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The magnetocatenary

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Abstract
A current-carrying wire hanging between two suspension points in a transverse magnetic field adopts a shape intermediate between a circle and a hyperbolic cosine. This magnetocatenary can be analytically calculated as a novel extension of the standard hanging chain problem in an intermediate mechanics course.

(Some figures may appear in colour only in the online journal)

1. Introduction

A freely hanging cable adopts the shape of a catenary between adjacent suspension points. Many interesting variations on catenary problems have been considered [1, 2 and references therein]. In this paper we analyze the change in shape of the cable when a magnetic force also acts on it, arising from a current in the wire responding to a uniform magnetic field. To maximize the effect, the magnetic field is everywhere perpendicular to the wire. An analytic solution is found as a function of the ratio of the gravitational and magnetic forces. The shape varies continuously from a catenary when the magnetic force is negligible to a circular arc when the magnetic force dominates. (A circular shape is obtained in the latter case because the magnetic force acts perpendicularly outward on every segment of the wire, just as a jumping wire [3] would do if its two ends were pinned so that it could not escape the field region.)

As one possible physical realization of the problem, the following setup is suggested. A small smooth board of width W and height H is hinged along its upper edge, so that it can be oriented at any desired angle between a horizontal and a vertical position. A flexible wire with a length exceeding W has its two ends A and C affixed to the midpoints of the two side edges of the board. Connecting leads are attached from those two ends to a power supply driving a dc current I through the wire. Helmholtz coils of diameter much larger than both W and H are positioned above and below the board, parallel to it, so that the wire is immersed in a uniform magnetic field B. The coils are attached by posts to the board, to ensure that they remain parallel to the board when it is reoriented, so that the field $\vec{B}$ is always perpendicular to the wire. Increasing the board's angle from horizontal to vertical has the effect of increasing

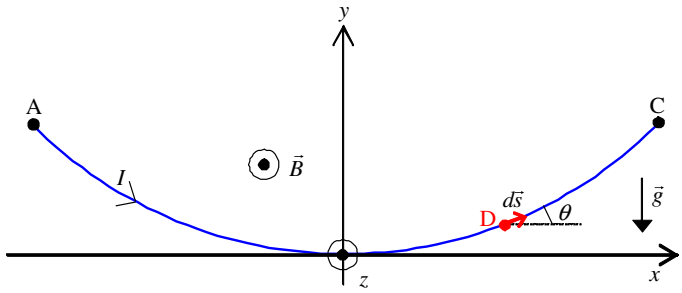

Figure 1. Current-carrying wire suspended between two points at the same height.

the magnitude of the apparent gravitational field[3] $\vec{g}$ directed toward the bottom edge of the board from 0 to 9.8 m s^{-2}.

2. Differential equation for the hanging wire

The geometry is sketched in figure 1. The wire is suspended between two points A and C at the same altitude above earth's surface. Choose a coordinate system with the origin at the lowest point of the wire, midway along its length. Points A and C lie along a line parallel to the x axis. The gravitational field $\vec{g}$ is vertically downward in the $-y$ direction, and the magnetic field $\vec{B}$ is initially assumed to be in the $+z$ direction. The current I runs from A to C. Consider an infinitesimal segment $\mathrm{d}\vec{s} = \mathrm{d}x\,\hat{i} + \mathrm{d}y\,\hat{j}$ directed along the current at arbitrary position D where the cable makes angle θ relative to the horizontal. The element of magnetic force on that segment is

$$\mathrm{d}\vec{F} = I\,\mathrm{d}\vec{s} \times \vec{B} = IB(\mathrm{d}y\,\hat{i} - \mathrm{d}x\,\hat{j}). \tag{1}$$

Integrating this expression, the magnetic force on the piece of the wire that runs from the origin to point D at coordinates $(x, y, 0)$ is then

$$\vec{F} = IB(y\,\hat{i} - x\,\hat{j}), \tag{2}$$

neglecting spatial variations in the magnitude and direction of the magnetic field (as would arise for a real Helmholtz coil, or if one included the magnetic field of the earth or that is created by neighboring segments of the wire). There are three additional forces acting on this piece of the cable: its weight $-W\,\hat{j}$, the tension $T\cos\theta\,\hat{i} + T\sin\theta\,\hat{j}$ at point D, and the tension $-T_0\,\hat{i}$ at the origin [4]. Balancing the vertical components of the forces leads to

$$T\sin\theta = W + IBx, \tag{3}$$

whereas the horizontal force balance implies

$$T\cos\theta = T_0 - IBy. \tag{4}$$

Dividing equation (3) by (4) eliminates the unknown T to give

$$\tan\theta \equiv \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{W + IBx}{T_0 - IBy}. \tag{5}$$

Letting s be the length of the piece of wire extending from the origin to point D, equation (5) can be rewritten as

$$(T_0 - IBy)\frac{\mathrm{d}y}{\mathrm{d}x} = \mu g s + IBx, \tag{6}$$

[3] This apparent field actually results from the sum of the gravitational and normal forces acting on the wire.

where μ is the linear mass density of the cable. According to the Pythagoras theorem,

$$\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}, \tag{7}$$

so that the x-derivative of equation (6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(T_0 - IBy)\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \mu g\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} + IB. \tag{8}$$

This second-order differential equation describes the shape of the cable, for which we coin the term 'magnetocatenary'.

3. Solving the differential equation

It is helpful to introduce the three dimensionless quantities $\beta \equiv \mu g/IB$ which gives the ratio of the gravitational to the magnetic force on any short segment, and $X \equiv IBx/T_0$ and $Y \equiv IBy/T_0$ which are scaled coordinates along the wire. For the directions of the current and magnetic field indicated in figure 1, the magnetic force pulls the cable downward. However, I and B can point in the opposite directions. Since they enter the problem only in the combination IB, it suffices to let just one of them have this freedom of direction. We will choose to define $+x$ to always point along the current at the origin. However the magnetic field can either point in the $+z$ direction which we will call a 'forward' field, or in the $-z$ direction defining a 'reverse' field. In contrast the combination μg can never be negative, although one could imagine it being zero. A forward or reverse magnetic field therefore implies that the coefficient β is positive or negative, respectively.

Equation (8) can be rewritten in terms of the three dimensionless variables as

$$(1 - Y)\frac{\mathrm{d}^2Y}{\mathrm{d}X^2} = \beta\sqrt{1 + \left(\frac{\mathrm{d}Y}{\mathrm{d}X}\right)^2} + 1 + \left(\frac{\mathrm{d}Y}{\mathrm{d}X}\right)^2. \tag{9}$$

The form of the right-hand side suggests the change in variable

$$U \equiv \sqrt{1 + \left(\frac{\mathrm{d}Y}{\mathrm{d}X}\right)^2}. \tag{10}$$

(The physical interpretation of U is that it is equal to $\sec\theta$ in figure 1.) Equation (10) implies

$$\frac{\mathrm{d}(U^2)}{\mathrm{d}X} = 2\frac{\mathrm{d}Y}{\mathrm{d}X}\frac{\mathrm{d}^2Y}{\mathrm{d}X^2} \tag{11}$$

and multiplying both sides of equation (11) by $\mathrm{d}X/\mathrm{d}Y$ results in

$$\frac{\mathrm{d}(U^2)}{\mathrm{d}Y} = 2\frac{\mathrm{d}^2Y}{\mathrm{d}X^2} \quad \Rightarrow \quad U\frac{\mathrm{d}U}{\mathrm{d}Y} = \frac{\mathrm{d}^2Y}{\mathrm{d}X^2}. \tag{12}$$

Substituting equations (10) and (12) into (9) leads to the separable equation

$$(1 - Y)\frac{\mathrm{d}U}{\mathrm{d}Y} = \beta + U, \tag{13}$$

whose solution is

$$(1 - Y)(\beta + U) = K, \tag{14}$$

where K is a constant. At the origin, $Y = 0$ and the chain is horizontal so that $U = 1$ according to equation (10), and thus equation (14) implies $K = 1 + \beta$. Substituting that value back into equation (14) solved for U and equating it to the right-hand side of equation (10) leads to

$$1 + \left(\frac{\mathrm{d}Y}{\mathrm{d}X}\right)^2 = \left(\frac{1 + \beta Y}{1 - Y}\right)^2. \tag{15}$$

This equation can be manipulated into the form

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \pm\frac{\sqrt{1+\beta}}{1-Y}\sqrt{2Y - Y^2 + \beta Y^2}, \tag{16}$$

which again can be separated and integrated. Provided $|\beta| < 1$, which corresponds to a strong magnetic field in the sense that $|IB| > \mu g$, one obtains

$$\pm X\sqrt{1+\beta} = \frac{1}{1-\beta}\left[\int \frac{1-Y+\beta Y}{\sqrt{2Y-Y^2+\beta Y^2}}\,\mathrm{d}Y - \beta\int\frac{\mathrm{d}Y}{\sqrt{2Y-Y^2+\beta Y^2}}\right] \tag{17}$$

and the last two integrals can both be done by change of variables: the first by $w \equiv (2Y - Y^2 + \beta Y^2)^{1/2}$ and the second by $Y \equiv (2\sin^2\phi)/(1-\beta)$. The final result is

$$\pm X = \frac{1}{1-\beta}\left[\sqrt{\frac{2Y-Y^2+\beta Y^2}{1+\beta}} - \frac{2\beta}{\sqrt{1-\beta^2}}\sin^{-1}\sqrt{\frac{(1-\beta)Y}{2}}\right], \tag{18a}$$

where the constant of integration is zero because the cable passes through the origin. Although this equation cannot be inverted to find $Y(X)$, it can be plotted in a spreadsheet such as Excel, as presented in section 4.

For any point on the chain with coordinates (X, Y) there is another point on the chain reflected symmetrically across the vertical axis which has opposite signs of X and $\mathrm{d}Y/\mathrm{d}X$, thus explaining the $\pm$ signs in equations (16) and (18).

Next consider the two cases when $|\beta| = 1$. First for $\beta = -1$, equation (16) implies $\mathrm{d}Y/\mathrm{d}X = 0$. The simplest solution is that the wire runs horizontally along the x axis and then vertically up or down to its two endpoints A and C, because the magnetic and gravitational forces are equal and opposite on any horizontal segment of the cable. (However, it is not clear what maintains a nonzero tension T_0 at the origin and patterns such as a 'staircase' of horizontal and vertical segments from A to the origin are also theoretically possible.) Second for $\beta = 1$, equation (16) rearranges into

$$\pm X = \int\frac{\mathrm{d}Y}{2\sqrt{Y}} - \int\frac{\sqrt{Y}}{2}\mathrm{d}Y = \sqrt{Y} - \tfrac{1}{3}Y^{3/2}, \tag{18b}$$

which passes through the origin. Alternatively, one can obtain equation (18b) directly from (18a) by taking the limit as $\beta \to 1$. For this purpose, define $\alpha \equiv 1-\beta$ to rewrite equation (18a) as

$$\pm X = \sqrt{\frac{2}{2-\alpha}}\frac{\sqrt{Y}}{\alpha}\left[\left(1-\frac{\alpha Y}{2}\right)^{1/2} - (1-\alpha)\left(\frac{2}{\alpha Y}\right)^{1/2}\sin^{-1}\left(\frac{\alpha Y}{2}\right)^{1/2}\right]. \tag{19}$$

Now use the first two terms of the binomial series and of the Taylor series for inverse sine to obtain

$$\pm X = \lim_{\alpha\to 0}\sqrt{\frac{2}{2-\alpha}}\cdot\lim_{\alpha\to 0}\frac{\sqrt{Y}}{\alpha}\left[\left(1-\frac{\alpha Y}{4}\right) - (1-\alpha)\left(\frac{2}{\alpha Y}\right)^{1/2}\right.$$
$$\left.\times\left\{\left(\frac{\alpha Y}{2}\right)^{1/2} + \frac{1}{6}\left(\frac{\alpha Y}{2}\right)^{3/2}\right\}\right] \tag{20}$$

in the limit that β approaches 1. Expanding all products inside the square bracket, equation (20) simplifies to (18b).

Finally, for a weak magnetic field such that $|\beta| > 1$, the identity

$$\sin^{-1}(\mathrm{i}w) = \mathrm{i}\ln\left[\sqrt{1+w^2} + w\right] \tag{21}$$

converts equation (18*a*) into

$$\pm X = \frac{1}{1-\beta}\left[\sqrt{\frac{2Y - Y^2 + \beta Y^2}{1+\beta}} - \frac{2\beta}{\sqrt{\beta^2 - 1}}\ln\left(\sqrt{1 + \frac{(\beta-1)Y}{2}} + \sqrt{\frac{(\beta-1)Y}{2}}\right)\right]. \quad (22)$$

When the magnetic field is weak, it makes sense to redefine the scaled coordinates as $\beta X \to X \equiv \mu g x/T_0$ and $\beta Y \to Y \equiv \mu g y/T_0$ so that equation (22) becomes

$$\pm X = \frac{1}{1-\beta}\left[\sqrt{\frac{2\beta Y - Y^2 + \beta Y^2}{1+\beta}} - \frac{2\beta}{\sqrt{1 - \beta^{-2}}}\ln\left(\sqrt{1 + \frac{(\beta-1)Y}{2\beta}} + \sqrt{\frac{(\beta-1)Y}{2\beta}}\right)\right]. \quad (18c)$$

Equations (18*a*), (18*b*), and (18*c*) describe the various possible analytic forms $X(Y)$ of a magnetocatenary.

4. Plotting the solutions

The shapes of the magnetocatenaries only need to be plotted for $X \geqslant 0$ because the curves have a mirror-reflected shape for negative X. There are a variety of cases to consider, depending on the strength and direction of the magnetic field.

4.1. Magnetocatenaries for a forward magnetic field

For a forward field, β, y, and Y are all positive. First consider the limit in which β approaches zero from positive values, which arises when the wire is massless or is in zero (apparent or real) gravity, or more generally whenever the magnetic force is much larger than the gravitational force in magnitude. Equation (18*a*) immediately becomes

$$\pm X = \sqrt{2Y - Y^2} \quad \Rightarrow \quad X^2 + (Y-1)^2 = 1 \quad (23)$$

and thus the cable has the shape of a circular arc centered at (0, 1) with unit radius, as plotted in figure 2(a).

The interpretation of any of the panels in this figure (or in figure 3 below) is that end C of the cable can lie anywhere along the curve, depending on the strength of the field and the length of the wire. That terminated graph can then be reflected across the vertical axis to display the entire shape of the cable.

Figure 2(b) plots the magnetocatenary from equation (18*a*) for $\beta = 1/2$. Compared to figure 2(a), there is now a gravitational force on the cable which causes it to sag downward. When $\beta = 1$, the gravitational and magnetic forces have equal magnitude and both act downward on a horizontal segment. The shape of the curve, given by equation (18*b*) and plotted in figure 2(c), is similar to that of figure 2(b), in agreement with the limiting behavior found from equations (19) and (20). Continuing the trend, figure 2(d) plots equation (18*c*) for the magnetocatenary with $\beta = 2$. By comparing figures 2(a)–(d), one sees that the upper Y intercept increases in value as β gets larger.

Finally, if β gets very large, only the bottom part of the graph is accessible for a wire of realistic length. For example, figure 3(a) plots equation (18*c*) for $\beta = 500$ as the continuous curve. For comparison, the dots plot a standard catenary that passes through the origin with zero slope, given by

$$Y = \cosh(\pm X) - 1. \quad (24)$$

With a bit of algebra, one can show that equation (24) is obtained from equation (18*c*) in the limit that $\beta \to \pm\infty$, as expected when the magnetic field is zero.

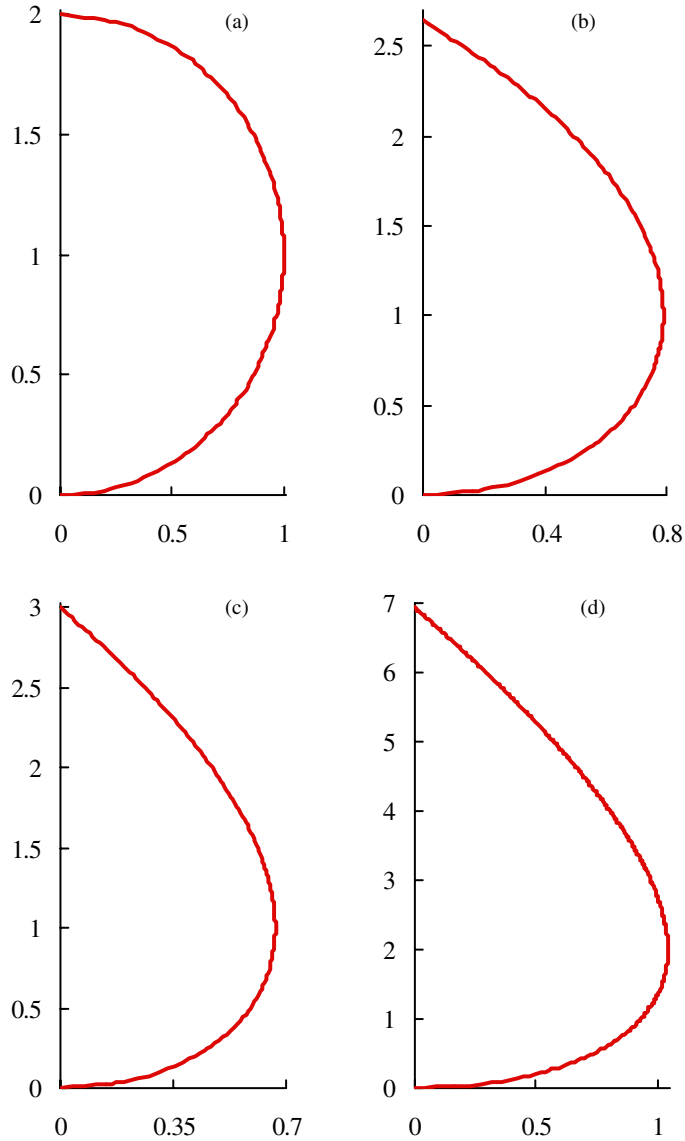

Figure 2. Plot of Y versus X when (a) $\beta = 0^+$, (b) $\beta = 0.5$, (c) $\beta = 1$, and (d) $\beta = 2$.

4.2. Magnetocatenaries for a reverse magnetic field

For a strong reverse field, β and y are negative, and thus it makes sense to reverse the sign of Y so that it is also negative. Starting again with the case of β approaching zero but this time from negative values, the wire has the shape of figure 2(a) upside-down, as plotted in figure 3(b) using equation (23).

Figure 3(c) plots the magnetocatenary for $\beta = -1/2$ from equation (18*a*) with the sign of Y reversed. There is a minimum (largest negative) value of Y such that the two terms in equation (18*a*) remain well defined, $2\,|Y| - Y^2 + \beta Y^2 > 0$ and $(1-\beta)\,|Y|/2 < 1$. Either of these inequalities rearranges into $Y_{\min} = -2/(1-\beta)$, which equals $-4/3$ for the case of

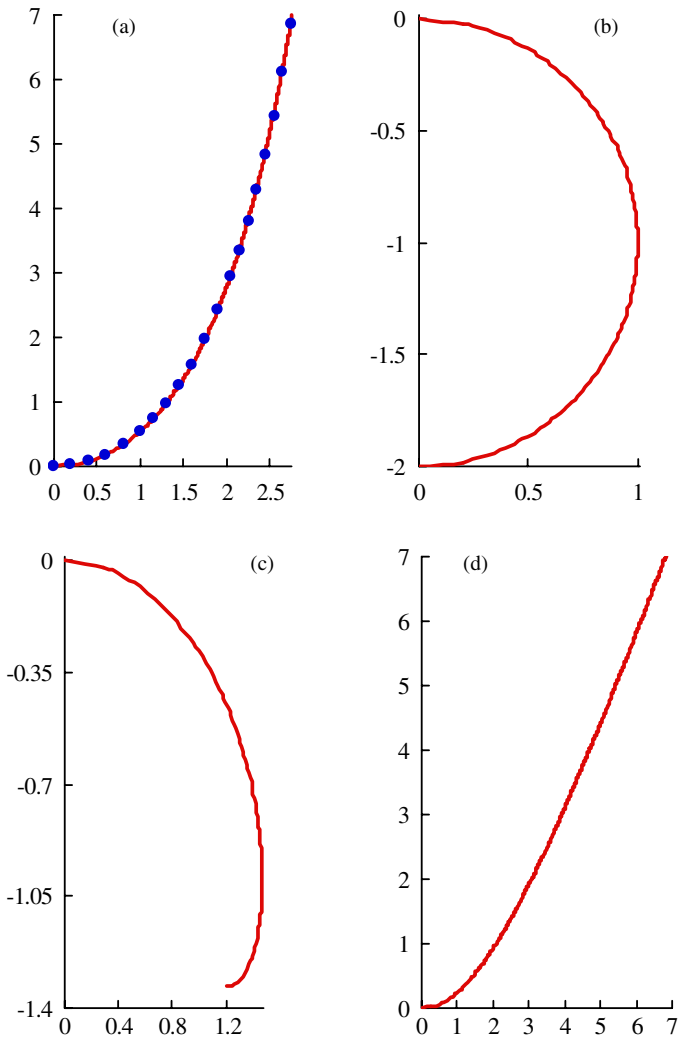

Figure 3. (a) The continuous curve is a plot of Y versus X when $\beta = 500$; the dots plot the limiting catenary given by equation (24). Plot of Y versus X when (b) $\beta = 0^-$, (c) $\beta = -0.5$, and (d) $\beta = -2$.

figure 3(c). Note from equation (16) that $\mathrm{d}Y/\mathrm{d}X = 0$ at this lowest point on the graph. If the wire were longer, its end would hook back up around to the left and cross over the segment plotted in figure 3(c), giving rise eventually to quite an ornate pattern (which might however be experimentally unstable).

As β approaches –1 from larger values (e.g., if equation (18*a*) with the sign of Y reversed is plotted for $\beta = -0.999$), then the graph approximately follows the X axis out past 5 in value before dropping down to $Y_{\min} = -1$. When $\beta < -1$, the upward magnetic force becomes weaker than the downward gravitational force, and we return to using the unreversed sign of Y. For example, if $\beta = -1.001$ the graph follows the X axis out past 5 in value before rising sharply upward. These results are consistent with the discussion in the sentences preceding equation (18*b*).

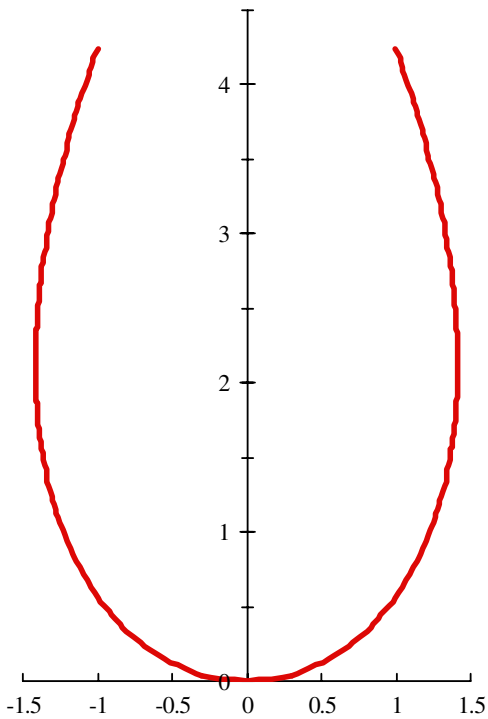

Figure 4. Geometry in horizontal and vertical units of centimeters for a 10-cm-long wire suspended between two endpoints horizontally separated by 2 cm if the fields, current, and wire gauge are chosen such that $IB = \mu g$.

Finally figure 3(d) plots equation (18*c*) for the magnetocatenary with $\beta = -2$. The curve is returning to the gravity-dominated shape of figure 3(a).

5. Closing remarks

Reviewing figures 2 and 3, any of these half magnetocatenaries exhibit a distinctive D shape, albeit sometimes inverted or cut off after some distance from the origin. The simplest case for classroom presentation which has this characteristic shape is $\beta = 1$, plotted in figure 2(c). A cubic equation is obtained by squaring equation (18*b*) and it can be inverted to find $Y(X)$, unlike equations (18*a*) and (18*c*). The shape of the curve can be explained either by beginning from figure 2(a) and turning on gravity, or by beginning from figure 3(a) and turning on a magnetic field. The two fields act in concert near the origin. In contrast, along a vertical segment, gravity pulls the cable down whereas the magnetic force pushes it outward.

These scaled graphs and equations can be converted into real units. Assume that values have been measured for the cable's linear mass density μ, the gravitational field g, the current I, the magnetic field B, the total length L of the wire, and the horizontal positions of the two suspension points $\pm x_C$. Substituting equation (16) into (6) and evaluating it at point C gives one equation in the two unknowns T_0 and y_C. A second equation is obtained by selecting the appropriate version of equation (18) and evaluating it at point C. Those two algebraic equations can then be simultaneously solved (numerically, if not analytically). This procedure is similar to the one used in a standard catenary problem, where T_0 and y_C are also unknown.

As a specific example, consider the case of $\beta = 1$ so that $IB = \mu g$. The simultaneous solution then becomes

$$\frac{T_0}{\mu g} = \frac{(L/2 + x_C)^2}{\sqrt{3L^2 - 12x_C^2}} \quad \text{and} \quad y_C = \frac{\sqrt{3L^2 - 12x_C^2}}{4}. \tag{25}$$

These two expressions have real values only if $L > 2x_C$, as expected because the wire extends horizontally from $-x_C$ to $+x_C$. For instance, suppose a wire of length $L = 10$ cm is suspended between two points A and C that are 2 cm apart, so that $x_C = 1$ cm. Then equation (25) implies that $T_0/\mu g \approx 2.12$ cm and $y_C \approx 4.24$ cm. The latter value implies that the lowest point of the hanging wire will be 4.24 cm below the suspension points A and C. Meanwhile, the ratio of the tension in the wire at the lowest point to the product of the wire's mass density and the gravitational field, $T_0/\mu g = T_0/IB$, implies that we can convert figure 2(c) into length units of centimeters by multiplying the values along both axes by 2.12. Doing so, cutting the curve off after it reaches endpoint C at (1 cm, 4.24 cm), and then mirror reflecting it across the vertical axis to show the entire 10-cm-long wire gives figure 4. A similar method can be used to find the actual geometry for any other values of μ, g, I, B, L, and x_C. In all cases, one ends up with a pair of equations similar to (25) that determine the endpoint coordinates and a scaling factor for the overall size of the curve.

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