

# Energy-based solution for the speed of a ball moving vertically with drag

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## Abstract

Determining the speed of a ball as a function of its height after it is launched straight upward in the absence of fluid resistance is a standard problem treated at all levels of introductory physics. Inclusion of drag in the problem is seldom covered even in the introductory course for majors. But, in fact, this problem can be solved via a straightforward application of the work–energy theorem for either linear or quadratic drag, and instructively plotted on graphs of kinetic versus potential energy.

## 1. Setting up the problem

Choose upward to be the positive  $y$ -axis and the origin to coincide with the launch position of the ball. Let  $v_i$  be the ball's upward launch speed,  $h$  the maximum height that the ball attains before turning around and beginning its descent, and  $v_f$  the ball's speed as it returns to its launch position. The problem consists in finding the speed  $v$  of the ball as a function of its height  $y$  between launch and return.

In the absence of drag, the solution can easily be found by conservation of mechanical energy  $E$ ,

$$K + U = E = \text{constant}, \quad (1)$$

where  $K = \frac{1}{2}mv^2$  is the ball's kinetic energy (assuming the ball has mass  $m$ ) and  $U = mgy$  is the gravitational potential energy (if the reference level is taken at the origin). The value of the constant energy can be conveniently expressed either in terms of the potential energy at the topmost point,  $U_{\text{top}} = mgh$ , or of the launch kinetic energy,  $K_i = \frac{1}{2}mv_i^2$ . Substituting these expressions into equation (1) and rearranging immediately gives  $v(y)$  as desired.

On the other hand, when fluid resistance is present, a dissipative drag force  $\mathbf{F}_D$  acts on the ball during its motion. Let us initially consider the specific case of a baseball moving at ordinary speeds of throwing in air (i.e., not at the turbulent launch speeds of professional baseball pitchers). In that case, as further discussed in section 4 below, the drag force is

quadratic in speed [1] so that it has a magnitude of the form  $F_D = bv^2$ , where  $b$  is a constant of proportionality (that depends on the cross-sectional area of the ball and the density of the air), and a direction opposite to the velocity  $v$  of the ball. If we drop the ball from a large enough height that it reaches terminal speed  $v_T$ , then the gravitational and drag forces acting on the ball must become equal and opposite, so that

$$bv_T^2 = mg \quad \Rightarrow \quad b = \frac{mg}{v_T^2} \quad \Rightarrow \quad F_D = mg \left( \frac{v}{v_T} \right)^2. \quad (2)$$

(In general, buoyancy can be neglected provided the ratio of the fluid density to the average density of the ball is much less than unity. Otherwise we can rigorously account for the buoyant force by reducing the effective gravitational acceleration  $g$  from its vacuum value; specifically, the fractional decrease in  $g$  is equal to this density ratio [2].)

It is convenient to normalize the energies by the kinetic energy the ball would have if it were moving at terminal speed,  $K_T = \frac{1}{2}mv_T^2$ . That is, we introduce the dimensionless kinetic and potential energies,  $\mathcal{K} \equiv K/K_T = (v/v_T)^2$  and  $\mathcal{U} \equiv U/K_T = 2gy/v_T^2$ . In terms of these new variables, equation (1) for the ball's motion in the absence of air resistance becomes

$$\mathcal{U} + \mathcal{K} = \mathcal{U}_{\text{top}}. \quad (3)$$

When subject to drag, the baseball's kinetic energy after launch will be smaller at a given height (or equivalently, a given potential energy) than it would be in the absence of air resistance for the same launch speed, since the drag force is dissipative. We therefore need to replace  $\mathcal{K}$  in this equation by some function of  $\mathcal{K}$  that varies more slowly than linearly. On the other hand, the function must reduce back to  $\mathcal{K}$  in the limit that  $\mathcal{K} \ll 1$  because this implies that  $v \ll v_T$  which in turn means that the drag force is negligibly weak (compared to  $mg$ ) according to equation (2). As is formally proven in the next section, the upward motion of the ball is in fact described by replacing  $\mathcal{K}$  by the simple function  $\ln(1 + \mathcal{K})$ , which by inspection satisfies both of these conditions. During the downward phase of the ball's motion, the drag force reverse direction (since it is always opposite to  $v$ ), and the relevant function is instead  $-\ln(1 - \mathcal{K})$ . (The insertion of two minus signs ensures that this expression still has the correct limiting value  $\mathcal{K}$  for  $\mathcal{K} \ll 1$ .) To summarize, in the presence of quadratic fluid resistance, equation (3) becomes

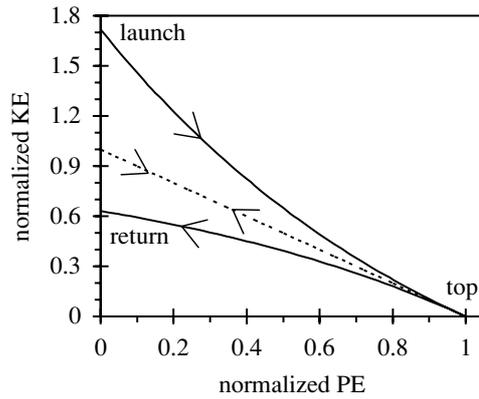
$$\mathcal{U} \pm \ln(1 \pm \mathcal{K}) = \mathcal{U}_{\text{top}}, \quad (4)$$

where the upper (lower) signs refer to upward (downward) motion of the ball. Equations (3) and (4) are compared in figure 1, taking  $\mathcal{U}_{\text{top}} = 1$  in both cases. This latter condition implies that the baseball was launched at terminal speed in the absence of air drag and at  $\sqrt{e-1}v_T$  (i.e., 31% faster than terminal speed, to allow for subsequent losses) in the presence of drag. It returns to the launch position with the terminal speed again, in the absence of drag, but with a speed of only  $\sqrt{1-1/e}v_T$  (i.e., 20% slower than terminal speed) in the presence of air resistance. (See section 3 for a derivation of these numerical values.) Experimental data using video analysis of dropped balls [3] or balloons falling onto a motion sensor [4] are in good agreement with the assumption of quadratic drag underlying equation (4).

## 2. Derivation of equation (4)

During an infinitesimal change  $dy$  in the ball's position, mechanical energy is lost due to the work done by the nonconservative drag force,

$$dK + dU = \mathbf{F}_D \cdot d\mathbf{r}, \quad (5)$$



**Figure 1.** Normalized kinetic versus potential energies for a ball launched straight up and subsequently returning to the origin, assuming it is thrown just fast enough to reach a maximum height of  $v_T^2/2g$ . In the absence of drag, the ball regains its original speed on the way down (dashed line), whereas quadratic drag results in a net loss in kinetic energy (solid curves).

where the ball's displacement is vertical,  $d\mathbf{r} = dy\hat{\mathbf{j}}$ . Noting that the drag force is  $\mathbf{F}_D = \mp mg(v/v_T)^2\hat{\mathbf{j}}$ , where we continue to use the convention that the upper (lower) sign refers to upward (downward) motion of the ball, we can rewrite equation (5) as

$$dK + dU = \mp mg \left( \frac{v}{v_T} \right)^2 dy. \quad (6)$$

Despite the double sign appearing on the right-hand side, that expression is always negative because  $dy > 0$  when the ball is travelling upwards, while  $dy < 0$  when the ball is descending. That is, mechanical energy is being dissipated in both the upward and downward segments of the motion, as expected.

As before, we normalize this energy expression by dividing every term through by  $K_T$  to get

$$d\mathcal{K} + d\mathcal{U} = \mp \mathcal{K} d\mathcal{U}. \quad (7)$$

which rearranges into

$$\frac{d\mathcal{K}}{1 \pm \mathcal{K}} = -d\mathcal{U}. \quad (8)$$

We integrate both sides over any macroscopic increment of motion (in either the upward or downward direction) to obtain

$$\pm \Delta \ln(1 \pm \mathcal{K}) = -\Delta\mathcal{U}, \quad (9)$$

where, as usual, 'delta' ( $\Delta$ ) refers to the change in the value of the quantity that follows it (e.g.,  $\Delta\mathcal{U}$  equals the value of  $\mathcal{U}$  at the end of the increment of motion minus its value at the beginning of the increment). In particular, for the increment between any arbitrary point on the ball's path and the topmost point,

$$\pm [\ln(1 \pm \mathcal{K}_{\text{top}}) - \ln(1 \pm \mathcal{K})] = -[\mathcal{U}_{\text{top}} - \mathcal{U}]. \quad (10)$$

But  $\mathcal{K}_{\text{top}} = 0$  since the ball instantaneously comes to rest at the topmost point and therefore the first logarithm on the left-hand side of this expression drops out. Rearranging what is left now gives equation (4), as we wished to show.

### 3. Applications of equation (4)

Apply equation (4) to the initial launch of the ball upwards to find

$$\ln(1 + \mathcal{K}_i) = \mathcal{U}_{\text{top}} \Rightarrow h = \frac{v_T^2}{2g} \ln \left[ 1 + \left( \frac{v_i}{v_T} \right)^2 \right], \quad (11)$$

which is the maximum height attained by the ball in terms of its launch speed. Next apply equation (4) to the return of the ball downwards to its launch position to obtain

$$-\ln(1 - \mathcal{K}_f) = \mathcal{U}_{\text{top}} \Rightarrow h = \frac{v_T^2}{2g} \ln \left[ \frac{1}{1 - (v_f/v_T)^2} \right]. \quad (12)$$

In contrast, equation (3) in the absence of drag implies that

$$\mathcal{K}_i = \mathcal{U}_{\text{top}} = \mathcal{K}_f \Rightarrow h = \frac{v_i^2}{2g} = \frac{v_f^2}{2g}, \quad (13)$$

which agrees with equations (11) and (12) using the Taylor expansion  $\ln(1+x) \approx x$  for  $|x| \ll 1$  (as is appropriate because the drag force can be ‘turned off’ by taking the limit  $v_T \rightarrow \infty$ ).

Equations (11) and (12) can be inverted to get the normalized launch and return speeds in terms of the maximum height,

$$\frac{v_i}{v_T} = \sqrt{\exp\left(\frac{2gh}{v_T^2}\right) - 1} \quad \text{and} \quad \frac{v_f}{v_T} = \sqrt{1 - \exp\left(-\frac{2gh}{v_T^2}\right)}. \quad (14)$$

Squaring these results and substituting  $h = v_T^2/2g$  gives the y-intercepts of the solid curves in figure 1, consistent with the speeds mentioned at the end of section 1. In general, dividing the second equality in equation (14) by the first and squaring gives the compact expression  $\mathcal{K}_f/\mathcal{K}_i = \exp(-\mathcal{U}_{\text{top}})$ . On the other hand, if one equates the expressions for  $h$  in equations (11) and (12) and rearranges, one finds the memorable result [5]

$$\frac{1}{v_f^2} = \frac{1}{v_i^2} + \frac{1}{v_T^2}, \quad (15)$$

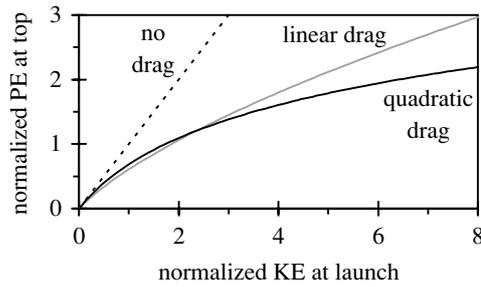
alternatively expressed as  $1/K_f = 1/K_i + 1/K_T$ . Equation (15) tells us that  $v_f$  is always smaller than both  $v_i$  and  $v_T$ , that we recover the drag-free result  $v_f = v_i$  in the limit that  $v_i \ll v_T$ , and that we return to the ground at terminal speed if  $v_i \gg v_T$ . Finally, one can calculate the cumulative amount of mechanical energy lost [6]  $\mathcal{E} \equiv E_{\text{lost}}/K_T$  in normalized form, as

$$\mathcal{E} = \mathcal{K}_i - \mathcal{K} - \mathcal{U} \quad (16)$$

since  $\mathcal{U}_i = 0$ , where the dimensionless kinetic energy  $\mathcal{K}$  can be expressed in terms of the height  $y$  using equation (4). In particular, at launch one finds  $\mathcal{K}_i = \exp(\mathcal{U}_{\text{top}}) - 1$  from equation (11). Note that  $E_{\text{lost}}$  becomes thermal energy of the atmosphere.

### 4. Comparison with linear drag

For a sphere of diameter  $D$  moving at speed  $v$  through a fluid of density  $\rho$  and viscosity  $\eta$ , the Reynolds number is defined as  $R \equiv \rho D v / \eta$ . The drag force is linear in  $v$  (Stokes’ law) when  $R < 1$ , while the drag force is quadratic in  $v$  (Newton’s law) when  $R > 1000$  [2]. (At intermediate values of  $R$ , one gets a mixture of both forms that is best treated numerically. Also, turbulence arises when  $R > 2 \times 10^5$ .) For air at room temperature,  $\rho = 1.21 \text{ kg m}^{-3}$  and  $\eta = 1.83 \times 10^{-5} \text{ kg m s}^{-1}$ , so that a baseball of diameter  $D = 7.4 \text{ cm}$  is clearly in the quadratic regime for speeds  $v$  greater than  $0.2 \text{ m s}^{-1}$  (i.e., the entire trajectory except within



**Figure 2.** Normalized potential energy at the maximum height versus normalized kinetic energy at launch. The curves in the presence of drag fall below the straight line in the absence of drag due to the dissipation of mechanical energy, but asymptotically approach the drag-free line in the limit of small launch speed (compared to the ball's terminal speed).

2 mm of the top turning point). On the other hand, if the ball is a dust grain of  $100 \mu\text{m}$  diameter falling at a speed of  $0.1 \text{ m s}^{-1}$ , then it is in Stokes' regime. In that case, we have a drag force of magnitude  $F_D = b'v$  where  $b' = 3\pi\eta D$ . At terminal speed  $v_T$ , the forces balance so that in analogy to equation (2),  $b'v_T = mg \Rightarrow F_D = mg(v/v_T) = mg\sqrt{\mathcal{K}}$ . Therefore, equation (8) becomes

$$\frac{d\mathcal{K}}{1 \pm \sqrt{\mathcal{K}}} = -d\mathcal{U}. \quad (8')$$

This is easily integrated by making the substitution  $z \equiv 1 \pm \sqrt{\mathcal{K}}$  to obtain

$$2\Delta[\pm\sqrt{\mathcal{K}} - \ln(1 \pm \sqrt{\mathcal{K}})] = -\Delta\mathcal{U}. \quad (9')$$

(For comparison purposes, all equations numbered with a prime for linear drag are analogous to the corresponding unprimed equations for quadratic drag.) Again considering an increment between any arbitrary point on the ball's path and the topmost point, one finds that

$$\mathcal{U} \pm 2\sqrt{\mathcal{K}} - 2\ln(1 \pm \sqrt{\mathcal{K}}) = \mathcal{U}_{\text{top}}. \quad (4')$$

Using the expansion  $\ln(1+x) \approx x - x^2/2$  for  $|x| \ll 1$ , equation (4') correctly reduces to (3) in the absence of drag.

Applying equation (4') to the initial launch of the ball upwards implies

$$2\sqrt{\mathcal{K}_i} - 2\ln(1 + \sqrt{\mathcal{K}_i}) = \mathcal{U}_{\text{top}} \Rightarrow h = \frac{v_T^2}{g} \left[ \frac{v_i}{v_T} - \ln\left(1 + \frac{v_i}{v_T}\right) \right]. \quad (11')$$

In the absence of drag, equation (3) implies that  $\mathcal{K}_i = \mathcal{U}_{\text{top}}$ ; together with the first equalities in equations (11) and (11'), this drag-free result is plotted in figure 2. The linear and quadratic curves track closely with each other up to surprisingly large launch speeds; in fact, the two curves cross when  $v_i = 1.539v_T$ . Evidently, a measurement of the maximum height is not an effective way to distinguish experimentally between linear and quadratic drag unless the ball is launched with a speed of at least  $2v_T$ . For this specific launch speed, equation (11) predicts that the ratio of the maximum height in the presence of quadratic drag to the height attained in the absence of drag is  $\frac{1}{4} \ln 5 = 0.402$ , while equation (11') predicts that the ratio of the maximum height in the presence of linear drag to the height attained in the absence of drag is  $1 - \frac{1}{2} \ln 3 = 0.451$ .

The analogue of equation (12) can next be written down for the downward return of the ball and that resulting expression for  $h$  equated to the second equality in (11') to obtain

$$\frac{v_i + v_f}{v_T} = \ln\left(\frac{v_T + v_i}{v_T - v_f}\right) \Rightarrow \frac{v_f}{v_T} = 1 + W\left[-\frac{1 + v_i/v_T}{\exp(1 + v_i/v_T)}\right], \quad (15')$$

where  $W[x]$  is the Lambert  $W$  function [7]. There are two negative solutions of this function for values of  $v_i/v_T > 0$  and one should take the solution with the smaller absolute value to ensure that  $v_f/v_T > 0$ . (The other solution merely gives the launch velocity  $v_f = -v_i$ .) The appearance of the Lambert  $W$  function nicely complements another recent work in which this function also arises in connection with a projectile subject to linear drag [8].

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### References

- [1] Lock J A 1982 The physics of air resistance *Phys. Teach.* **20** 158–60 (also see the follow-up letter and reply on pp 400–1 of the September 1982 issue)
- [2] Timmerman P and van der Weele J P 1999 On the rise and fall of a ball with linear or quadratic drag *Am. J. Phys.* **67** 538–46
- [3] Greenwood M S, Hanna C and Milton J 1986 Air resistance acting on a sphere: numerical analysis, strobe photographs, and videotapes *Phys. Teach.* **24** 153–9
- [4] Takahashi K and Thompson D 1999 Measuring air resistance in a computerized laboratory *Am. J. Phys.* **67** 709–11
- [5] Barger V and Olsson M 1995 *Classical Mechanics* (New York: McGraw-Hill) p 32 problem 1–11
- [6] Varriano J A 1996 Energy diagrams with drag forces *Phys. Teach.* **34** 546–8
- [7] Hayes B 2005 Why  $W$ ? *Am. Sci.* **93** 104–8
- [8] Warburton R D H and Wang J 2004 Analysis of asymptotic projectile motion with air resistance using the Lambert  $W$  function *Am. J. Phys.* **72** 1404–7 (also see the follow-up letter on p 199 of the March 2005 issue)