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LETTERS AND COMMENTS

Another comment on 'Eccentricity as a vector'

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Abstract
Defining the eccentricity vector as the dimensionless Runge–Lenz vector provides an alternative elementary derivation of the Keplerian polar orbit equation.

In a recent issue of this journal, Bringuier [1] provides an elegant derivation of Keplerian orbits at an elementary calculus-based physics level by introducing the concept of eccentricity as a *vector*. In this comment, a similar derivation is reviewed, using an alternative definition of the eccentricity vector, as presented in programme 22 of the Mechanical Universe video series [2].

Let the orbiting body have mass m and position vector $\mathbf{r}$ relative to a stationary (heavy) mass M at the origin. Then Newton's second law reads

$$m\mathbf{a} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \tag{1}$$

where $\mathbf{a}$ is the acceleration of the orbiting body and G is the gravitational constant. Since this gravitational force is radial, it exerts no torque about the origin, and hence the angular momentum $\mathbf{L}$ of the orbiting body is constant. Let the direction of this constant angular momentum vector define the z-axis, with the motion of body m confined to the r–θ plane in polar coordinates, and thus

$$\mathbf{L} = mr^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\mathbf{k}}. \tag{2}$$

Taking the cross-product of equations (1) and (2) gives

$$\mathbf{a} \times \mathbf{L} = GMm\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\mathbf{k}} \times \hat{\mathbf{r}}. \tag{3}$$

But the radial unit vector in the plane is $\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$, and it therefore follows that

$$\hat{\mathbf{k}} \times \hat{\mathbf{r}} = \cos\theta\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{i}} = \frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}\theta}. \tag{4}$$

Noting that the acceleration of the orbiting body is the time derivative of its velocity $\boldsymbol{v}$ and that its angular momentum is constant, one can rewrite equation (3) as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v} \times \mathbf{L}) = GMm\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} \tag{5}$$

where the right-hand side was simplified using equation (4). Integrate this to obtain

$$\boldsymbol{v} \times \mathbf{L} = GMm(\hat{\mathbf{r}} + \mathbf{e}) \tag{6}$$

where the integration constant is $\mathbf{e}$, the eccentricity vector, whose direction is chosen to define the x-axis.

As an aside, Bringuier chose his eccentricity vector—call it $\tilde{\mathbf{e}}$—to define the y-axis, which is not as intuitively appealing. The eccentricity vector $\mathbf{e}$ in equation (6) is in fact what Bringuier calls the Laplace–Runge–Lenz vector $\mathbf{A}$ divided by GMm to make it properly dimensionless. Note that the connection between the two eccentricity vectors is simply

$$\tilde{\mathbf{e}} = \hat{\mathbf{k}} \times \mathbf{e}, \tag{7}$$

so they have the same magnitude and differ only in choice of direction. (I would prefer to define the y-axis to be the direction of a vector $\mathbf{c}$, whose magnitude is defined below. We then have three physically meaningful constant vectors $\{\mathbf{e}, \mathbf{c}, \mathbf{L}\}$ in the directions of $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ respectively.) By vector multiplying $\hat{\mathbf{k}}$ by equation (6), one obtains

$$\boldsymbol{v} = \frac{GMm}{L}(\hat{\theta} + \tilde{\mathbf{e}}), \tag{8}$$

since the middle expression in equation (4) is the azimuthal unit vector $\hat{\theta}$. Equation (8) is the same as Bringuier's equation (5). The present approach can therefore reproduce all of the merits of Bringuier's formulation. This ends the aside.

To complete the derivation of the Keplerian orbital equation, take the dot product of the position vector with equation (6) to get

$$\mathbf{r} \cdot (\boldsymbol{v} \times \mathbf{L}) = GMm\mathbf{r} \cdot (\hat{\mathbf{r}} + \mathbf{e}). \tag{9}$$

Since the 'dot' and 'cross' can be interchanged in a triple scalar product, this becomes

$$\frac{m}{m}(\mathbf{r} \times \boldsymbol{v}) \cdot \mathbf{L} = GMm(r + er\cos\theta). \tag{10}$$

Noting that the left-hand side can be rewritten as L^2/m, this gives the polar equation of the Keplerian orbits,

$$r = \frac{c}{1 + e\cos\theta} \tag{11}$$

where $c \equiv L^2/(GMm^2)$ is a constant called the semilatus rectum. If $e = 0$ then $r = c$ for all θ, which defines a circular orbit. On the other hand, note that r diverges (implying an unbounded orbit) if $\cos\theta = -1/e$, which occurs if $e \geqslant 1$. A geometric interpretation of the two constants e and c in equation (11) is that c is the y-intercept and $r_{\min} \equiv c/(1+e)$ is the x-intercept of the orbit. The polar equation can be recast in the rectangular form of an ellipse, parabola or hyperbola by cross-multiplying to get $\sqrt{x^2 + y^2} + ex = c \Rightarrow x^2 + y^2 = (c - ex)^2$ and then (if $e \neq 1$) completing the square in x.

The preceding derivation assumed that the heavier mass is stationary at the origin. However, it is straightforward to generalize the derivation to the exact case where both masses orbit their common centre of mass at the origin. Simply reinterpret m as the reduced mass, $\mathbf{r}$ as the relative coordinate (with time derivatives $\boldsymbol{v}$ and $\mathbf{a}$), M as the total mass, and $\mathbf{L}$ as the total angular momentum.

The Mechanical Universe series is available in streaming video online at no charge [3] which makes it convenient for presentation in 30 minutes of class time. Retention is enhanced by having students fill in a worksheet as they watch and summarizing the derivation afterward. The videos also animate the circular velocity hodograph [4], although it is not called that name.

References

[1] Bringuier E 2004 Eccentricity as a vector: a concise derivation of the orbit equation in celestial mechanics *Eur. J. Phys.* **25** 369–72
[2] 1987 *The Mechanical Universe . . . and Beyond* (Santa Barbara: Annenberg/CPB Project)
[3] http://www.learner.org/resources/series42.html
[4] Butikov E I 2004 Comment on 'Eccentricity as a vector' *Eur. J. Phys.* **25** L41–3