

Collision of a ball with a barbell and related impulse problems

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Abstract

The collision of a ball with the end of a barbell illustrates the combined conservation laws of linear and angular momentum. This paper considers the instructive but unfamiliar case where the ball's incident direction of travel makes an acute angle with the barbell's connecting rod. The analysis uses the coefficient of restitution generalized to extended objects (in contrast to the usual point particles of introductory physics). The treatment becomes a springboard to studying impulsive two-dimensional collisions between other rigid objects.

(Some figures in this article are in colour only in the electronic version)

1. Barbell collisional dynamics

Many introductory physics books discuss the problem of a ball striking a barbell (or more generally a stick with some specified mass distribution along it) perpendicularly at one end [1]. However, the results are richer if the ball's initial line of motion instead is at an arbitrary angle to the rod connecting the ends of the barbell.

Consider a ball of mass m and initial velocity \vec{v}_i that strikes a barbell at rest consisting of two spherical masses M (labelled A and B) separated by a massless rigid rod of length $2L$, as sketched in figure 1. Assume the ball strikes sphere A on centre, so that the overall motion of the system is two-dimensional if it is on a frictionless horizontal tabletop or in deep space. For further simplicity, the ball and two spheres will be treated as point particles. Define a coordinate system with the origin fixed in the laboratory at the point of contact between the ball and barbell, and with the x -axis chosen along the ball's initial direction of motion. Let θ be the initial angle between the x -axis and the rod.

Suppose the collision is impulsive, i.e., the ball and barbell attain their final translational and angular velocities (immediately after the collision) before the barbell has had time to significantly rotate or translate. In that case, the impact force applied to the barbell is solely in the $+x$ direction, so that the final velocity \vec{V} of its centre of mass (CM) must be in that direction, as figure 1 indicates. Similarly, the ball receives an impulse in the $-x$ direction,

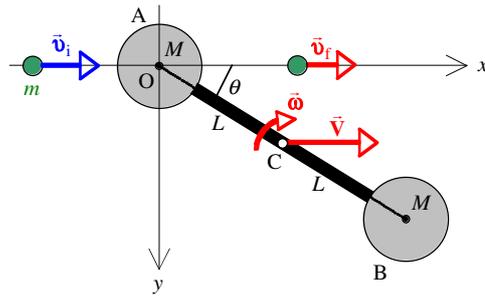


Figure 1. Key parameters describing the collision of a ball with a barbell. The initial velocity of the ball, and the final translational and angular velocities of the two objects are indicated by arrows. The barbell's centre of mass is at its geometric centre C. The spheres A and B at the two ends of the barbell are shown enlarged for clarity. For specificity the ball's final velocity has been drawn in the positive x direction, although it could be zero or negative. The z -axis is into the page.

so that its final velocity \vec{v}_f has no y component but only an x component (which could be positive, zero, or negative depending on the magnitude of the impulse it receives). In general, the barbell will also begin to rotate about its CM with an angular velocity $\vec{\omega}$ (which will be in the $+z$ direction—into the page—for $0^\circ < \theta < 180^\circ$) as a result of the kick given to it by the ball.

Given the ball's initial velocity v_{ix} and the values of the constants m , M , L , and θ , there are three unknowns: v_{fx} , V_x and ω_z . They can be determined as follows. One equation is provided by conservation of linear momentum of the system

$$mv_{ix} = 2MV_x + mv_{fx}. \quad (1)$$

Secondly, the system's angular momentum is conserved. In fact, an even stronger statement can be made by choosing to compute the angular momentum about the z -axis passing through the origin O rather than about the system's CM [2]. In that case, the collisional impulses on the ball and barbell have zero moment arms so that they exert no torques. Therefore the angular momenta of the ball and barbell about O are *separately* conserved. That of the ball is trivially zero both before and after the collision. The barbell is at rest before the collision; hence its final angular momentum must also be zero. This angular momentum can be expressed as the sum of two equal and opposite terms. First the barbell has a 'spin' angular momentum $I\vec{\omega}$ about its centre (labelled point C in figure 1), for which its moment of inertia is $I = 2ML^2$. Secondly it has an 'orbital' angular momentum of $\vec{R} \times \vec{P}$ where $\vec{P} = 2M\vec{V}$ is the total linear momentum of the barbell and \vec{R} is the position of the barbell's CM relative to O. The z component of the barbell's final angular momentum is therefore

$$0 = 2ML^2\omega_z - 2MV_xL \sin \theta \quad \Rightarrow \quad \omega_z L = V_x \sin \theta \quad (2)$$

since \vec{R} has magnitude L and makes angle θ relative to the direction of \vec{V} .

The third equation for the final velocities describes the elastic nature of the interaction between the ball and barbell. The coefficient of restitution e for a collision between any two objects 1 and 2 at a single point of contact O is defined in general as the negative of the ratio of the normal components of the final and initial relative contact velocities

$$e \equiv -\frac{v_{2fx} - v_{1fx}}{v_{2ix} - v_{1ix}}, \quad (3)$$

where the x -axis has been defined to pass through O perpendicularly to the tangent plane of the surfaces of the two objects at that point, assuming that neither object collides with a sharp corner or edge of the other object. (This normal axis is called the 'line of impact' [3].

Note carefully that it is *not* always the direction of the relative initial contact velocity of the bodies.) Equation (3) is a generalization of the familiar one-dimensional definition of e and was originally proposed by Newton [4]. (More formally it is called the *kinematic* coefficient of restitution. Other forms have been proposed, such as an *energetic* coefficient [5] which reduces to the kinematic value for low-speed collisions between smooth, rigid objects.) The value of e ranges from 0 for a perfectly inelastic (or ‘plastic’) collision up to 1 for an elastic one. In the present problem, take object 1 to be the ball and object 2 the barbell. The final velocity of the barbell at the point of contact O is equal to the velocity of sphere A, $\vec{v}_{2f} = \vec{V} + \vec{\omega} \times (-\vec{R})$. Consequently equation (3) becomes

$$V_x + \omega_z L \sin \theta - v_{fx} = e v_{ix}. \quad (4)$$

The simultaneous solution of equations (1), (2) and (4) is

$$v_{fx} = \frac{m - 2Mf}{m} v_{ix}, \quad V_x = f v_{ix} \quad \text{and} \quad \omega_z = \frac{f \sin \theta}{L} v_{ix} \quad (5)$$

where $f \equiv (e + 1)m/[2M + m(1 + \sin^2 \theta)]$ is a dimensionless ratio of constants. These results reduce to the expected forms for a one-dimensional head-on collision when $\theta = 0$, in which case $\omega_z = 0$. For example, $v_{fx} = V_x = m v_{ix}/(2M + m)$ in the perfectly inelastic case $e = 0$, whereas v_{fx} and V_x reproduce the standard textbook expressions [6] for a 1D elastic collision of a mass m with another mass $2M$ initially at rest when $e = 1$.

2. Implications of equation (5)

The total (translational plus rotational) kinetic energy lost in the collision is

$$\Delta K_{\text{lost}} = K_i - K_f = \frac{1}{2} m v_{ix}^2 - \left[\frac{1}{2} m v_{fx}^2 + \frac{1}{2} (2M) V_x^2 + \frac{1}{2} (2ML^2) \omega_z^2 \right]. \quad (6)$$

Substituting equation (5), the fractional kinetic energy lost becomes

$$\frac{\Delta K_{\text{lost}}}{K_i} = \frac{1 - e^2}{1 + \frac{m}{2M} (1 + \sin^2 \theta)}. \quad (7)$$

As expected, the loss in mechanical energy increases as the coefficient of restitution decreases. Specifically, no kinetic energy is lost in an elastic collision, and a maximum amount of kinetic energy is lost in a perfectly inelastic collision. On the other hand, for any fixed value of the coefficient of restitution $e < 1$, the dissipation of kinetic energy is minimized when $\theta = 90^\circ$. (Define the impact parameter $b = L \sin \theta$ as the distance between point C and the x -axis in figure 1. Then note that the amount of mechanical energy dissipated decreases with increasing b/L , i.e., as the collision becomes more off-centre. A similar conclusion has recently been reached by Newburgh and Peidle for the perpendicular collision of a ball and stick [7].) The main reason the final value of the total kinetic energy increases is that the final rotational energy increases with increasing angle θ . That is, by plotting ω_z as a function of θ for any given values of e and M/m , one finds that the angular speed increases with the angle of impact up to 90° .

Notice that for $0^\circ < \theta < 90^\circ$, sphere A has a velocity component in the $-y$ direction immediately after the collision (cf figure 1) as a result of the rotation of the barbell. This motion seems counter-intuitive because the ball only exerts a horizontal impulse on the sphere. But consider replacing the bar connecting the spheres with a spring. Then sphere A will at first move only in the $+x$ direction. That compresses the spring, resulting in an upward force on sphere A (and a downward force on sphere B) relative to the centre point C. Now if the spring is made very stiff, it becomes an essentially rigid rod exerting a compressive reaction force along its length on the two spheres during the collision. By similar reasoning, *any*

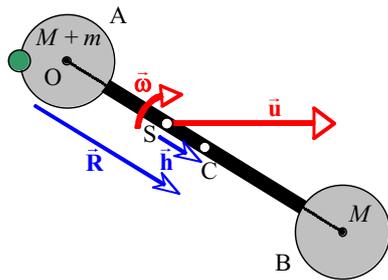


Figure 2. Final translational velocity of and angular velocity about the system's centre of mass S if the ball sticks to sphere A of the barbell. The position of the geometric centre C of the barbell is labelled relative either to the origin O of the coordinate system or to the position of the centre of mass S of the combined system of ball and barbell at the instant of impact.

object will experience a change in the tangential component of its contact velocity during a collision, unless a line drawn from the object's CM to the point of contact is either parallel or perpendicular to the direction of the external impulse applied to the object. (One concludes that the so-called tangential coefficient of restitution [8] will in general depend on the angles of impact.) Surprisingly, this italicized statement is not found in engineering dynamics textbooks. They apparently do not analyse eccentric collisions (i.e., ones for which the line of impact does not pass through both centres of mass) of a particle with an unconstrained thick bar (or box), even for the simplest case of direct impact (for which the relative contact velocity is parallel to the line of impact). Only bars (or plates) which are thin and/or hinged are typically discussed.

Unlike in the one-dimensional case, a perfectly inelastic two-dimensional collision does *not* in general imply that the objects stick together afterward. All equation (3) says is that the ball and sphere A have the same x component of velocity *immediately* after the collision. But because of the barbell's rotation, its spheres have x and y velocity components that vary in time. Therefore, sphere A will not remain in contact with the ball (unless θ is equal to 0° or 180°); a tangential 'sticking' force is required if the ball is to follow the barbell's rotation. Incidentally, if $e = 0$ one can show that some portion of the barbell will later strike the ball from behind for any initial angle θ of the barbell between 0° and a critical value $\theta_c \approx 19.7^\circ$. This critical angle is the solution of the equation $\tan \theta_c = 1/(\pi - \theta_c)$ in radians (easily solved graphically), and corresponds to sphere B striking the ball after the barbell rotates through an angle of $2(\pi - \theta_c)$, just short of one revolution. For larger initial angles (i.e., $\theta_c < \theta < \pi$) the ball will outrace the extremities of the barbell; for example, if $\theta = 90^\circ$ then $v_{fx} = 2V_x$ from equation (5) when $e = 0$.

It is an instructive exercise to determine the final velocities if the ball does stick (e.g., suppose it is made out of clay) as in figure 2. The CM of the barbell-ball combination after sticking is located a distance $h = mL/(2M + m)$ along the bar toward sphere A relative to the geometric centre C of the barbell. To avoid confusion, let's continue to denote the final velocities of the ball and of point C on the barbell by \vec{v}_f and \vec{V} respectively (recognizing that neither are in the x direction in general any longer) and call \vec{u} the (constant) final velocity of the CM (located at point S) of the barbell-ball system. Conservation of linear momentum implies that

$$mv_{fx} = (2M + m)u_x. \quad (8)$$

On the other hand, the z component of the final angular momentum of the *system* about point O equals that of the barbell *alone* and is determined by the sum of its spin about C and

the orbit of point C

$$0 = 2ML^2\omega_z - 2M[u_x L \sin\theta - \omega_z h L] \Rightarrow \omega_z(L+h) = u_x \sin\theta. \quad (9)$$

(The term in square brackets comes from writing \vec{V} as the sum of \vec{u} and $\vec{\omega} \times \vec{h}$.) Many textbooks [9] work out this kind of problem by computing the spin angular momentum of the system using the parallel-axis theorem to calculate the moment of inertia of the barbell about point S and adding to it the final rotational inertia of the ball about that point, and then equating the result to the initial orbital angular momentum of the ball about point S; the reader may wish to verify that such an approach reproduces equation (9) but with considerably more algebra.

Equations (8) and (9) give rise to

$$u_x = \frac{m}{2M+m} v_{ix} \quad \text{and} \quad \omega_z = \frac{m \sin\theta}{2(M+m)L} v_{ix}. \quad (10)$$

An alternative way to get this solution is less direct but enlightening. Equations (1) and (4) remain valid, provided $e = 0$ is substituted into the latter. Equation (2), however, requires a small modification: the z component of $\vec{R} \times \vec{V}$ is no longer simply $-V_x L \sin\theta$ but is instead $-V_x L \sin\theta + V_y L \cos\theta$ because the geometric centre of the barbell is now rotating about point S in figure 2 with a y component of its velocity of $V_y = \omega_z h \cos\theta$. Simultaneous solution of these three equations leads to the motion described by equation (10). Note that for the important special case of $\theta = 90^\circ$, this alternative approach implies that one gets the same values for v_{fx} , V_x and ω_z immediately after a perfectly inelastic collision *whether or not the ball sticks to the barbell*. That makes sense because no tangential sticking force is initially required at this angle.

3. Impact between two extended objects

The basic theory of impulsive planar collisions between two rigid bodies is now straightforward to describe. Consider two objects (with masses m_1 and m_2) that momentarily impact each other at a single point O. Define the normal direction to their surfaces in contact to be the x -axis and the tangent direction as the y -axis of a coordinate system with origin at O. Suppose the two bodies are smooth, meaning that there is no tangential ('frictional') component of the contact force. The velocity of either object at its point of contact is labelled \vec{v} (with appropriate subscripts). In addition, label the velocity of the centre of mass of either object as \vec{V} , and the angular velocity of either body about its CM as $\vec{\omega}$. The translational velocities each have an x and y component, but only the z components of the angular velocities are assumed to be relevant to the collisional dynamics. Since the bodies are rigid, there is a constraining relation between these velocities for each body

$$\vec{v} = \vec{V} - \vec{\omega} \times \vec{R} \quad (11)$$

both initially and finally, where \vec{R} is the position of the CM relative to the impact point on each object. Equation (11) specifies the initial values of \vec{v}_1 and \vec{v}_2 , plus four expressions for v_{1fx} , v_{1fy} , v_{2fx} and v_{2fy} .

Suppose that \vec{R}_1 , \vec{R}_2 , and the initial values of \vec{V}_1 , \vec{V}_2 , $\vec{\omega}_1$ and $\vec{\omega}_2$ are known (along with the masses and moments of inertia), and that the final values of these latter four velocities (corresponding to six components) are to be determined. Equation (3) gives one equation. Another equation expresses conservation of the x component of linear momentum

$$m_1 V_{1ix} + m_2 V_{2ix} = m_1 V_{1fx} + m_2 V_{2fx}. \quad (12)$$

Two more equations come from the fact that the y component of the linear momentum of each object cannot change

$$V_{1iy} = V_{1fy} \quad \text{and} \quad V_{2iy} = V_{2fy} \quad (13)$$

because the impulses are purely in the x direction. The final pair of equations is conservation of the z component of the angular momentum of each object about the origin O, expressed as the sum of the spin–angular momentum of each object about its CM and the orbital–angular momentum of the CM of each object about O

$$I_1 \vec{\omega}_{1i} + m_1 \vec{\mathbf{R}}_1 \times \vec{\mathbf{V}}_{1i} = I_1 \vec{\omega}_{1f} + m_1 \vec{\mathbf{R}}_1 \times \vec{\mathbf{V}}_{1f} \quad (14)$$

and likewise for body 2. If the impact is central (i.e., $\vec{\mathbf{R}}$ for each object only has an x component), then equations (13) and (14) together imply that

$$\omega_{1iz} = \omega_{1fz} \quad \text{and} \quad \omega_{2iz} = \omega_{2fz} \quad (15)$$

since there is no mechanism to alter these spins.

As a simple application of these equations, consider a collision between two smooth, equal-mass spheres or discs. (The more general problem of rough, unequal-mass discs has also been investigated theoretically and experimentally [10].) The impact is necessarily central, so if neither sphere is initially rotating then neither will be afterward, and equation (11) then implies that $\vec{v} = \vec{\mathbf{V}}$ for each sphere both initially and finally. In that case, equations (12) and (3) can be simultaneously solved to obtain

$$V_{1fx} = \frac{(1 - e)V_{1ix} + (1 + e)V_{2ix}}{2} \quad (16)$$

and similarly for V_{2fx} by interchanging subscripts 1 and 2 in this expression. This leads to some familiar results. For example, if sphere 2 is initially at rest and the collision is elastic then $V_{1fx} = 0$ and $V_{2fx} = V_{1ix}$. If the collision is on centre ('direct') that means the two spheres exchange their velocities; on the other hand, if the impact is oblique (i.e., the impact parameter $b = (R_1 + R_2) \sin \theta$ is nonzero, where θ is the angle between \vec{v}_{1i} and the line of impact) then $\vec{\mathbf{V}}_{1f}$ is in the y direction and $\vec{\mathbf{V}}_{2f}$ is in the x direction, perpendicular to each other [11].

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