

# NOTES AND DISCUSSIONS

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## More about the falling raindrop

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A simple strategy is presented for solving the “inverse rocket” problem of a particle accumulating material from a medium through which it falls vertically. Some forms of drag can also be easily included, thereby changing the constant acceleration to a more realistic value.

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Sokal<sup>1</sup> has introduced a general version of the problem of a raindrop accumulating mass as it falls through mist and solved it using the chain rule and an integrating factor. In this Note, it is shown that his solution can be simplified by using momentum as the dependent variable, making the solution more accessible to introductory physics majors. In addition, the problem is further generalized to include a drag force.

In the absence of drag, Newton’s second law for the drop is

$$\frac{dp}{dt} = mg, \quad (1)$$

where  $p = mv$  is its momentum and the gravitational field  $g$  is assumed to be constant. The mass accretion rate is assumed to scale with powers of the mass (or size) and speed of the drop,

$$\frac{dm}{dt} = \lambda m^\alpha v^\beta = \lambda m^{\alpha-\beta} p^\beta, \quad (2)$$

with values of the parameters  $\lambda > 0$ ,  $1 > \alpha \geq 0$ , and  $\beta \geq 0$  chosen to ensure stability. Dividing Eq. (1) by Eq. (2) gives

$$p^\beta dp = \frac{g}{\lambda} m^{1+\beta-\alpha} dm, \quad (3)$$

which has the general solution

$$\frac{p^{1+\beta}}{1+\beta} = \frac{gm^{2+\beta-\alpha}}{\lambda(2+\beta-\alpha)} + \frac{C}{1+\beta}, \quad (4)$$

where the last term is a constant of integration. If we substitute  $p = mv$ , Eq. (4) becomes the same as Sokal’s solution.

Drag can be included by adding a term to the right-hand side of Eq. (1),

$$\frac{dp}{dt} = mg - \varepsilon m^\alpha v^\gamma = mg - \varepsilon m^{\alpha-\gamma} p^\gamma, \quad (5)$$

with  $\varepsilon \geq 0$  and  $\gamma > 0$  and where the same exponent  $\alpha$  for  $m$  is used for the drag and mass accretion terms because both effects are expected to scale similarly with the size of the drop.<sup>2</sup> Solving Eqs. (2) and (5) simultaneously is straightforward when  $\gamma = 1 + \beta$ , which includes the important special

cases  $\gamma = 1$  and  $\beta = 0$  (linear drag with speed-independent mass accretion) and  $\gamma = 2$  and  $\beta = 1$  (quadratic drag with linear speed accretion). The solution when  $\gamma = 1 + \beta$  can be obtained by dividing Eq. (5) by Eq. (2), which leads to

$$\frac{dp}{dm} + \frac{\varepsilon p}{\lambda m} = \frac{g}{\lambda} m^{1+\beta-\alpha} p^{-\beta}. \quad (6)$$

The left-hand side of Eq. (6) suggests changing the dependent variable to  $u = m^{\varepsilon/\lambda} p$ , which yields<sup>3</sup>

$$u^\beta du = \frac{g}{\lambda} m^{(1+\beta)(1+\varepsilon/\lambda)-\alpha} dm \quad (7)$$

as a generalization of Eq. (3). This separated equation can be integrated. Rewriting the resulting solution in terms of  $v$  gives

$$v^{1+\beta} = \frac{g(1+\beta)m^{1-\alpha}}{\lambda(2+\beta-\alpha) + \varepsilon(1+\beta)} + \frac{C}{m^{(1+\beta)(1+\varepsilon/\lambda)}}. \quad (8)$$

We let  $C = 0$  (assuming  $v = 0$  when  $m = 0$ ) in Eq. (8), differentiate it with respect to time  $t$ , and then substitute Eq. (2) into the right-hand side and find the acceleration  $a \equiv dv/dt$  to be a constant,

$$a = \frac{ng}{1 + \varepsilon(1-n)/\lambda}, \quad (9)$$

where  $n \equiv (1-\alpha)/(2+\beta-\alpha)$ . This acceleration reduces to Sokal’s drag-free result for  $\varepsilon = 0$ . By including a value for  $\varepsilon$  of the order of  $100\lambda$ , the acceleration of a raindrop can be reduced to a more realistic value<sup>4</sup> of a few thousandths of  $g$  (for typical values of  $n$  of a few tenths).

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<sup>1</sup>A. D. Sokal, “The falling raindrop, revisited,” *Am. J. Phys.* **78**, 643–645 (2010).

<sup>2</sup>M. H. Partovi and D. R. Aston, “The generalized raindrop problem,” *Am. J. Phys.* **57**, 913–920 (1989).

<sup>3</sup>A similar observation applied to Eq. (8) divided by  $v^\beta$  in Ref. 1 suggests a change of dependent variable to  $p = mv$ .

<sup>4</sup>B. F. Edwards, J. W. Wilder, and E. E. Seime, “Dynamics of falling raindrops,” *Eur. J. Phys.* **22**, 113–118 (2001).