

Babylonian resistor networks

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Abstract

The ancient Babylonians had an iterative technique for numerically approximating the values of square roots. Their method can be physically implemented using series and parallel resistor networks. A recursive formula for the equivalent resistance R_{eq} is developed and converted into a nonrecursive solution for circuits using geometrically increasing numbers of identical resistors. As an example, 24 resistors R are assembled into a second-order network and R_{eq}/R is measured to equal $\sqrt{2}$ to better than 0.2%, as could be done in an introductory physics laboratory.

(Some figures may appear in colour only in the online journal)

1. Aim

Given a bucket of identical resistors R , construct a circuit whose equivalent resistance approximates $\sqrt{2}R$ (to any desired degree of accuracy, limited only by the precision of the resistors). To ensure that the construction is accessible to introductory physics students, the resistors are to be combined either in series or in parallel at each stage of the circuit [1].

2. Theoretical analysis

To simplify the equations that follow, choose units of resistance so that $R = 1$. In the next section, it will be explained how to restore conventional units of ohms to the terms.

Start by putting two resistors in series to get a resistance of 2. Denote that combination as S_0 where the ‘ S ’ refers to putting two initial resistors in series and the subscript ‘0’ denotes a zeroth-order approximation to $\sqrt{2}$. However, S_0 is larger than $\sqrt{2}$ and is thus an overestimate. On the other hand, one gets an underestimate by constructing

$$P_0 = \frac{2}{S_0} = 1 \quad (1)$$

where in later steps, ‘ P ’ will refer to putting two previous resistor combinations in parallel. These two zeroth-order circuits are illustrated in figure 1.

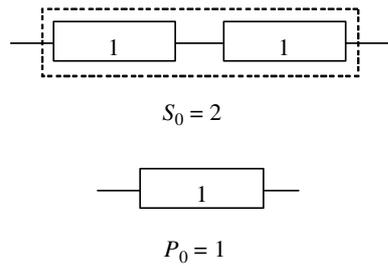


Figure 1. Starting circuits to construct a resistor network whose equivalent resistance approximates $\sqrt{2}$.

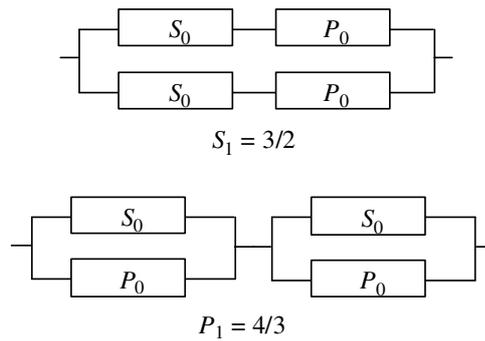


Figure 2. The two circuits that can be constructed using six unit resistors whose equivalent resistances are nearest in value to $\sqrt{2}$.

To achieve an equivalent resistance that is nearer in value to $\sqrt{2}$, one can average together S_0 and P_0 to get

$$S_1 = \frac{1}{2} (S_0 + P_0) = \frac{3}{2}. \quad (2)$$

To accomplish that, first put S_0 and P_0 in series with each other. Then put that trio of unit resistors in parallel with an identical trio to halve the total resistance. On the other hand, if one first puts S_0 and P_0 in parallel, and then puts that trio in series with an identical trio, one doubles the resistance of each trio to end up with

$$P_1 = 2 \left(\frac{1}{S_0} + \frac{1}{P_0} \right)^{-1} = 2 \frac{S_0 P_0}{S_0 + P_0} = \frac{S_0 P_0}{S_1} \quad (3)$$

using equation (2) in the last step. Consequently

$$S_1 P_1 = S_0 P_0 = 2 \quad (4)$$

from equation (1) and therefore

$$P_1 = \frac{2}{S_1} = \frac{4}{3}. \quad (5)$$

These two first-order resistor combinations are sketched in figure 2. Again S_1 is an overestimate and P_1 is an underestimate to the value of $\sqrt{2}$.

Table 1. Values predicted by equation (9) for a starting value of $S_0 = 2$.

n	S_n
0	2
1	$3/2 = 1.5$
2	$17/12 \approx 1.417$
3	$577/408 \approx 1.414\ 22$

These two constructions can now be iterated to successively approach an equivalent resistance of $\sqrt{2}$. To form S_2 and P_2 , simply replace every occurrence of S_0 and P_0 in figure 2 with S_1 and P_1 , respectively. Then equation (2) becomes

$$S_2 = \frac{1}{2} (S_1 + P_1) = \frac{1}{2} \left(S_1 + \frac{2}{S_1} \right) \tag{6}$$

using equation (5) in the last step, and equation (4) becomes

$$S_2 P_2 = S_1 P_1 = 2 \tag{7}$$

so that

$$P_2 = \frac{2}{S_2}. \tag{8}$$

It takes 24 unit resistors to build either of the second-order networks S_2 or P_2 .

More generally for $n \geq 1$ one can use 1.5×4^n unit resistors to construct the two circuits

$$S_n = \frac{1}{2} \left(S_{n-1} + \frac{2}{S_{n-1}} \right) \tag{9}$$

and

$$P_n = \frac{2}{S_n}. \tag{10}$$

Equivalent resistances computed recursively using equation (9) are listed in table 1. One sees that S_n rapidly converges to $\sqrt{2} \approx 1.41421$.

Recursion relation (9) can be converted into a nonrecursive formula valid for $n \geq 0$,

$$S_n = \sqrt{2} \frac{1 + r^{f(n)}}{1 - r^{f(n)}} \tag{11}$$

where $f(n) \equiv 2^n$ and $r \equiv (\sqrt{2} - 1)/(\sqrt{2} + 1) \approx 0.1716$, as verified in appendix A. It follows immediately from this formula that $S_n \rightarrow \sqrt{2}$ as $n \rightarrow \infty$. One can more compactly write this result as

$$S_n = \sqrt{2} \coth \left(2^n \operatorname{acoth} \sqrt{2} \right) \tag{12}$$

where \coth and acoth are the forward and inverse hyperbolic cotangent functions. (This formula again implies $S_n \rightarrow \sqrt{2}$ as $n \rightarrow \infty$ because $\coth \infty = 1$.) This explicit formula for S_n is plotted in figure 3 as the solid curve, treating n as a continuous variable. Its values for the integers $n = 0$ to 5 are indicated by the dots. The asymptotic value $\sqrt{2}$ is denoted by the horizontal dashed line.

To prove that equation (9) converges to $\sqrt{2}$ without using equations (11) or (12), assume [2] that a limiting value of S_{n-1} exists and call it S . That must also be the limiting value of S_n and hence equation (9) in the limit $n \rightarrow \infty$ becomes

$$S = \frac{1}{2} \left(S + \frac{2}{S} \right) \Rightarrow S^2 = 2 \tag{13}$$

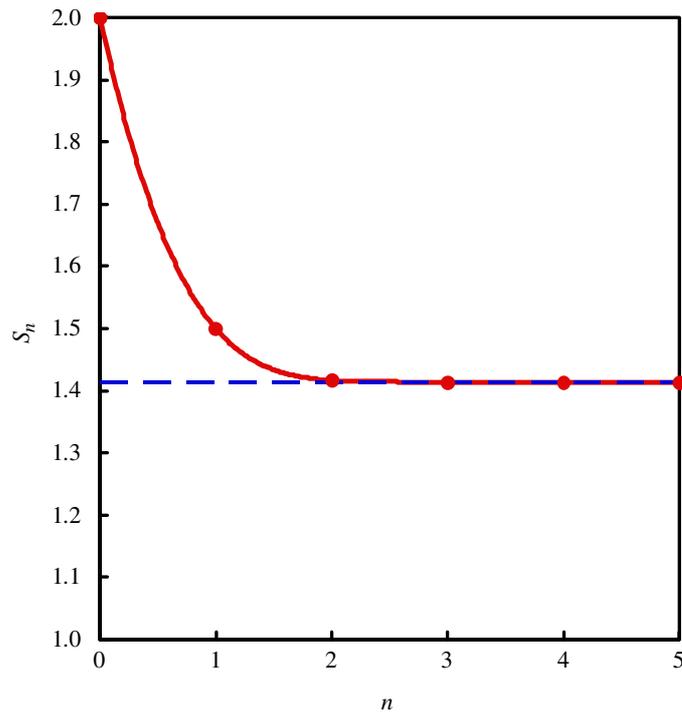


Figure 3. Explicit and recursive values of S_n asymptotically approaching the value $\sqrt{2}$.

whose positive solution is $S = \sqrt{2}$. (For another approach, see appendix B.) This limiting value can be understood intuitively by writing equation (9) as

$$S_n = \frac{1}{2} (S_{n-1} + P_{n-1}). \quad (14)$$

Since S_{n-1} is slightly larger than $\sqrt{2}$, it follows that $P_{n-1} = 2/S_{n-1}$ will be slightly smaller than $\sqrt{2}$. Therefore averaging together these two values gives an improved estimate for $\sqrt{2}$. This iterative method of approximating $\sqrt{2}$ was known to the ancient Babylonians [3], motivating the title of this paper.

3. Experimental verification

Students can assemble one of these Babylonian resistor circuits and measure its resistance. Noting from figure 3 that good accuracy is already obtained for $n = 2$ and that it would be difficult to correctly wire 96 or more resistors together, it is reasonable to build a second-order network. Since P_n is closer in value to $\sqrt{2}$ than is S_n (for any value of n), it makes sense to construct P_2 rather than S_2 . All of the equations in section 2 assumed unit resistors; for laboratory resistors of resistance R , one simply has to multiply any value or expression for S_n or P_n by R . (For example, S_1 in figure 2 then has a resistance of $1.5R$.)

The required components are 24 identical resistors R and five short jumper wires. First wire the resistors together as four parallel lines of $6R$ each by twisting their ends together (and securing them with solder), and then add the jumper wires indicated in red in figure 4.

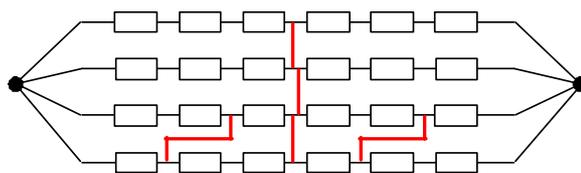


Figure 4. Circuit P_2 redrawn to make it easy to assemble.

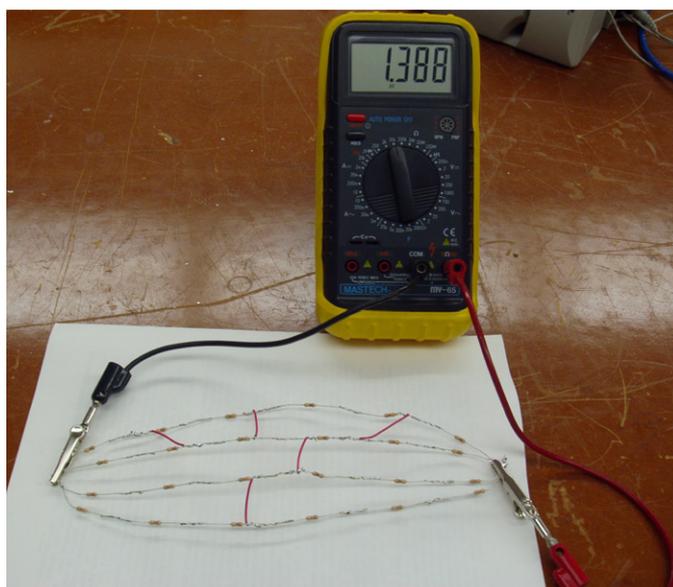


Figure 5. Measurement of the equivalent resistance of circuit P_2 .

The equivalent resistance between the left and right ends is predicted to be

$$\frac{24}{17}R \approx \sqrt{2}R. \tag{15}$$

The difference between $24/17$ and $\sqrt{2}$ is less than 0.2%.

The circuit in figure 4 was constructed using 24 nominally 1 k Ω resistors. The actual resistances were measured using a multimeter prior to assembly and were found to be $R = 983 \pm 5\Omega$. Thus, either the resistors are systematically low or the multimeter is a bit out of calibration, but that will not affect the ratio R_{eq}/R as long as the same multimeter is used to measure the equivalent resistance R_{eq} . As can be seen in the photograph of the setup in figure 5, we measured $R_{eq} = 1388\Omega$, which differs by less than 0.2% from $\sqrt{2}R = 1390\Omega$ and is well within the 5 Ω variation in the individual resistors.

4. Conclusions

Discussion and assembly of these resistor networks provide an interesting alternative to routine series and parallel circuits in an introductory physics course. An extension of the results presented here is to keep $P_0 = R$ (in laboratory units) but replace S_0 with any resistance kR

instead of $2R$. The equivalent resistance of networks iterated in the style of figure 2 will then asymptotically approach the value $\sqrt{k}R$ rather than $\sqrt{2}R$. The number k need not even be an integer but can have any real, positive value. (For example, if $P_0 = 200\Omega$ and $S_0 = 4.7\text{ k}\Omega$, then $k = S_0/P_0 = 23.5$.) Such Babylonian networks can thus be thought of as an analogue calculator of square roots. Another option is to replace the resistors with capacitors or springs because they too obey series and parallel combination rules.

Appendix A. Verification of equations (11) and (12)

Since $f(n+1) = 2 \cdot 2^n$, equation (11) implies that

$$S_{n+1} = \sqrt{2} \frac{1 + r^{2f}}{1 - r^{2f}}. \quad (\text{A.1})$$

But according to equations (9) and (11) this result is supposed to be equal to

$$\frac{1}{2} \left(S_n + \frac{2}{S_n} \right) = \frac{1}{\sqrt{2}} \frac{1 + r^f}{1 - r^f} + \frac{1}{\sqrt{2}} \frac{1 - r^f}{1 + r^f}. \quad (\text{A.2})$$

In other words, one needs to show that

$$\sqrt{2} \frac{1 + r^{2f}}{1 - r^{2f}} = \frac{1}{\sqrt{2}} \frac{(1 + r^f)^2}{1 - r^{2f}} + \frac{1}{\sqrt{2}} \frac{(1 - r^f)^2}{1 - r^{2f}}. \quad (\text{A.3})$$

But this equality is verified by expanding the two squares in the numerators of the right-hand side. Different values of r give rise to different initial values for S_0 . In particular, $r \equiv (\sqrt{2} - 1)/(\sqrt{2} + 1)$ results in $S_0 = 2$.

To directly obtain equation (12), run the recursion solver command

$$\text{RSolve}[\{S[n+1] == S[n]/2 + 1/S[n], S[0] == 2\}, S[n], n] \quad (\text{A.3})$$

in Mathematica. Alternatively one can convert between equations (11) and (12) using standard exponential and logarithmic identities for coth and acoth .

Appendix B. A physics-based approach to finding the limiting value of S_n

Rewrite equation (9) as

$$\frac{S_{n+1} - S_n}{(n+1) - n} = \frac{1}{S_n} - \frac{S_n}{2}. \quad (\text{B.1})$$

The left-hand side is $\Delta S_n / \Delta n \approx dS_n / dn$ by treating S_n and n as continuous variables. Substituting that derivative into equation (B.1) and multiplying through by $2S_n$, one obtains

$$\frac{d(S_n^2)}{dn} \approx 2 - S_n^2. \quad (\text{B.2})$$

By viewing S_n as a velocity, equation (B.2) can be interpreted as the change in kinetic energy of a particle falling in a uniform gravitational field while subject to a quadratic drag force. The differential equation is separated and integrated to get

$$S_n \approx \sqrt{2 - e^{-n}}. \quad (\text{B.3})$$

The limit $n \rightarrow \infty$ corresponds to a terminal speed of $\sqrt{2}$ in which case the left-hand side of equation (B.2) is zero since the velocity is no longer changing. Equation (B.3), unlike equations (11) and (12), is only approximate for finite n , however. In contrast to the red curve in figure 3, equation (B.3) describes a graph which starts at 1 and *increases* towards an asymptotic value of $\sqrt{2}$.

References

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