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Internal ballistics of a pneumatic potato cannon

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Abstract
Basic laws of thermodynamics and mechanics are used to analyse an air gun. Such devices are often employed in outdoor physics demonstrations to launch potatoes using compressed gas that is here assumed to expand reversibly and adiabatically. Reasonable agreement is found with reported muzzle speeds for such homebuilt cannons. The treatment is accessible to undergraduate students who have taken calculus-based introductory physics.

Potato cannons are a popular construction project for physics demonstrations and science fairs [1]. They can be powered in three different ways: by a compressed gas (which is the type analysed in the present paper), by an explosive propellant [2] or by a sudden vacuum breaking [3, 4]. In the first, pneumatic case, the projectile can be modelled as a piston accelerating down the bore of a horizontal cylinder under the action of pressurized gas, taken to be ideal for simplicity. Three equations are used for the analysis: Newton's second law, the ideal gas law and the first law of thermodynamics. Additional simplifying approximations are that the piston slides frictionlessly and no gas leaks around its edges (as should be appropriate for a thick, wet potato slice that is forcibly fit to the bore); atmospheric pressure is negligible compared to the gas pressure (while the piston is in the bore); the gas expansion occurs quasistatically (which will be valid provided the piston's speed is small compared to the speed of sound) and adiabatically (i.e., without heat leakage to the surroundings) [5]. Some of these assumptions can be lifted by performing a numerical rather than an analytic analysis [6], and such extensions could become attractive classroom projects with different possible levels of sophistication.

The ideal gas law for n moles of gas at pressure p, volume V and temperature T is

$$pV = nRT \tag{1}$$

where $R = 8.314$ J mol^{-1} K^{-1} is the universal gas constant. The force on a piston of cross-sectional area A is $F = pA$, as sketched in figure 1. Solving for p and substituting it into the left-hand side of equation (1) along with the volume of gas, $V = Ax$ where x is the distance from the sealed end of the cylinder to the piston, leads to

$$max = nRT, \tag{2}$$

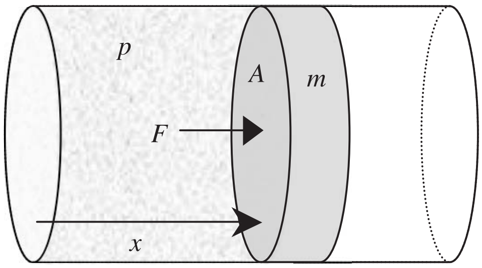

Figure 1. Sketch of the projectile moving down the bore of the cannon.

after replacing F by ma where m is the mass and a is the acceleration of the piston. The initial conditions are that the piston is held fixed in place (say by a pin) at position x_0 (so that its initial speed is $\upsilon_0 = 0$) with the gas at temperature T_0. Therefore the initial acceleration of the piston after the pin is pulled out is

$$a_0 = \frac{nRT_0}{mx_0} \tag{3}$$

from equation (2).

Now according to the first law of thermodynamics, in the absence of heat exchange with the surroundings, the internal energy U of the gas decreases as it does work on the piston. Noting that $U = ncT$ for an ideal gas, where c is the molar specific heat at constant volume (equal for example to $3R/2$ for a monatomic and to $5R/2$ for a diatomic gas near room temperature), and that the infinitesimal work done on the piston is $F\,\mathrm{d}x = p\,\mathrm{d}V = (nRT/x)\,\mathrm{d}x$, the balance between the rates of energy lost and work done implies

$$nc\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{nRT}{x}\upsilon. \tag{4}$$

Replacing nRT by max from equation (2) on both sides of equation (4) gives

$$\frac{c}{R}\frac{\mathrm{d}}{\mathrm{d}t}(ax) = -a\frac{\mathrm{d}x}{\mathrm{d}t}. \tag{5}$$

This equation can be separated and integrated to get

$$a = kx^{-(R/c+1)}. \tag{6}$$

Fit the constant of integration k to equation (3), so that

$$k = \frac{nRT_0}{m}x_0^{R/c}. \tag{7}$$

Next, writing $a = \upsilon\,\mathrm{d}\upsilon/\mathrm{d}x$, one can again separate and integrate equation (6) to find

$$\upsilon^2 = \frac{2ncT_0}{m}\left[1 - \left(\frac{x_0}{x}\right)^{R/c}\right] \tag{8}$$

using the initial condition $\upsilon_0 = 0$. Take the positive square root because the piston always moves away from the sealed end of the cylinder shown in figure 1 and separate yet again to obtain the integral result,

$$\int_{x_0}^{x} \frac{\mathrm{d}x}{\sqrt{1 - (x_0/x)^{R/c}}} = \sqrt{\frac{2ncT_0}{m}}\,t. \tag{9}$$

Make the change of variable $x = x_0 \csc^{2c/R}\theta$, where $\csc\theta \equiv 1/\sin\theta$ is the cosecant function, to obtain

$$\sqrt{\frac{2ncT_0}{m}}\, t = \frac{2c\,x_0}{R}\int_\phi^{\pi/2} \csc^{2c/R+1}\theta\,\mathrm{d}\theta \qquad \text{where} \quad \sin\phi \equiv (x_0/x)^{R/2c}. \tag{10}$$

This result can be rewritten in terms of dimensionless position and time variables,

$$\tilde{x} \equiv \frac{x}{x_0} \qquad \text{and} \qquad \tilde{t} \equiv \sqrt{\frac{2ncT_0}{mx_0^2}}\, t, \tag{11}$$

and in terms of the integer $N \equiv 2c/R$ (equal to the number of degrees of freedom per gas molecule) as

$$\tilde{t} = N\int_\phi^{\pi/2} \csc^{N+1}\theta\,\mathrm{d}\theta \qquad \text{where} \quad \phi \equiv \csc^{-1}\tilde{x}^{1/N}. \tag{12}$$

A recursive analytic solution for the indefinite integral exists. But rather than treating the general situation, consider two cases of practical interest: a monatomic and a diatomic gas. For the monatomic case, the required integral is

$$\int \csc^4\theta\,\mathrm{d}\theta = \frac{-\cot\theta}{3}(2+\csc^2\theta). \tag{13}$$

Substituting the limits of integration, equation (12) remarkably simplifies to

$$\tilde{t} = \sqrt{\tilde{x}^2 + 3\tilde{x}^{4/3} - 4}, \tag{14}$$

which is a solution for the time it takes the projectile to move any given distance down the bore. Recognizing the argument of the square root as a cubic polynomial in $\tilde{x}^{2/3}$, equation (14) can be inverted to obtain

$$\tilde{x} = \left[\left(1+\tfrac{1}{2}\tilde{t}^2 + \sqrt{\tilde{t}^2+\tfrac{1}{4}\tilde{t}^4}\right)^{1/3} + \left(1+\tfrac{1}{2}\tilde{t}^2 - \sqrt{\tilde{t}^2+\tfrac{1}{4}\tilde{t}^4}\right)^{1/3} - 1\right]^{3/2}. \tag{15}$$

In the diatomic case, the indefinite integral in equation (12) is

$$\int \csc^6\theta\,\mathrm{d}\theta = \frac{-\cot\theta}{15}(8+4\csc^2\theta+3\csc^4\theta) \tag{16}$$

so that

$$\tilde{t} = \tfrac{1}{3}\sqrt{9\tilde{x}^2 + 15\tilde{x}^{8/5} + 40\tilde{x}^{6/5} - 64} \tag{17}$$

and the radicand is a quintic polynomial in $\tilde{x}^{2/5}$. Equation (17) cannot be analytically inverted to find the position as a function of time. However, time increases monotonically with distance, as expected intuitively, and hence one can always find the inverse graphically (cf figure 2).

By separating and integrating equation (4), the temperature of the propellant gas can be written in dimensionless form as

$$\tilde{T} \equiv \frac{T}{T_0} = \tilde{x}^{-R/c} \tag{18}$$

and therefore since $px \propto T$ from equation (1), the dimensionless pressure is

$$\tilde{p} \equiv \frac{p}{p_0} = \tilde{x}^{-(R/c+1)}, \tag{19}$$

which can alternatively be written in the familiar adiabatic form pV^γ = constant, where $\gamma = (c+R)/c$ is the ratio of the specific heats at constant pressure and at constant volume. Note that $\tilde{p} = a/a_0$ since $ma = pA$ and hence the pressure is directly proportional to the acceleration of the projectile. Equation (19) is a useful conversion formula between pressure

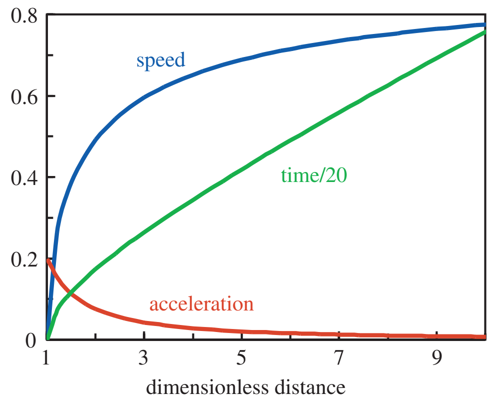

Figure 2. Dimensionless speed, acceleration and time as a function of distance for a projectile propelled along a cylinder by a diatomic gas. The time has been divided by 20 to keep it on scale. (This figure is in colour only in the electronic version)

and distance (and thus speed according to the next equation) since in practice pneumatic cannons are pressurized to some desired (and hopefully safe) initial value p_0. For example, suppose the initial pressure is $p_0 = 10$ atm (i.e., 1 MPa). Then equation (19) implies that the lowest useful pressure of $p = 1$ atm is attained when $\tilde{x} = 5.2$ for a diatomic gas such as dry air. But equation (8) can be rewritten as

$$\upsilon = \upsilon_\infty \sqrt{1 - \tilde{x}^{-R/c}} \tag{20}$$

where the limiting muzzle speed is

$$\upsilon_\infty = \sqrt{\frac{2ncT_0}{m}}. \tag{21}$$

Substituting say $n = 0.1$ mol, $c = 5R/2$, $T_0 = 300$ K, $m = 100$ g and $\tilde{x} = 5.2$, these two equations imply $\upsilon = 78$ m s^{-1}, comparable to reported values for homemade air cannons [7]. Ignoring air drag, that muzzle speed gives an optimal range of υ^2/g which is a little over 600 m.

Defining $\tilde{\upsilon} \equiv \mathrm{d}\tilde{x}/\mathrm{d}\tilde{t} = \upsilon/\upsilon_\infty$ and $\tilde{a} = \mathrm{d}^2\tilde{x}/\mathrm{d}\tilde{t}^2 = a/Na_0$, the preceding results correctly imply at $\tilde{t} = 0$ that $\tilde{x}_0 = 1$, $\tilde{\upsilon}_0 = 0$ and $\tilde{a}_0 = R/2c$. On the other hand, as $\tilde{t} \to \infty$ one obtains $\tilde{x}_\infty \to \tilde{t}$, $\tilde{\upsilon}_\infty = 1$ and $\tilde{a}_\infty = 0$. These initial and final values of the position, speed and acceleration give physical meaning to the scaling constants in equation (11). The dimensionless time, speed and acceleration are plotted versus distance along the bore in figure 2 for the diatomic case.

The key result is that the speed of the projectile levels off after the volume of the gas has expanded to a few times its initial, compressed value and so there is no point in making the bore longer than that. (In fact, a longer bore would be disadvantageous because of the effects of friction and atmospheric pressure.) To enhance the muzzle speed, one should maximize the product of the initial pressure and volume of the propellant gas and decrease the projectile mass, according to equation (21).

To compare the theoretical results obtained in this paper to experiment, one would need to make measurements while the projectile is internally travelling down the bore of the cannon. One possibility is to measure the temperature or pressure of the propellant gas for comparison to equations (18) and (19). That would require a fast sensor because the potato leaves the barrel in less than 50 ms after firing [2]. A more direct technique consists in measuring the speed of the projectile at several points along the bore for comparison to equation (20). The simplest way to do that would be to use photogates mounted along the length of a transparent cannon, such as one made out of an acrylic tube [4]. Additional useful experimental measurements include estimating the frictional loss between a potato and the bore, and the leakage of compressed gas around the potato. On the theoretical side, one could account for the acceleration of the air column behind the projectile [3] or determine by how much the gas pressure (and hence piston speed) increases if the expansion occurs not adiabatically but isothermally [8] (as might be appropriate if the cannon is made out of metal with a large thermal conductivity).

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