: } |d\,M\rangle^\dagger = d^*\,\langle M|.}$$

This statement is consistent with the dagger operation returning the complex conjugates of scalars. Complex conjugation and the dagger operation are distinct operations that can be used to interchange BRAs and KETs.

***** Important Exercise:** Follow the rules above and apply the properties of the inner product to show that: $\left[|A\rangle - \frac{\langle B|A\rangle|B\rangle}{\langle B|B\rangle}\right]^\dagger = \langle A| - \frac{\langle A|B\rangle\langle B|}{\langle B|B\rangle}$. $\langle B|A\rangle$ is a complex scalar.

Digression: Why is $|C\rangle = |A\rangle - \frac{\langle B|A\rangle|B\rangle}{\langle B|B\rangle}$ important? First consider the analogous expression in a more familiar setting. $\vec{C} = \vec{A} - \frac{(\vec{B}\cdot\vec{A})\,\vec{B}}{(\vec{B}\cdot\vec{B})} = \vec{A} - (\hat{B}\cdot\vec{A})\,\hat{B}$. Start with $\vec{A}$ and compute the component of $\vec{A}$ in the direction of $\vec{B}$ which is $(\hat{B}\cdot\vec{A})\,\hat{B}$. The vector $\vec{C}$ is what remains of $\vec{A}$ after the part parallel to $\vec{B}$ has been removed. What can be said about $\vec{C}$ and $\vec{B}$? What can be said about their directions?

In the abstract setting, it follows that $|C\rangle$ is what remains of $|A\rangle$ after its behavior $|B\rangle$ behavior has been removed. That is: Start with $|A\rangle$; remove all the $|B\rangle$ behavior in $|A\rangle$; the remainder $|C\rangle$ is orthogonal to $|B\rangle$ as it has no $|B\rangle$ behavior.

Exercise: Consider the set of all objects of the form $\vec{v} = a\,\hat{i} + b\,\hat{j}$ where *a* and *b* are any real numbers. This collection is a linear vector space. Propose the inner product: $\langle \vec{v}_1 | \vec{v}_2 \rangle = \vec{v}_1 \cdot \vec{v}_2 = (a_1\ a_2) + (b_1\ b_2)$. Demonstrate that this proposed inner product meets the requirements that it input two vectors and outputs a scalar value and that the process satisfies axioms I1 t I4.

If *a* and *b* are complex numbers rather than real, the inner product must be modified to be: $\langle \vec{v}_1 | \vec{v}_2 \rangle = (a_1^*\ a_2) + (b_1^*\ b_2)$. Consider the vectors $\vec{v}_1 = (2i)\,\hat{i} + (1)\,\hat{j}$ and $\vec{v}_2 = (2)\,\hat{i} + (1)\,\hat{j}$. Identify the properties of the inner product that would not be met without this change to the definition.

Matrix notation can be used to represent vectors in terms of their components. With this approach, the vectors and operations in the exercise above become:

$$|v_1\rangle = a_1\hat{i} + b_1\hat{j} \rightarrow \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}; \quad |v_2\rangle \rightarrow \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}; \quad |v_2\rangle^\dagger = \langle v_2| \rightarrow \begin{bmatrix} a_2^* & b_2^* \end{bmatrix};$$

$$\vec{v}_2 \cdot \vec{v}_1 = \langle v_2 | v_1 \rangle \rightarrow \begin{bmatrix} a_2^* & b_2^* \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = a_2^*\, a_1 + b_2^*\, b_2$$

In this matrix picture, the *dagger* operation has a concrete representation as the transpose operation coupled with complex conjugation. The inner product is just the matrix multiplication of the row (bra) and column (ket) representations of the vectors.

SAMPLE CALCULATION: Certifying a candidate for an inner product

Mathematical vectors can many things; they can be functions like *f*(*t*). A definition for the inner product of two functions that are periodic with period *T* is used in to develop Fourier series expansions. $\langle g | f \rangle = \left(1/T\right) \int_{-T/2}^{T/2} \left(g(t)\right)^* f(t)dt$ No attempt is to be made to relate this operation to a vector space at this time, but it will be tested to see if it can be certified as a possible inner product definition.

I0. The procedure acts on two vectors (functions), and it yields a scalar value $\left(\frac{1}{T}\right)\int_{-T/2}^{T/2}\left(g(t)\right)^{*} f(t)dt$ as required. The first inner product requirement is satisfied.

I1. $\langle g \mid f+h\rangle = \langle g \mid (\mid f\rangle + \mid h\rangle) = \langle g \mid f\rangle + \langle g \mid h\rangle$?

$$\langle g \mid f+h\rangle = \left(\frac{1}{T}\right)\int_{-T/2}^{T/2}\left(g(t)\right)^{*}\left[f(t)+h(t)\right]dt$$
$$= \left(\frac{1}{T}\right)\int_{-T/2}^{T/2}\left(g(t)\right)^{*}\left[f(t)\right]dt + \left(\frac{1}{T}\right)\int_{-T/2}^{T/2}\left(g(t)\right)^{*}\left[h(t)\right]dt = \langle g \mid f\rangle + \langle g \mid h\rangle$$

I2. $\langle g \mid (d \mid f\rangle) = d\,\langle g \mid f\rangle$?

$$\langle g \mid d\ \ f\rangle = \langle g \mid (d \mid f\rangle) = \left(\frac{1}{T}\right)\int_{-T/2}^{T/2}\left(g(t)\right)^{*}\left[d\ f(t)\right]dt$$
$$= d\left(\frac{1}{T}\right)\int_{-T/2}^{T/2}\left(g(t)\right)^{*}\left[f(t)\right]dt = d\,\langle g \mid f\rangle$$

I3. $\langle I \mid M\rangle = \langle M \mid I\rangle^{*}$?

$$\langle g \mid f\rangle^{*} = \left[\left(\frac{1}{T}\right)\int_{-T/2}^{T/2}\left(g(t)\right)^{*}\left[f(t)\right]dt\right]^{*} = \left[\left(\frac{1}{T}\right)^{*}\int_{-T/2}^{T/2}\left(\left(g(t)\right)^{*}\right)^{*}\left[f(t)\right]^{*}\ dt^{*}\right]$$
$$= \left[\left(\frac{1}{T}\right)\int_{-T/2}^{T/2} g(t)\left[f(t)\right]^{*}\ dt\right] = \left[\left(\frac{1}{T}\right)\int_{-T/2}^{T/2}\left[f(t)\right]^{*} g(t)\ dt\right] = \langle f \mid g\rangle$$

Facts used: The complex conjugate of a real is just that same real. The complex conjugate of the complex conjugate of any complex number is that same number.

I4. $\langle I \mid I\rangle \geq 0$ and $\langle I \mid I\rangle = 0$ if and only if $\mid I\rangle = 0$?

$$\langle f \mid f\rangle = \left(\frac{1}{T}\right)\int_{-T/2}^{T/2}\left(f(t)\right)^{*}\left[f(t)\right]dt = \left(\frac{1}{T}\right)\int_{-T/2}^{T/2}\left|f(t)\right|^{2}dt \geq 0$$

and $\langle f \mid f\rangle$=0 only if $\left|f(t)\right|^{2}$= 0 for all t or only if *f(t)* is the *zero* function.

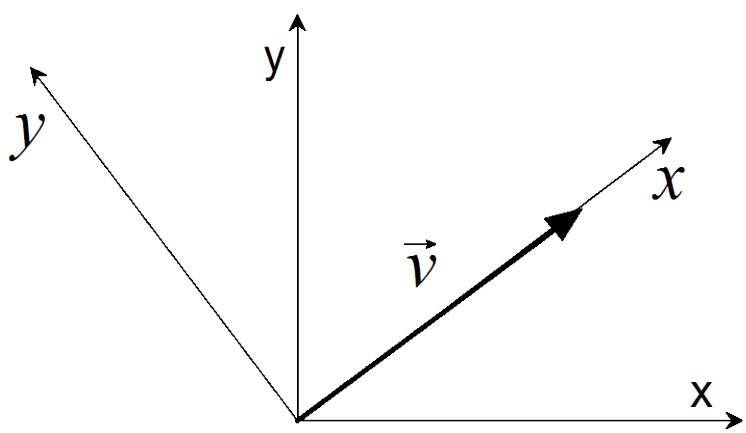
Fact used: The product of a complex number and its complex conjugate is a real positive (≥ 0) number. ($\forall\, z \in \mathbb{C}$), $z\, z^* \geq 0$; $z\, z^* = 0$ only if $z = 0$. Mathematicians would say $|f(t)|^2 = 0$ except for at a set of points of measure zero.

Conclusion: The proposed integral definition satisfies the requirements to be an inner product. It's a valid candidate. Perhaps a use can be found for it.

How are components of vectors computed?

The task under consideration is to **find the components of a vector** once a particular basis has been chosen for the space $\mathbb{V}$. The inner product is to be used, and the initial result is to be horrifying. **Don't panic.** A simpler solution is to be presented later.

Initial assumption: A basis for the space has been chosen. The components are the components for a representation of the space as the span of that basis. If the basis is changed, the components change.



Note that the components of the vector *v* in the *x*-*y* system are different from those in the x-y system. The vector itself is the same; the components and basis vectors are different.

$$\vec{v} = a_{vx}\,\hat{e}_x + a_{vy}\,\hat{e}_y$$

$$= \vec{v} = a_{v\mathrm{x}}\,\hat{e}_{\mathrm{x}} + a_{v\mathrm{y}}\,\hat{e}_{\mathrm{y}}$$

Computing vector components given a particular basis. Beginning with the representation of a general vector: $|V_i\rangle = a_{i1}\,|B_1\rangle + a_{i2}\,|B_2\rangle + \ldots + a_{in}\,|B_n\rangle$, a set of *n* (same as the dimension) equations is generated by taking the inner product of *each basis vector* with the $|V_i\rangle$ in succession.

$$\langle B_1|V_i\rangle = a_{i1}\langle B_1|B_1\rangle + a_{i2}\langle B_1|B_2\rangle + \ldots + a_{in}\langle B_1|B_n\rangle$$

$$\langle B_2|V_i\rangle = a_{i1}\langle B_2|B_1\rangle + a_{i2}\langle B_2|B_2\rangle + \ldots + a_{in}\langle B_2|B_n\rangle$$

$$\ldots\ldots$$

$$\langle B_n|V_i\rangle = a_{i1}\langle B_n|B_1\rangle + a_{i2}\langle B_n|B_2\rangle + \ldots + a_{in}\langle B_n|B_n\rangle$$

These *n* equations in *n* unknowns can be written in matrix form.

$$\begin{bmatrix} \langle B_1|B_1\rangle & \langle B_1|B_2\rangle & \ldots & \langle B_1|B_n\rangle \\ \langle B_2|B_1\rangle & \langle B_2|B_2\rangle & \ldots & \langle B_2|B_n\rangle \\ \ldots & \ldots & \ldots & \ldots \\ \langle B_n|B_1\rangle & \langle B_n|B_2\rangle & \ldots & \langle B_n|B_n\rangle \end{bmatrix} \begin{bmatrix} a_{i1} \\ a_{i2} \\ \ldots \\ a_{in} \end{bmatrix} = \begin{bmatrix} \langle B_1|V_i\rangle \\ \langle B_2|V_i\rangle \\ \ldots \\ \langle B_n|V_i\rangle \end{bmatrix}$$

A form that is horrifying until the inner products are identified as *just* scalars. Cramer's rule or row reduction techniques can be used to complete the calculation of the components.

Trivial Example: $|V\rangle = 4\hat{i} + 3\hat{j};\ |B_1\rangle = \hat{i};\ |B_2\rangle = \hat{i} + \hat{j} \ \Rightarrow\ |V\rangle = \alpha\ |B_1\rangle + \beta\ |B_2\rangle$

$$\langle B_1|V\rangle = \alpha\langle B_1|B_1\rangle + \beta\langle B_1|B_2\rangle \qquad 4 = 1\alpha + 1\beta$$

$$\langle B_2|V\rangle = \alpha\langle B_2|B_1\rangle + \beta\langle B_2|B_2\rangle \qquad 7 = 1\alpha + 2\beta$$

Using Cramer's rule: $\alpha = \dfrac{\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 1 \qquad \beta = \dfrac{\begin{vmatrix} 1 & 4 \\ 1 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 3$

Hence, we find: $4\hat{i} + 3\hat{j} = (1)\ \hat{i} + 3(\hat{i} + \hat{j})$. The expansion coefficients in terms of a linearly independent set are unique as proven previously.

The problem would be much simpler if the inner products of one basis vector with any other member of the basis set were zero.

$$\langle B_i|B_j\rangle = \langle B_i|B_i\rangle\,\delta_{ij} = N_i\ \delta_{ij}$$

As the inner product gauges the degree to which the vectors share a common behavior or character, the goal is to choose a set of vectors as the basis in which each vector adds its unique new behavior (dimension) to the set, but which does not share the behavior

of any other member of the set. If each vector adds at least some behavior not found in any of the others, then it is linearly independent of the others. If its inner product with another vector is zero, it is said to be *orthogonal* to the other vector. If all the vectors in a set are mutually orthogonal, then the basis set is said to be *an orthogonal basis*. Enough talk. What benefit is reaped if an orthogonal basis is chosen? The equations for the components become:

$$\begin{bmatrix} \langle B_1|B_1\rangle & 0 & \dots & 0 \\ 0 & \langle B_2|B_2\rangle & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \langle B_n|B_n\rangle \end{bmatrix} \begin{bmatrix} a_{i1} \\ a_{i2} \\ \dots \\ a_{in} \end{bmatrix} = \begin{bmatrix} \langle B_1|V_i\rangle \\ \langle B_2|V_i\rangle \\ \dots \\ \langle B_n|V_i\rangle \end{bmatrix}$$

These equations are easily solved to yield the *component calculator* for expansions $|V_i\rangle = a_{i1}|B_1\rangle + a_{i2}|B_2\rangle + \dots + a_{in}|B_n\rangle$ in terms of an orthogonal basis $\{|B_1\rangle, |B_2\rangle, \dots, |B_n\rangle\}$.

component calculator (*requires orthogonal basis*)	$a_{im} = \dfrac{\langle B_m|V_i\rangle}{\langle B_m|B_m\rangle}$

The moral is that one should work hard to find and use orthogonal basis sets.

Trivial Example: $|V\rangle = 4\hat{i} + 3\hat{j}$; $|b_1\rangle = \hat{i} + \hat{j}$; $|b_2\rangle = \hat{i} - \hat{j} \;\Rightarrow\; |V\rangle = \gamma\,|b_1\rangle + \delta\,|b_2\rangle$

Checking the orthogonality: $\langle b_2|b_1\rangle = (\hat{i} - \hat{j})\cdot(\hat{i} + \hat{j}) = 0$. It checks.

Using the component calculator, $\gamma = \dfrac{\langle b_1|V\rangle}{\langle b_1|b_1\rangle} = \dfrac{7}{2}$ $\qquad \delta = \dfrac{\langle b_2|V\rangle}{\langle b_2|b_2\rangle} = \dfrac{1}{2}$.

Hence, we find: $4\hat{i} + 3\hat{j} = (7/2)\,(\hat{i} + \hat{j}) + (1/2)(\hat{i} - \hat{j})$. The expansion coefficients in terms of a linearly independent set are unique as proven previously.

Sample component calculation:

The set of all Fourier series with coefficients any real number form a vector space.

$$\left|FS_f\right\rangle = f(t) = c_{f0}\,(1) \;+\; \sum_{n=1}^{\infty} a_{fn}\cos\left[\omega_n t\right] \;+\; \sum_{p=1}^{\infty} b_{fp}\sin\left[\omega_p t\right]$$

The inner product: $\langle g|f\rangle = \left(1/T\right)\int_{T/2}^{-T/2} (g(t))^* f(t)\,dt$. The coefficient a_{fm} is conjugate to the

basis function $|B_m\rangle = \cos(\omega_m t)$. Substituting into the component calculator*:

$$a_{fm} = \frac{\langle B_m | f \rangle}{\langle B_m | B_m \rangle} = \frac{\left(\frac{1}{T}\right)\int_{T/2}^{-T/2} (\cos[\omega_m t])^* \, f(t)\, dt}{\left(\frac{1}{T}\right)\int_{T/2}^{-T/2} (\cos[\omega_m t])^* \cos[\omega_m t] \, dt} = \frac{\left(\frac{1}{T}\right)\int_{T/2}^{-T/2} (\cos[\omega_m t])^* \, f(t)\, dt}{\left(\frac{1}{2}\right)}$$

$$a_{fm} = \left(\frac{2}{T}\right)\int_{T/2}^{-T/2} \left(\cos[\omega_m t]\right)^* f(t)\, dt \quad \text{The result presented in standard texts.}$$

As cosine is a real-valued function, $\left[\cos(\omega t)\right]^* = \cos(\omega t)$. The average value of cosine squared is ½. The result above is the standard result for Fourier trig series.

*The expansion set $\{1;\{\ \ldots, \cos[\omega_m t], \ldots\};\{\ \ldots, \sin[\omega_p t], \ldots\}\}$ is mutually orthogonal. This property is proven in the Fourier series handout.

How can an orthogonal basis be developed from just any basis?

The inner product may be used to reduce a spanning set or a basis set to a **basis** set of mutually orthogonal vectors satisfying a normalization condition that is also a spanning set for the same space. Starting with a spanning set, $\{|B_1\rangle, |B_2\rangle, |B_3\rangle, \ldots\}$, take the first member of the set and apply a normalization (scaling) procedure. The result is $|b_1\rangle$. [A normalization procedure scales a vector (multiplies it by a scalar constant). Hence $|b_1\rangle = k\,|B_1\rangle$ for some $(k \neq 0) \in \mathbb{K}$. A possible, but perhaps not very practical, normalization rule would be to set $\langle b_1 | b_1 \rangle = 1, \langle b_2 | b_2 \rangle = 2$, …., $\langle b_n | b_n \rangle = n$.] (The most convenient choice is: $\langle b_i | b_i \rangle = 1 \ \forall\ i$. Unfortunately, this choice is not universal as illustrated by Fourier series and Legendre polynomials.)

The next member of the new basis is formed by removing the component of $|B_2\rangle$ *along* $|b_1\rangle$ before normalizing. For the remaining vectors, components along all the previously normalized basis vectors are subtracted before the new member is normalized. This procedure is the **Gram-Schmidt** orthogonalization procedure.

$$\left|E_2\right\rangle = \left|B_2\right\rangle - \frac{\left\langle b_1 \middle| B_2\right\rangle \left|b_1\right\rangle}{\left\langle b_1 \middle| b_1\right\rangle}\text{; then normalize } \left|E_2\right\rangle \to \left|b_2\right\rangle$$

First remove the part of $\left|B_2\right\rangle$ that is dependent on $\left|b_1\right\rangle$ by subtracting the component of $\left|B_2\right\rangle$ along $\left|b_1\right\rangle$. Then apply the normalization procedure. Continue to generate the rest of the basis set. For each new candidate member, remove all components of that candidate along previously added members. Note that it is the projections with the previously orthogonalized $\left|b_i\right\rangle$ that are removed from the subsequent $\left|B\right\rangle$'s.

$$\left|E_3\right\rangle = \left|B_3\right\rangle - \frac{\left\langle b_1 \middle| B_3\right\rangle \left|b_1\right\rangle}{\left\langle b_1 \middle| b_1\right\rangle} - \frac{\left\langle b_2 \middle| B_3\right\rangle \left|b_2\right\rangle}{\left\langle b_2 \middle| b_2\right\rangle}\text{; then normalize } \left|E_3\right\rangle \to \left|b_3\right\rangle$$

Every $\left|b_i\right\rangle$ is orthogonal to every other $\left|b_i\right\rangle$ so that removing $\frac{\left\langle b_j \middle| B_k\right\rangle \left|b_j\right\rangle}{\left\langle b_j \middle| b_j\right\rangle}$ the component along a particular $\left|b_j\right\rangle$ from the next $\left|B_k\right\rangle$ does not remove any part of its component along any other $\left|b_i\right\rangle$. That is: $\frac{\left\langle b_j \middle| B_k\right\rangle \left|b_j\right\rangle}{\left\langle b_j \middle| b_j\right\rangle}$ is orthogonal to every other $\left|b_i\right\rangle$ in the set of basis vectors that has been constructed.

$$\left|E_j\right\rangle = \left|B_j\right\rangle - \sum_{i<j} \frac{\left\langle b_i \middle| B_j\right\rangle \left|b_i\right\rangle}{\left\langle b_i \middle| b_i\right\rangle}\text{; then normalize } \left|E_j\right\rangle \to \left|b_j\right\rangle$$

If $\left|B_k\right\rangle$ is not linearly independent of the previously added vectors, then $\left|E_k\right\rangle$ will vanish. The dimension N of the space is the number of vectors in the final Gram-Schmidt set $\left\{\left|b_1\right\rangle, \left|b_2\right\rangle, \left|b_3\right\rangle, ..., \left|b_N\right\rangle\right\}$. (All zero vectors are discarded.)

Note: The Gram-Schmidt method applied to our standard 3D problems produced an orthonormal basis that may **not obey the right-hand rule**. Multiply the last vector in a three dimensional set by -1 if necessary to bring the set into right hand rule order.

Exercise: Show that $\langle b_1 | E_2 \rangle = 0$ and hence $\langle b_1 | b_2 \rangle = 0$.

Exercise: Show that $\langle b_3 | b_1 \rangle = 0$ and $\langle b_3 | b_2 \rangle = 0$. Recall that $\langle b_3 | b_1 \rangle = \langle b_1 | b_3 \rangle^*$

Exercise: A not uncommon error distorts the Gram-Schmidt (G-S) method by removing behaviors of the previous vectors $|B_i\rangle$ in stead of behaviors of the vectors previously added to the new G-S basis set that have been made orthogonal. That is the faulty procedure:

$$|E_j\rangle = |B_j\rangle - \sum_{i<j} \frac{\langle B_i | B_j \rangle | B_i \rangle}{\langle B_i | B_i \rangle} \text{ ; then normalize}$$

was used rather than the correct procedure:

$$|E_j\rangle = |B_j\rangle - \sum_{i<j} \frac{\langle b_i | B_j \rangle | b_i \rangle}{\langle b_i | b_i \rangle}; \text{ then normalize } |E_j\rangle \to |b_j\rangle$$

Compare the results using the two procedures for the initial set $\{\hat{i}, \hat{i}+\hat{j}, \hat{j}+\hat{k}\}$. The distorted method removes all the components along each of the $|B_i\rangle$. The correct procedure removes all the x pointing behavior from $|B_2\rangle$ before adding the remaining independent pointing behavior ($|b_2\rangle = \hat{j}$) to the G-S set. In the next step, the first subtraction removes all the x pointing behavior from $|B_3\rangle$. The next subtraction using $|b_2\rangle = \hat{j}$ only removes behavior that was not removed in the first subtraction. Incorrectly using $|b_2\rangle$ would removed the $\hat{j}$ behavior, but it would also remove more x pointing behavior from $|B_3\rangle$. As a result, the incorrect third vector actually has a negative x component rather than being orthogonal to the x direction.

Correct method answer: $\{\hat{i}, \hat{j}, \hat{k}\}$ Incorrect method result: $\{\hat{i}, \hat{j}, \frac{-\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}}\}$

USE the previously added G-S vectors, not the members of the original set.

A common, *but not universal*, normalization is to divide a vector by its norm. The result is that the vectors in the basis will all be normalized to have length one (unity). This choice is so common that the term normalized without modifiers is usually assumed to mean the unity normalization just discussed. We will use e_i rather than b_i when we wish to indicate ***unity normalization***.

$$\left|e_1\right\rangle = \frac{\left|B_1\right\rangle}{\sqrt{\left\langle B_1 \middle| B_1\right\rangle}} \qquad \left|E_2\right\rangle = \left|B_2\right\rangle - \left\langle e_1 \middle| B_2\right\rangle \left|e_1\right\rangle; \text{ then } \left|e_2\right\rangle = \frac{\left|E_2\right\rangle}{\sqrt{\left\langle E_2 \middle| E_2\right\rangle}}.$$

$$\left|E_j\right\rangle = \left|B_j\right\rangle - \sum_{i<j} \left\langle e_i \middle| B_j\right\rangle \left|e_j\right\rangle; \text{ then } \left|e_j\right\rangle = \frac{\left|E_j\right\rangle}{\sqrt{\left\langle E_j \middle| E_j\right\rangle}}.$$

SAMPLE CALCULATION: A Gram-Schmidt basis construction:

Start with the spanning set $\mathbb{S} = \left\{ \vec{B}_1 = 2\,\hat{\imath} + 2\,\hat{\jmath} - 2\,\hat{k},\ \vec{B}_2 = 4\,\hat{\imath} + 3\,\hat{\jmath} - 3\,\hat{k},\ \vec{B}_3 = -\,\hat{\jmath} + \hat{k} \right\}$. Pick the first member of the set and normalize it. Unity normalization is to be adopted.

$$\hat{e}_1 = \frac{2\,\hat{\imath} + 2\,\hat{\jmath} - 2\,\hat{k}}{\sqrt{\left(2\,\hat{\imath} + 2\,\hat{\jmath} - 2\,\hat{k}\right)\cdot\left(2\,\hat{\imath} + 2\,\hat{\jmath} - 2\,\hat{k}\right)}} = \frac{2\,\hat{\imath} + 2\,\hat{\jmath} - 2\,\hat{k}}{\sqrt{12}} = \frac{\hat{\imath} + \hat{\jmath} - \hat{k}}{\sqrt{3}}$$

Next, the component of $\vec{B}_2$ along $\hat{e}_1$ is to be subtracted from $\vec{B}_2$, and the result normalized. $\vec{E}_2 = \vec{B}_2 - \left(\hat{e}_1 \cdot \vec{B}_2\right)\hat{e}_1 = \left(4\,\hat{\imath} + 3\,\hat{\jmath} - 3\,\hat{k}\right) - \left[\left(\frac{\hat{\imath} + \hat{\jmath} - \hat{k}}{\sqrt{3}}\right)\cdot\left(4\,\hat{\imath} + 3\,\hat{\jmath} - 3\,\hat{k}\right)\right]\frac{\hat{\imath} + \hat{\jmath} - \hat{k}}{\sqrt{3}}$

$$\vec{E}_2 = \vec{B}_2 - \left(\hat{e}_1 \cdot \vec{B}_2\right)\hat{e}_1 = \left(4\,\hat{\imath} + 3\,\hat{\jmath} - 3\,\hat{k}\right) - \left[\frac{10}{3}\right]\left(\hat{\imath} + \hat{\jmath} - \hat{k}\right) = \frac{2}{3}\hat{\imath} - \frac{1}{3}\hat{\jmath} + \frac{1}{3}\hat{k} \text{ or } \hat{e}_2 = \frac{2\,\hat{\imath} - \hat{\jmath} + \hat{k}}{\sqrt{6}}$$

Finally, the components of $\vec{B}_3$ along $\hat{e}_1$ and $\hat{e}_2$ are to be subtracted from $\vec{B}_3$, and the result normalized.

$$\vec{E}_3 = \vec{B}_3 - \left(\hat{e}_1 \cdot \vec{B}_3\right)\hat{e}_1 - \left(\hat{e}_2 \cdot \vec{B}_3\right)\hat{e}_2 =$$

$$\left(-\,\hat{\jmath} + \hat{k}\right) - \left[\left(\frac{\hat{\imath} + \hat{\jmath} - \hat{k}}{\sqrt{3}}\right)\cdot\left(-\,\hat{\jmath} + \hat{k}\right)\right]\left(\frac{\hat{\imath} + \hat{\jmath} - \hat{k}}{\sqrt{3}}\right) - \left[\left(\frac{2\hat{\imath} - \hat{\jmath} + \hat{k}}{\sqrt{6}}\right)\cdot\left(-\,\hat{\jmath} + \hat{k}\right)\right]\left(\frac{2\hat{\imath} - \hat{\jmath} + \hat{k}}{\sqrt{6}}\right)$$

$$\vec{E}_3 = \left(-\hat{j}+\hat{k}\right) - \left[\frac{-2}{3}\right]\left(\hat{i}+\hat{j}-\hat{k}\right) - \left[\frac{2}{6}\right]\left(2\hat{i}-\hat{j}+\hat{k}\right) = \left[\frac{2}{3}-\frac{4}{6}\right]\hat{i} + \left[-1+\frac{2}{3}-\frac{-2}{6}\right]\hat{j} + \left[1-\frac{2}{3}-\frac{2}{6}\right]\hat{k} = \vec{0}$$

The last member of the set does not survive the Gram-Schmidt procedure, and the set of orthogonal and normalized vectors which span the same space contains only two members. Therefore, the spanned space has two dimensions.

$$\mathbb{S} = \left\{ \hat{e}_1 = \frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}, \hat{e}_2 = \frac{2\,\hat{i}-\hat{j}+\hat{k}}{\sqrt{6}} \right\}$$

Exercise: Consider the spanning set $\left\{\hat{i}+2\,\hat{j}+\hat{k}, 2\,\hat{i}-2\,\hat{j}+2\,\hat{k}, 2\,\hat{i}+\hat{j}+2\,\hat{k}\right\}$. Apply the Gram-Schmidt procedure to find an orthonormal (orthogonal and unity normalized) basis. What is the dimension of the space spanned by this basis set?

Exercise: A set has members of the form $|I\rangle = a_i\,|e_1\rangle + b_i\,|e_2\rangle + c_i\,|e_3\rangle$ where the constants a_i, b_i, and c_i are arbitrary real constants, the components of the vector. Addition is defined by the relation: $|I\rangle + |J\rangle = \left(a_i + a_j\right)|e_1\rangle + \left(b_i + b_j\right)|e_2\rangle + \left(c_i + c_j\right)|e_3\rangle$ and multiplication by $d\,|I\rangle = \left(d\,a_i\right)|e_1\rangle + \left(d\,b_i\right)|e_2\rangle + \left(d\,c_i\right)|e_3\rangle$

a.) Show that this set satisfies the requirements to be a vector space. Assume and use the properties of scalar addition and multiplication (associative, etc.).

b.) The basis vectors $|e_i\rangle$ have the following inner products $\langle e_i|e_j\rangle = \delta_{ij}$ as they are orthogonal and normalized. Find an expression for $\langle J|I\rangle$ where $|I\rangle = a_i\,|e_1\rangle + b_i\,|e_2\rangle + c_i\,|e_3\rangle$ and $|J\rangle = a_j\,|e_1\rangle + b_j\,|e_2\rangle + c_j\,|e_3\rangle$. Note $\langle J| = a_j^*\,\langle e_1| + b_j^*\,\langle e_2| + c_j^*\,\langle e_3|$ for a complex vector space. See property I3 of the inner product.

c.) Find the component of $|J\rangle$ along $|e_2\rangle$. *What is a component?*

Mega-Exercise: Given any two vectors in an inner product space $|A\rangle$ and $|B\rangle$, prove the Schwarz Inequality: $\left|\langle A|B\rangle\right| \le \left\|A\rangle\right\|\left\|B\rangle\right|$. Use the vector $|A\rangle - \left[\langle B|B\rangle\right]^{-1}\langle B|A\rangle\,|B\rangle$ and

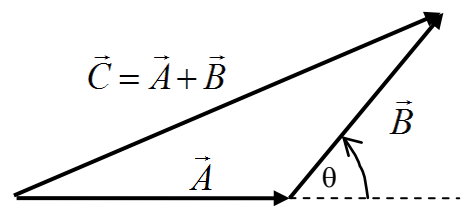
the properties of an inner product.

Mega-Exercise: Use the Schwarz Inequality: $\left|\langle A|B\rangle\right| \le \|A\rangle\|\|B\rangle|$ to establish the triangle inequality: The magnitude of the sum of two vectors is between the magnitude of the difference of the magnitudes of the two vectors and their sum.

$\left|\,|A\rangle| - |B\rangle|\,\right| \le \left|\,|A\rangle + |B\rangle\,\right| \le |A\rangle| + |B\rangle|$ Begin with the expressions

$\left|\,|A\rangle| + |B\rangle|\,\right|^2 = \langle A|A\rangle + 2\,|A\rangle|\,|B\rangle| + \langle B|B\rangle$ and $\left|\,|A\rangle| - |B\rangle|\,\right|^2$.

The triangle inequality is that $\left|\,|\vec{A}| - |\vec{B}|\,\right| \le |\vec{A} + \vec{B}| \le |\vec{A}| + |\vec{B}|$ stating that the length of the third side of a triangle is bounded by the magnitude of the difference and of the sum of the magnitudes of the other two sides.



$C^2 = A^2 + B^2 + 2\,A\,B\cos\theta$

$$C^2 = A^2 + B^2 + 2\,A\,B\cos\theta \quad \text{and} \quad |A - B|^2 \le A^2 + B^2 + 2\,A\,B\cos\theta \le |A + B|^2$$

The Schwarz inequality states that the magnitude of the inner product of two vectors is less than or equal to the product of the magnitudes (norms) of the two vectors. In the familiar inner product case, the dot product, $|\vec{A}\cdot\vec{B}| = |\vec{A}||\vec{B}||\cos\theta| \le |\vec{A}||\vec{B}|$. Clearly it works. The point is that the result holds for all inner product spaces.

Scale-Free Gauge of Common Behavior: From the Schwarz inequality, it follows that: $\dfrac{|\langle A|B\rangle|}{\|A\rangle\|\|B\rangle|} \le 1$. This scale-free combination is to be called the Schwarz measure of shared behavior in this series of handouts.

Exercise: Find an expression for the Schwarz measure for two of our *standard* vectors based on the relation: $|\vec{A}\cdot\vec{B}| = |\vec{A}||\vec{B}||\cos\theta_{AB}| \le |\vec{A}||\vec{B}|$. The Schwarz measure gauges the degree to which the two vectors point in the same direction independent of their

magnitudes.

Representations: Our vector space notation is abstract; representations lower the degree of the abstraction. Consider the inner product. An integral representation for the inner product is used in introductory quantum mechanics is: $\langle\psi|\phi\rangle = \int_{-\infty}^{\infty}[\psi(x)]^* \phi(x)\,dx$.

A robust inner product representation provides the instructions required to *actually compute* the scalar value of the inner product.

Exercise: Show that the inner product definition $\langle\psi|\phi\rangle = \int_{-\infty}^{\infty}[\psi(x)]^* \phi(x)\,dx$ satisfies the four axioms for inner products.

Matrix Representation of a Linear Vector Space: Matrices may be used as a convenient representation of vectors and vector operations. It begins by specifying a basis set for the vector space and determining the expansion coefficients for the vectors using this basis. A vector is represented by the column vector of its coefficients.

Most of our examples are for three dimensions, but they could be for *N* dimensions.

Basis: Orthogonal and Unity Normalized:

Consider the vector: $|V_i\rangle = a_i|e_1\rangle + b_i|e_2\rangle + c_i|e_3\rangle$. The coefficients: a_i, b_i, c_i are often called components. So, for example, the $|e_1\rangle$ component of $|V_i\rangle$ can be **projected** out using the inner product.

$$\langle e_1|V_i\rangle = a_i\langle e_1|e_1\rangle + b_i\langle e_1|e_2\rangle + c_i\langle e_1|e_3\rangle = a_i \quad \text{after using} \quad \langle e_i|e_j\rangle = \delta_{ij}$$

Once a specific set of basis vectors has been chosen that is orthogonal and unity normalized ($\langle e_i|e_j\rangle = \delta_{ij}$), any vector can be represented by a column matrix of its components. The vector is the linear combination of the basis vectors using those components and the basis is to remain the specified set with no changes.

$$|V_i\rangle \rightarrow \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} \qquad \text{and} \qquad |V_j\rangle^\dagger = \langle V_j| \rightarrow \begin{bmatrix} a_j^* & b_j^* & c_j^* \end{bmatrix}$$

Dagger $\Rightarrow$ Complex conjugate of the transpose

The actual vector is $\left\|V_i\right\rangle = a_i \left\|\hat{e}_1\right\rangle + b_i \left\|\hat{e}_2\right\rangle + c_i \left\|\hat{e}_3\right\rangle$, but the basis vectors need not be displayed ***as long as*** that set is specified and understood and immutable.

A robust representation is capable of supporting all the rules obeyed by the vectors. Addition is the standard matrix addition. Scalar multiplication is the standard scalar multiplication for matrices. The **DAGGER Operation** (converting Kets to Bras and vice versa) is the complex conjugation of the transpose. Finally, the inner product is realized as matrix multiplication of the bra and ket.

$$\left\langle V_j \middle| V_i \right\rangle = \begin{bmatrix} a_j^* & b_j^* & c_j^* \end{bmatrix} \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = a_j^* a_i + b_j^* b_i + c_j^* c_i$$

Inner Product $\Rightarrow$ Matrix Product

Basis: Orthogonal and normalized (not unity normalized)

A basis expansion set is chosen to be three orthogonal vectors that are not unity normalized. $\left\|V_i\right\rangle = c_{1i} \left\|b_1\right\rangle + c_{2i} \left\|b_2\right\rangle + c_{3i} \left\|b_3\right\rangle$ where $\left\langle b_i \middle| b_j \right\rangle = N_j \, \delta_{ij}$

As above, $\left\|V_i\right\rangle \rightarrow \begin{bmatrix} c_{1i} \\ c_{2i} \\ c_{3i} \end{bmatrix}$ and $\left\langle V_j \right| \rightarrow \begin{bmatrix} c_{1j}^* & c_{2j}^* & c_{3j}^* \end{bmatrix}$ but the inner product is rather more complicated: (the matrix in the middle is the *metric*)

$$\left\langle V_j \middle| V_i \right\rangle = \begin{bmatrix} c_{1j}^* & c_{2j}^* & c_{3j}^* \end{bmatrix} \begin{bmatrix} N_1 & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{bmatrix} \begin{bmatrix} c_{1i} \\ c_{2i} \\ c_{3i} \end{bmatrix} = c_{1j}^* c_{1i} \, N_1 + c_{2j}^* c_{2i} \, N_2 + c_{3j}^* c_{3i} \, N_3$$

Basis: Just linearly independent (not orthogonal or unity normalized)

A basis expansion set is chosen to be three vectors $\{s_1, s_2, s_3\}$ which are neither orthogonal nor unity normalized. (the *metric* now has off diagonal elements)

$$\left|V_i\right\rangle = c_{1i}\left|s_1\right\rangle + c_{2i}\left|s_2\right\rangle + c_{3i}\left|s_3\right\rangle$$

Recalling

$$\begin{bmatrix} \langle B_1|V_i\rangle \\ \langle B_2|V_i\rangle \\ \dots \\ \langle B_n|V_i\rangle \end{bmatrix} = \begin{bmatrix} \langle B_1|B_1\rangle & \langle B_1|B_2\rangle & \dots & \langle B_1|B_n\rangle \\ \langle B_2|B_1\rangle & \langle B_2|B_2\rangle & \dots & \langle B_2|B_n\rangle \\ \dots & \dots & \dots & \dots \\ \langle B_n|B_1\rangle & \langle B_n|B_2\rangle & \dots & \langle B_n|B_n\rangle \end{bmatrix} \begin{bmatrix} a_{i1} \\ a_{i2} \\ \dots \\ a_{in} \end{bmatrix}$$

Then where $\langle s_i|s_j\rangle = n_{ij}$

$$\begin{bmatrix} \langle s_1|V_i\rangle \\ \langle s_2|V_i\rangle \\ \dots \\ \langle s_n|V_i\rangle \end{bmatrix} = \begin{bmatrix} \langle s_1|s_1\rangle & \langle s_1|s_2\rangle & \dots & \langle s_1|s_n\rangle \\ \langle s_2|s_1\rangle & \langle s_2|s_2\rangle & \dots & \langle s_2|s_n\rangle \\ \dots & \dots & \dots & \dots \\ \langle s_n|s_1\rangle & \langle s_n|s_2\rangle & \dots & \langle s_n|s_n\rangle \end{bmatrix} \begin{bmatrix} a_{i1} \\ a_{i2} \\ \dots \\ a_{in} \end{bmatrix}$$

As above,

$$\left|V_i\right\rangle \rightarrow \begin{bmatrix} c_{1i} \\ c_{2i} \\ c_{3i} \end{bmatrix} \quad \text{and} \quad \left\langle V_j\right| \rightarrow \begin{bmatrix} c_{1j}^* & c_{2j}^* & c_{3j}^* \end{bmatrix}$$

A vector has a unique expansion in terms of any linearly independent set. Basis sets are linearly independent. The values for (c_{1i}, c_{2i}, c_{3i}) for a vector are unique for a given basis set.

The inner product is much more complicated: Define: $n_{ij} = \langle s_i|s_j\rangle$

$$\langle V_j|V_i\rangle = \begin{bmatrix} c_{1j}^* & c_{2j}^* & c_{3j}^* \end{bmatrix} \begin{bmatrix} n_{11} & n_{21} & n_{31} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix} \begin{bmatrix} c_{1i} \\ c_{2i} \\ c_{3i} \end{bmatrix} = \sum_{i,j=1}^{3} c_{\ell j}^* n_{\ell m} c_{mi}$$

The moral is to work hard when you build your basis. Make sure that it is orthogonal and unity normalized. At the very least, make sure that the basis is orthogonal.

The expansion basis for Fourier series is orthogonal, but not unity normalized.

$$\frac{1}{T}\int_{-T/2}^{T/2} [1]\ [1]\, dt = 1; \qquad \frac{1}{T}\int_{-T/2}^{T/2} [1]\sin[\omega_m t]\, dt = 0; \qquad \frac{1}{T}\int_{-T/2}^{T/2} [1]\cos[\omega_m t]\, dt = 0$$

$$\frac{1}{T}\int_{-T/2}^{T/2} \sin\left[\omega_p t\right]\cos\left[\omega_m t\right] dt = 0; \quad \frac{1}{T}\int_{-T/2}^{T/2} \sin\left[\omega_p t\right]\sin\left[\omega_m t\right] dt = ½\,\delta_{pm}; \quad \frac{1}{T}\int_{-T/2}^{T/2} \cos\left[\omega_p t\right]\cos\left[\omega_m t\right] dt = ½\,\delta_{pm}$$

The sines and cosines are normalized to an inner product value of ½.

A Gram-Schmidt Sample Calculation for a Function Space:

A space of polynomials is defined with domain the interval [-1,1]. The inner product for functions on that interval is defined to be:

$$\langle g \,|\, f \rangle = \int_{-1}^{1} g(x)\, f(x)\, dx$$

A possible basis set for the space is { 1, x, x^2, x^3, … , x^n, … }. Choose $|B_1\rangle = 1$ and then work upward power by power. Use the Gram-Schmidt process to develop an orthogonal set of vectors (polynomials). Apply the ***standard*** arbitrary normalization condition the $P_n(1) = 1$ to set the overall scaling (normalization of the polynomials. Mathematica: LegendreP[2,x] = $-\frac{1}{2}+\frac{3}{2}\,x^2$; LegendreP[3,x] = $-\frac{3}{2}x+\frac{5}{2}\,x^3$.

BASIS VECTOR 1: Just take $|B_1\rangle = 1$ as the first vector to be added to the orthogonal set and apply the arbitrary normalization condition the $P_1(1) = 1$. $|B_1\rangle = 1$ is the constant 1 so it is already normalized $\Rightarrow |b_1\rangle = P_0(x) = 1$.

BASIS VECTOR 2: Start with $|B_2\rangle = x$ and remove the 1-behavior from it.

$$|B_2\rangle - \frac{\langle b_1 | B_2\rangle}{\langle b_1 | b_1\rangle}|b_1\rangle = x - \frac{\int_{-1}^{1}(1)(x)dx}{\int_{-1}^{1}(1)(1)dx}(1) = x - \frac{0}{2}(1) = x$$

We note that x evaluated for $x = 1$ is 1 so the normalization condition is already met.

$$|b_2\rangle = P_1(x) = x$$

BASIS VECTOR 3: Start with $|B_3\rangle = x^2$ and remove the P_o-behavior and P_1-behavior.

$$\left|B_3\right\rangle - \frac{\langle b_1|B_3\rangle}{\langle b_1|b_1\rangle}\left|b_1\right\rangle - \frac{\langle b_2|B_3\rangle}{\langle b_2|b_2\rangle}\left|b_2\right\rangle = x^2 - \frac{\int_{-1}^{1}(1)(x^2)dx}{\int_{-1}^{1}(1)(1)dx}(1) - \frac{\int_{-1}^{1}(x)(x^2)dx}{\int_{-1}^{1}(x)(x)dx}(x) = x^2 - \tfrac{1}{3}$$

This polynomial is not 1 for $x = 1$. $\Rightarrow$ Scale it to get the desired normalization.

$$a\,(x^2 - {}^1/_3) = P_2(x) \text{ so } P_2(1) = a\,({}^2/_3) = 1 \text{ or } a = {}^3/_2$$

With scaling, the final form is $\left|b_3\right\rangle = P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2}$.

BASIS VECTOR 4: Start with $\left|B_4\right\rangle = x^3$ and remove the P_o, P_1 and P_2 behaviors.

$$\left|B_4\right\rangle - \frac{\langle b_1|B_4\rangle}{\langle b_1|b_1\rangle}\left|b_1\right\rangle - \frac{\langle b_2|B_4\rangle}{\langle b_2|b_2\rangle}\left|b_2\right\rangle - \frac{\langle b_3|B_4\rangle}{\langle b_3|b_3\rangle}\left|b_3\right\rangle$$

$$= x^3 - \frac{\int_{-1}^{1}(1)(x^3)dx}{\int_{-1}^{1}(1)(1)dx}(1) - \frac{\int_{-1}^{1}(x)(x^3)dx}{\int_{-1}^{1}(x)(x)dx}(x) - \frac{\int_{-1}^{1}(\tfrac{3}{2}x - \tfrac{1}{2})(x^3)dx}{\int_{-1}^{1}(\tfrac{3}{2}x - \tfrac{1}{2})(\tfrac{3}{2}x - \tfrac{1}{2})dx}(\tfrac{3}{2}x - \tfrac{1}{2})$$

$$= x^3 - \frac{0}{\int_{-1}^{1}(1)(1)dx}(1) - \frac{\tfrac{2}{5}}{\tfrac{2}{3}}(x) - \frac{0}{\int_{-1}^{1}(\tfrac{3}{2}x - \tfrac{1}{2})(\tfrac{3}{2}x - \tfrac{1}{2})dx}(\tfrac{3}{2}x - \tfrac{1}{2}) = x^3 - \tfrac{3}{5}x$$

This polynomial is not 1 for $x = 1$. Scale it to get the desired normalization.

$$a\,(x^3 - {}^3/_5\,x) = P_3(x) \text{ so } P_3(1) = a\,({}^2/_5) = 1 \text{ or } a = {}^5/_2$$

With scaling, the final form is $\left|b_4\right\rangle = P_3(x) = \tfrac{5}{2}x^3 - \tfrac{3}{2}\,x$.

The general result is that the $(n + 1)^{\text{th}}$ vector is the nth order Legendre polynomial.

Hermite Polynomials: A space of polynomials is defined with domain the interval [-∞,∞]. The inner product for functions on that interval is defined to be:

$$\langle g|f\rangle = \int_{-\infty}^{\infty} g(x)\,f(x)\;e^{-x^2}\,dx \qquad \text{Note the weight function: } e^{-x^2}$$

A possible basis set for the space is { 1, x, x^2, x^3, … , x^n, … }. Choose $\left|B_1\right\rangle = 1$ and then work upward power by power. Use the Gram-Schmidt process to develop an orthogonal set of vectors (polynomials). The resulting polynomials are the Hermite polynomials that use yet another weird normalization. The nth order polynomial is scaled such that the lead term is $2^n\,x^n$. $\Rightarrow$ Multiply the Gram-Schmidt result by

whatever factor is necessary to yield the desired highest order term.

From Mathematica: HermiteH[2,x] = $4x^2$ - 2; HermiteH[3,x] = $8x^3 - 12x$.

$\int_{-\infty}^{\infty} x^{2n}\, e^{-x^2}\, dx = \sqrt{\pi}\,\left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\ldots\left(\tfrac{1}{2}\right)$. The integral vanishes for odd powers of x.

Warmup Exercises:

VSWUP1. Memorize the next two sentences. *A vector space is a set of elements with two defined operations. The set of elements must have an associated rule to decide whether or not a particular element is a member of that set*. What are the required defined operations? Give the vector space axioms that each operation must obey. Provide an equation representation and a prose statement for each axiom.

VSWUP2. Give the definition of linear dependence of a set of vectors. Use the definition to class and to justify the classification of the following sets as linearly independent or as linearly dependent. $\left\{\hat{i}, \hat{j}, 3\hat{i}+\hat{k}\right\}$, $\left\{\hat{i}, \hat{j}, 3\hat{i}-\hat{j}\right\}$, $\left\{\hat{i}-j, \hat{i}+\hat{j}\right\}$ and $\left\{\hat{i}-j, \hat{i}+\hat{j}, 3\hat{i}-3\hat{j}+\hat{k}\right\}$.

VSWUP3. Develop the Gram-Schmidt basis sets from: $\left\{\hat{i}, \hat{j}, 3\hat{i}+\hat{k}\right\}$, $\left\{\hat{i}, \hat{j}, 3\hat{i}-\hat{j}\right\}$, and $\left\{\hat{i}-j, \hat{i}+\hat{j}, 3\hat{i}-3\hat{j}+\hat{k}\right\}$ working with each vector in the order that it appears. Give the dimension of the space spanned in each case.

VSWUP4. Show that the collection of elements of the form $\left|v_1\right\rangle = a_1\,\hat{i} + b_1\,\hat{j} + c_1\,\hat{k}$ where a_i, b_i and c_i can be any real number is a vector space if addition is defined as: $\left|v_1\right\rangle + \left|v_2\right\rangle = (a_1+a_2)\,\hat{i} + (b_1+b_2)\,\hat{j} + (c_1+c_2)\,\hat{k}$ and scalar multiplication is defined as: $d\left|v_1\right\rangle = d\,a_1\,\hat{i} + d\,b_1\,\hat{j} + d\,c_1\,\hat{k}$ where d is any real number.

VSWUP5. State the axioms for an inner product. Show that an inner product defined as $\left\langle A \middle| B\right\rangle = A_x B_x + A_y B_y + A_z B_z$ satisfies those axioms for the space of vectors of the form $A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ where A_x, A_y, and A_z can be any real number. Use:

$$\vec{A}+\vec{B}=\left(A_x\hat{i}+A_y\hat{j}+A_z\hat{k}\right)+\left(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right)=(A_x+B_x)\hat{i}+(A_y+B_y)\hat{j}+(A_z+B_z)\hat{k}$$

$$d\,\vec{B}=d\left(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right)=(d\,B_x)\hat{i}+(d\,B_y)\hat{j}+(d\,B_z)\hat{k}$$

$$\left\langle e_i \middle| e_j \right\rangle=\hat{e}_i\cdot\hat{e}_j=\delta_{ij} \text{ where } \{\hat{i},\hat{j},\hat{k}\}\Leftrightarrow\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}.$$

VSWUP6. Define: span, spanning set and basis. Contrast basis and spanning set. What is the dimension of a vector space?

VSWUP7. Consider a finite dimensional vector space. Prove that any set of elements of that vector space that includes the zero vector is linearly dependent.

VSWUP8. A vector has the representation $\vec{A}=A_x\,\hat{i}+A_y\,\hat{j}$ in an original coordinate system. A rotated coordinate system rotated by θ in the *CCW* sense has axis directions $\hat{i}'$ and $\hat{j}'$. In that system, $\vec{A}=A'_x\,\hat{i}'+A'_y\,\hat{j}'$. We conclude that:

$$\vec{A}=A_x\,\hat{i}+A_y\,\hat{j}=A'_x\,\hat{i}'+A'_y\,\hat{j}'$$

a.) Prepare a drawing and evaluate: $\hat{i}\cdot\hat{i}',\ \hat{i}\cdot\hat{j}',\ \hat{j}\cdot\hat{i}'$ and $\hat{j}\cdot\hat{j}'$.

b.) The inner product of which pair of vectors provides information about A_x?
Compute that inner product.

c.) The inner product of which pair of vectors provides information about A'_y?
Compute that inner product. Repeat for A'_x.

d.) Substitute the values of $\hat{i}\cdot\hat{i}',\ \hat{i}\cdot\hat{j}',\ \hat{j}\cdot\hat{i}'$ and $\hat{j}\cdot\hat{j}'$ into the expressions. Summarize the results in matrix form. Provide explicit values of the λ_{ij}.

$$\begin{bmatrix} A'_x \\ A'_y \end{bmatrix}=\begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}\begin{bmatrix} A_x \\ A_y \end{bmatrix}$$

VSWUP9. Vectors in a space are of the form $\vec{A}=c_A\,\hat{i}+d_A\,\hat{j}$ where the *c*'s and *d*'s are any complex number. The plan is to define the inner product of two such vectors as: $\left\langle \vec{A} \middle| \vec{B} \right\rangle=c_A^*\,c_B+d_A^*\,d_B$. Show that this definition has the correct inputs and outputs to be

an inner product. Show that the procedure obeys the 4 axioms for inner products. Conclude that the procedure defines an acceptable inner product.

VSWUP10. State the axioms for an inner product. Show that an inner product candidate defined as $\langle A|B\rangle = A_x B_x^2 + A_y B_y^2 + A_z B_z^2$ fails to satisfy all of the axioms for a good inner product.

VSWUP10. **PROJECTION (A MOST WONDERFUL PROCESS)**: Consider a vector represented as $|V_i\rangle = a_{i1}|B_1\rangle + a_{i2}|B_2\rangle + ... + a_{in}|B_n\rangle$ in terms of the basis set { .. $|B_i\rangle$ .. }. Further, assume that the basis set is mutually orthogonal. $\Rightarrow \langle B_j|B_i\rangle = \langle B_j|B_i\rangle\delta_{ij}$. The coefficient a_{ik} represents the amplitude of the $|B_k\rangle$ behavior in the vector $|V_i\rangle$.

a.) Use the inner product to develop a general expression for a_{ik} in terms of the ratio of inner products for vectors expanded in an orthogonal basis. Note that while the basis is mutually orthogonal, the vectors are not necessarily unity normalized.

b.) Exercise this equation for the case of $\vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$ and where the inner product is are standard *dot* product to find an expression for A_y. Does it work properly?

Problems:

1.) Prove that if a set of *n* (numbered 1 to *n*; left to right) *non-zero* vectors is linearly dependent, that one of the vectors can be expressed as a linear combination of the previous members (ones numbered lower). Further that member can be removed from the set without the loss of any vectors from the span of the original set of *n* vectors. You may assume that the span of any set of vectors is a vector space.

2.) Suppose that you have a space $\mathbb{V}$ which is the span of a set of *n* vectors $\{|V_1\rangle, |V_2\rangle, ..., |V_n\rangle\}$ and you have a set of *m* linearly independent vectors $\{|W_1\rangle, |W_2\rangle, ..., |W_m\rangle\} \in \mathbb{V}$. Show that the Ws can be used to replace *m* members of the original set to yield a set $\{|W_1\rangle, |W_2\rangle, ..., |W_m\rangle, and\ (n-m)\ of\ the\ Vs\}$ that is a spanning set for

the entire space $\mathbb{V}$. Conclude that $m \le n$. APPROACH: Study the definitions of all the terms. Use the axioms and previously established results related to linear independence. Base your solution on the Replacement Lemma. When you substitute the second $|W\rangle$ into the spanning set you cast out a 'current' member. What member must you argue is not cast out? Assume that one (or more) $|W\rangle$'s remains outside the spanning set after the n $|V\rangle$'s have been replaced by n $|W\rangle$'s. What property of the $|W\rangle$'s would be contradicted?

One concludes that the number of members is a linearly independent set of vectors in a space is less than or equal to the number of members in a spanning set for that space.

3.) Prove that if your have a basis set $\boldsymbol{B_1}$ with m members and a basis set $\boldsymbol{B_2}$ with n members for the same space $\mathbb{V}$ then $m = n$. APPROACH: Use problem VS2.

4.) Prove that the largest number of vectors in a linearly independent set of vectors in an n dimensional space is n. That is: If you have a set of n+1 vectors in an n dimensional space, then the set must be linearly dependent. APPROACH: Use the properties of a basis set and problem VS2.

5.) Consider a spanning set $\left\{\hat{i}+2\,\hat{j}+\hat{k},\, 2\,\hat{i}+\hat{j}+2\,\hat{k},\, 2\,\hat{i}-2\,\hat{j}+2\,\hat{k}\right\}$. Apply the Gram-Schmidt procedure to find an orthonormal (orthogonal and unity normalized) basis. What is the dimension of the space spanned by this basis set?

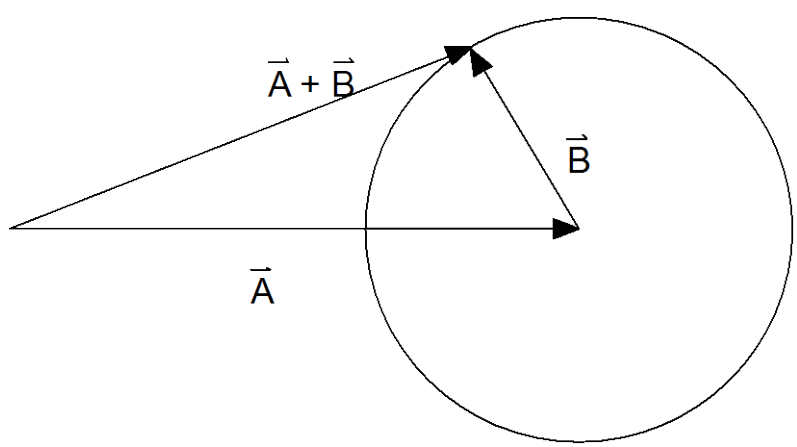
6.) A set has members of the form $|I\rangle = a_i\,|e_1\rangle + b_i\,|e_2\rangle + c_i\,|e_3\rangle$ where the constants a_i, b_i, and c_i are arbitrary real numbers, the components of the vector. Addition is defined by the relation $|I\rangle + |J\rangle = \left(a_i + a_j\right)|e_1\rangle + \left(b_i + b_j\right)|e_2\rangle + \left(c_i + c_j\right)|e_3\rangle$ and scalar multiplication by $d\,|I\rangle = (d\,a)\,|e_1\rangle + (d\,b)\,|e_2\rangle + (d\,c)\,|e_3\rangle$. This set satisfies the requirements to be a vector space.

a.) The basis vectors $|e_i\rangle$ have the following inner products $\langle e_i|e_j\rangle = \delta_{ij}$ as they are orthogonal and normalized. Find an expression for $\langle J|I\rangle$ where $|I\rangle = a_i\,|e_1\rangle + b_i\,|e_2\rangle + c_i\,|e_3\rangle$ and $|J\rangle = a_j\,|e_1\rangle + b_j\,|e_2\rangle + c_j\,|e_3\rangle$. Note $\langle J| = a_j^*\,\langle e_1| + b_j^*\,\langle e_2| + c_j^*\,\langle e_3|$ for a complex vector space. See property I3 of the inner product.

b.) Find the component of $|J\rangle$ along $|e_2\rangle$. *What is a component*? You should create a definition!

7.) Given any two vectors in an inner product space $|A\rangle$ and $|B\rangle$, prove the Schwarz Inequality: $|\langle A|B\rangle| \le \|A\rangle\|\|B\rangle|$. Use the vector $|C\rangle = |A\rangle - [\langle B|B\rangle]^{-1}\langle B|A\rangle|B\rangle$ and the properties of an inner product. The form of the vector $|C\rangle$ reminds you of a procedure studied in this handout. Name that procedure. Study the dagger operation as it provides the rule for computing the *BRA* vector conjugate to a given *KET* vector. Recall that $|V|^2 = \langle V|V\rangle$.

8.) Argue that $|\langle A|B\rangle + \langle B|A\rangle| \le 2|\langle A|B\rangle|$. Hint: Regard $\langle A|B\rangle$ as an arbitrary complex number.



9.) Use the Schwarz Inequality: $|\langle A|B\rangle| \le \|A\rangle\|\|B\rangle|$, the previous problem and $|V|^2 = \langle V|V\rangle$ to establish the triangle inequality: The magnitude of the sum of two vectors is between the magnitude of the difference of the magnitudes of the two vectors and the sum of

their magnitudes. $\left|\,|A\rangle\right| - \left|\,|B\rangle\right| \le \left|\,|A\rangle + |B\rangle\right| \le \left|\,|A\rangle\right| + \left|\,|B\rangle\right|$ Compare the expressions

$\left|\,\left|\,|A\rangle\right| + \left|\,|B\rangle\right|\,\right|^2 = \langle A|A\rangle + 2\left|\,|A\rangle\right|\left|\,|B\rangle\right| + \langle B|B\rangle$ and $\left|\,\left|\,|A\rangle\right| - \left|\,|B\rangle\right|\,\right|^2$ with $\left|\,|A\rangle + |B\rangle\,\right|^2$.

10.) A vector is expanded in terms of a basis set as $|I\rangle = a_i\,|e_1\rangle + b_i\,|e_2\rangle + c_i\,|e_3\rangle$ where the constants a_i, b_i, and c_i are scalar values and $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$ is the basis. Show that the components of the vector, the values a_i, b_i, and c_i, are unique. HINT: Review the definition of a basis set.

11.) A space of 3rd order polynomials is defined with domain the interval [-1,1]. An inner product for functions on that interval is defined to be:

$$\langle g\,|\,f\rangle = \int_{-1}^{1} g(x)\,f(x)\,dx$$

A possible basis set for the space is { 1, x, x^2, x^3 }. Choose $|B_1\rangle = 1$ and then work upward power by power. Use the Gram-Schmidt process to develop an orthogonal set of vectors (polynomials). Apply the normalization condition the $P_n(1) = 1$ to set the overall scaling (normalization of the polynomials. From Mathematica:
LegendreP[2,x] = $-\frac{1}{2} + \frac{3}{2}\,x^2$; LegendreP[3,x] = $-\frac{3}{2}x + \frac{5}{2}\,x^3$.(See Boas p. 485.)

12.) A space of 3rd order polynomials is defined with domain the interval [-∞,∞]. An inner product for functions on that interval is defined to be:

$$\langle g\,|\,f\rangle = \int_{-\infty}^{\infty} g(x)\,f(x)\;e^{-x^2}\,dx \qquad \text{Note the weight function: } e^{-x^2}$$

A possible basis set for the space is { 1, x, x^2, x^3 }. Choose $|B_1\rangle = 1$ and then work upward power by power. Use the Gram-Schmidt process to develop an orthogonal set of vectors (polynomials). The resulting polynomials are the Hermite polynomials that use yet another weird normalization. For the nth order polynomial, the lead term is $2^n\,x^n$. Multiply the Gram-Schmidt result by whatever factor is necessary to yield the desired highest order term. (See Boas p. 530.)

From Mathematica: HermiteH[2,x] = $4\,x^2-2$; HermiteH[3,x] = $8x^3-12x$.

$\int_{-\infty}^{\infty} x^{2n}\ e^{-x^2}\,dx=\sqrt{\pi}\ (n-\tfrac{1}{2})(n-\tfrac{3}{2})\ldots(\tfrac{1}{2})$; The integral vanishes for odd powers of *x*.

13.) An infinite dimensional function space is defined with domain the interval [-∞,∞]. An inner product for functions on that interval is defined to be:

$$\langle g\,|\,f\rangle=\int_{-\infty}^{\infty} g(x)\,f(x)\ \ dx$$

A possible basis set for the space is: $\left\{1\ e^{-\left(x^2/2\right)},\ x\ e^{-\left(x^2/2\right)},\ x^2\ e^{-\left(x^2/2\right)},\ x^3\ e^{-\left(x^2/2\right)},\ldots\right\}$.

Choose $|B_1\rangle=1\ e^{-\left(x^2/2\right)}$ and then work upward using $|B_{n+1}\rangle=x^n\ e^{-\left(x^2/2\right)}$ as an initial basis function. Use the Gram-Schmidt process to develop an orthogonal set of vectors (functions). The resulting functions are the *Hermite functions* and are to be *unity* normalized using the prescribed inner product. Find the first four Hermite functions. (Compare with the harmonic oscillator wave functions found in Harris or Eisberg.)
Hint: One could just apply the unity normalization to the results of the Hermite polynomial problem above. *What happened to the weight function?*

Use: $\int_{-\infty}^{\infty} e^{-x^2}\,dx=\sqrt{\pi}$ and $\int_{-\infty}^{\infty} x^{2n}\ e^{-x^2}\,dx=\sqrt{\pi}\ (n-\tfrac{1}{2})(n-\tfrac{3}{2})\ldots(\tfrac{1}{2})$

The integral vanishes for odd powers of *x*.

14.) **a.)** Use the Gram-Schmidt procedure to create an ortho-normal basis from the spanning set $\left\{\hat{i}+\hat{j}+\hat{k};\ \hat{i}-2\,\hat{j}+\hat{k};\ \hat{k}\right\}$. What is the dimension of the space spanned by the set? Start with $|B_1\rangle=\hat{i}+\hat{j}+\hat{k}$ and work to the right.

b.) Consider the space spanned by the set of vectors: $\left\{\hat{i}+\hat{j}+\hat{k},\hat{i}+\hat{j},\hat{k}\right\}$. Use the Gram-Schmidt procedure to create an ortho-normal basis for the space. What is the dimension of the space? Start with $|B_1\rangle=\hat{i}+\hat{j}+\hat{k}$ and work to the right.

c.) Consider the components of each of the vectors in the initial spanning set as the row of a matrix. Compute the determinant of that matrix for the case of exercise one and for the case of exercise two. Comment on the outcomes.

d.) The Gram-Schmidt method applied to our standard 3D problems produced an orthonormal basis that may **not obey the right-hand rule**. Multiply the last vector in a three dimensional set by -1 if necessary to bring the set into right hand rule order.

15.) a.) Starting with $|B_1\rangle$, apply the Gram-Schmidt procedure to develop an orthonormal basis set from: $\left\{|B_1\rangle = \hat{i} + \hat{k};\ \ |B_2\rangle = \hat{i} + \hat{j};\ \ |B_3\rangle = \hat{k}\right\}$

b.) Compare with the results for the set $\left\{|B_1\rangle = \hat{i} + \hat{k};\ \ |B_2\rangle = 2\left(\hat{i} + \hat{j}\right);\ \ |B_3\rangle = 3\,\hat{k}\right\}$.

c.) Compare with the results for the set $\left\{|B_1\rangle = \hat{i} + \hat{j};\ \ |B_2\rangle = \hat{i} + \hat{k};\ \ |B_3\rangle = \hat{k}\right\}$.

d.) Compare with the results for the set $\left\{|B_1\rangle = \hat{i} + \hat{k};\ \ |B_2\rangle = \hat{i} + \hat{j};\ \ |B_3\rangle = \hat{i}\right\}$.

e.) The Gram-Schmidt method applied to our standard 3D problems produced an orthonormal basis that may **not obey the right-hand rule**. For each case above, multiply the last vector in a three dimensional set by -1 if necessary to bring the set into right hand rule order.

NOT PART OF THE PROBLEM: **The Replacement Lemma**: Suppose that $\left\{|v_1\rangle, |v_2\rangle, ..., |v_m\rangle\right\}$ spans a vector space $\mathbb{V}$. Given any non-zero $|w\rangle \in \mathbb{V}$, the vector $|w\rangle$ can be substituted into the spanning set for one of the $|v\rangle$'s (say $|v_i\rangle$) and the resulting set $\left\{|w\rangle, |v_1\rangle, |v_2\rangle, \ldots, |v_{i-1}\rangle, |v_{i+1}\rangle, \ldots, |v_m\rangle\right\}$ still spans the space $\mathbb{V}$. This lemma is the anchor for shifting to the new Gram-Schmidt (G-S) basis. The Gram-Schmidt process generates vectors that are linear combinations of the current set and hence they are in its span. Each can be substituted into the spanning set without losing any vectors from the span. Each new G-S vector is orthogonal to all previous G-S vectors indicating that the one of the *old* set can be removed, not one of the new G-S vectors.

16.) The set of vectors $\{\hat{i}, \hat{j}, \hat{k}\} = \{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$ is a complete orthonormal basis set for three

dimensional vectors such as $\vec{A} = A_x\,\hat{e}_x + A_y\,\hat{e}_y + A_z\,\hat{e}_z$ and $\vec{B} = B_x\,\hat{e}_x + B_y\,\hat{e}_y + B_z\,\hat{e}_z$. Adopt the *new* notation where {|1⟩, |2⟩, |3⟩} represents $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$ and ⟨A|B⟩ represents $\vec{A}\cdot\vec{B}$.

a.) What are ⟨2|B⟩ and ⟨1|A⟩? What are ⟨1|2⟩, ⟨3|3⟩ and ⟨*m*|*n*⟩?

b.) Evaluate $\sum_{m=1}^{3} \hat{e}_m\ (\hat{e}_m \cdot \vec{A})$ and $\sum_{m=1}^{3} \hat{e}_m\ (\hat{e}_m \cdot \vec{B})$.

c.) Write out $\sum_{m=1}^{3} \hat{e}_m\ (\hat{e}_m \cdot \vec{A})$ *in the new notation. Repeat for* $\sum_{m=1}^{3} \hat{e}_m\ (\hat{e}_m \cdot \vec{B})$.

d.) Consider $\sum_{m=1}^{3} |m\rangle\langle m|$ as an operator on the three dimensional space of vectors. Let |*V*⟩ be an arbitrary three dimensional vector in this space. What is $\{\sum_{m=1}^{3} |m\rangle\langle m|\}|V\rangle$?

e.) It is said that $\sum_{m=1}^{3} |m\rangle\langle m|$ is equal to 1. What is $1|V\rangle$?

f.) What is $\{\sum_{n=1}^{3} |n\rangle\langle n| \sum_{m=1}^{3} |m\rangle\langle m|\}|V\rangle$?

g.) Compare ⟨*A*|*B*⟩ and $\langle A|\{\sum_{n=1}^{3} |n\rangle\langle n| \sum_{m=1}^{3} |m\rangle\langle m|\}|B\rangle$.

h.) $\langle A|\{\sum_{m,n=1}^{3} |n\rangle\langle n|m\rangle\langle m|\}|B\rangle\}$ Evaluate $\langle n|m\rangle$. Execute the sum over *n*. Finally, execute the sum over *m*.

i.) Rewrite $\langle A|\{\sum_{n=1}^{3} |n\rangle\langle n| \sum_{m=1}^{3} |m\rangle\langle m|\}|B\rangle$ using summation convention. That is with summations over repeated indices being implicit removing the need for the Σ's.

j.) Would it make sense to say that $\sum_{m=1}^{3} |m\rangle\langle m|$ is equal to 1? Would it make sense to say that $\sum_{m=1}^{2} |m\rangle\langle m|$ is equal to 1. Why not? The sum should be over the vectors in a complete orthonormal basis set. The operator $\sum_{m=1}^{2} |m\rangle\langle m|$ maps vectors into their projections onto the *x*-*y* (1-2) plane.

17. **The span of a set of vectors** is all the vectors that can be represented as $\sum a_i |I\rangle$ where the a_i are arbitrary scalar coefficients. The set $\mathbb{S} = \{..., |I\rangle, ...\}$ is said to ***span*** the space of elements $\sum a_i |I\rangle$. Span($\{..., |I\rangle, ...\}$) $= \{...., \sum a_i |I\rangle, ...\}$ for any $\{\ldots a_i \ldots\} \in \mathbb{K}$.

a.) The members of the set $\mathbb{S} = \{..., |I\rangle, ...\}$ are linearly independent. This linearly independence immediately identifies one vector that cannot be a member of the set. What is that excluded vector?

b.) The representation of a vector as a linear combination of vectors in a linearly independent set is unique. Prove this proposition. Begin by assuming that there are two distinct linear combinations of the vectors that equal the vector $|v\rangle$.

$$|v\rangle = a_1|1\rangle + a_2|2\rangle + ... + a_N|N\rangle \text{ and } |v\rangle = b_1|1\rangle + b_2|2\rangle + ... + b_N|N\rangle$$

Show that this assumption leads to a contradiction. It follows immediately that $a_1|1\rangle + a_2|2\rangle + ... + a_N|N\rangle = b_1|1\rangle + b_2|2\rangle + ... + b_N|N\rangle$ requires that $a_i = b_i$ for $1 \le i \le N$. <u>Compare this problem with FS3 and LS9.</u> Add the result from the next problem and compare again!

Consider displacements of lengths 2 meters and 3 meters whose direction make an angle of 75^0 with respect to one another. Are these displacements linearly independent? A third displacement is added to the set. What character must it possess if the set of the three displacements is to be linearly independent?

18.) Consider a set of vectors that is mutually orthogonal. That is: every member has a vanishing inner product with every other member of the set. Prove that the set of vectors is linearly independent. Well, slow down a little. There is one trivial little bothersome vector that can never appear in a set that is to be linearly independent. a.) What is that irksome little vector? b.) Add the statement that the irksome vector is not a member of our set of mutually orthogonal vectors and prove that such a set of mutually

orthogonal vectors is linearly independent. The conclusion is that, absent $|\text{zero}\rangle$, a mutually orthogonal set is linearly independent so all propositions proven for linearly independent sets are also true for mutually orthogonal sets. Note that $|\text{zero}\rangle$ cannot be a member of an orthonormal set.

Consider a set of three mutually perpendicular (orthogonal) displacements. Are they linearly independent?

19.) a.) Use the Gram-Schmidt procedure to create an ortho-normal basis from the spanning set: $\left\{\hat{i}+\hat{j}+\hat{k};\ \hat{i}-2\,\hat{j}+\hat{k};\ \hat{k}\right\}$. What is the dimension of the space spanned by the set? Start with $|B_1\rangle = \hat{i}+\hat{j}+\hat{k}$ and work to the right.

b.) Repeat the process, but start with the rightmost vector and work to the left.

c.) Comment on the results.

23.) It is proposed that the inner product of complex-valued functions with the domain of the real line (∞, - ∞) be defined as: $\langle\phi|\psi\rangle = \int_{-\infty}^{\infty}[\phi(x)]^*\psi(x)\,dx$. The operation returns a scalar value as required. Show that the proposed definition meets the requirements of the four axioms for inner products. The space of function is restricted to functions $\zeta(x)$ such that $\int_{-\infty}^{\infty}|\zeta(x)|^2\,dx$ is defined (finite) $\Rightarrow$ that is: functions that are square integrable.

24.) Suppose that $\langle\phi|\psi\rangle = \int_{-\infty}^{\infty}[\phi(x)]^*\psi(x)\,dx$ is a valid inner product for complex-valued functions with the domain of the real line (∞, - ∞). What can be said about the values $\langle\phi|\psi\rangle$, $\langle\psi|\psi\rangle$ and $\langle\phi|\phi\rangle$? What name is associated with the relationship that you propose?

25.) Show that $\langle V|\text{ZERO}\rangle = 0$ for all vectors $|V\rangle$. Build your proof on the properties of the inner product.

26.) Consider the set of all complex-valued functions that are periodic with period T and that have a finite square integral. $\int_{-T/2}^{T/2} |f(t)|^2\, dt$ finite. The sum of two functions is: Sum[$f(t)$, $g(t)$] = $f(t) + g(t)$. The scalar multiple of f is: ScalarMult[d, $f(t)$] = $d\, f(t)$.
For complex valued functions, complex numbers are the appropriate scalar field. (If the set is restricted to real-valued functions, then the real numbers should be chosen.) Show that the set defined with the prescribed operations is a vector space.

27.) A definition for the inner product of two functions that are periodic with period T and finite square integral has been proposed. $\langle g | f \rangle = \left(1/T\right) \int_{-T/2}^{T/2} \left(g(t)\right)^* f(t) dt$

a.) What type of value does this operation return?
b.) Verify that the operation satisfies the axioms required of an inner product for the set of functions prescribed in the previous problem.

28.) Consider the set of all complex-valued functions that can be formed as linear combinations of the functions { 1, { .. , cos[$m\omega_o t$], … }, { … , sin[$n\omega_o t$], … }} where $\omega_o = {}^{2\pi}/_{T}$ and $\omega_m = m\omega_o$ subject the restriction * below is a vector space with the operations on the representative members $g(t)$ and $f(t)$:

$$f(t) = c_{f0}(1) + \sum_{m=1}^{\infty} a_{fm} \cos[\omega_m t] + \sum_{m=1}^{\infty} b_{fm} \sin[\omega_m t]$$

$$g(t) = c_{g0}(1) + \sum_{m=1}^{\infty} a_{gm} \cos[\omega_m t] + \sum_{m=1}^{\infty} b_{gm} \sin[\omega_m t]$$

$$\text{Sum}[f(t), g(t)] = \left(c_{f0} + c_{g0}\right)(1) + \sum_{m=1}^{\infty} \left(a_{fm} + a_{gm}\right) \cos[\omega_m t] + \sum_{n=1}^{\infty} \left(b_{fn} + b_{gn}\right) \sin[\omega_m t]$$

The scalar multiple of a function is:

$$\text{ScalarMult}[d, f(t)] = d\, c_{f0}(1) + \sum_{m=1}^{\infty} \left(d\, a_{fm}\right) \cos[\omega_m t] + \sum_{n=1}^{\infty} \left(d\, b_{fn}\right) \sin[\omega_n t]$$

Retriction *: $\left|c_{f0}{}^2\right| + \tfrac{1}{2} \sum_{m=1}^{\infty} \left|a_{fm}\right|^2 + \tfrac{1}{2} \sum_{m=1}^{\infty} \left|b_{fm}\right|^2$ is finite.

Show that the set defined with the prescribed operations is a vector space. The scalars can be the real numbers or the complex numbers.

29.) A definition for the inner product of the vectors in the previous problem has been proposed. $\langle g(t)|f(t)\rangle = c_{g0}^{*}c_{f0}(1) + \frac{1}{2}\sum_{m=1}^{\infty} a_{gm}^{*}a_{fm} + \frac{1}{2}\sum_{n=1}^{\infty} b_{gn}^{*}b_{fn}$ s

a.) What type of value does this operation return?

b.) Verify that the operation satisfies the axioms required of an inner product for the set of functions prescribed in the previous problem.

30.) **The span of a set of vectors** $\mathbb{S}$ is all the vectors that can be represented as $\sum a_i |S_i\rangle$ where the a_i are arbitrary scalar coefficients. The set $\mathbb{S} = \{..., |S_i\rangle, ...\}$ is said to ***span*** the space of elements $\sum a_i |S_i\rangle$. Span($\{..., |S_i\rangle, ...\}$) $= \left\{ .... , \sum a_i |S_i\rangle , ... \right\}$ for any $\{\ldots a_i \ldots\} \in \mathbb{K}$.

a.) Verify that, for any set of vectors $\mathbb{S} = \{..., |S_i\rangle, ...\}$ contained in the space $\mathbb{V}$, the span of that set is a vector space. Further, that every vector in that space is also a vector in the space $\mathbb{V}$.

b.) Wikipedia (linear span) defines the span of a set of vectors by the following.

> Given a vector space $\mathbb{V}$ over a field $\mathbb{K}$, the span of a set $\mathbb{S}$ (not necessarily finite) is defined to be the intersection $\mathbb{W}$ of all subspaces of $\mathbb{V}$ which contain $\mathbb{S}$. When $\mathbb{S}$ is a finite set, then $\mathbb{W}$ is referred to as the subspace spanned by the vectors in $\mathbb{S}$.

Assume that the set $\mathbb{S}$ has a finite number of members. Note that the span of $\mathbb{S}$ is a vector subspace. Argue by closure that it is the smallest space that includes the set $\mathbb{S}$. Conclude that for a finite set $\mathbb{S}$, that the two definitions agree.

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