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LETTERS AND COMMENTS

Comment on 'The two-capacitor problem revisited: a mechanical harmonic oscillator approach'

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Abstract
An expression is developed for the energy dissipated when a constant external force is suddenly applied to the end of a particle moving in 1D subject to a conservative restoring force and a general damping force. Assuming the particle was initially at rest and ends up in static equilibrium, the fraction of the mechanical energy lost depends only on the conservative force and on the net displacement of the particle. Mechanical models are suggested to illustrate the ideas and their capacitor circuit analogies. The treatment is appropriate for an intermediate-level undergraduate mechanics course.

(Some figures in this article are in colour only in the electronic version)

In a recent paper, Lee discusses a constant force suddenly applied to the end of a Hookean spring in the presence of a damping force that can depend on both velocity and acceleration as a mechanical analogue to a battery suddenly connected to a capacitor and returning to static equilibrium dissipatively [1]. The present comment extends Lee's results in three ways:

(1) The form of the damping force is broadened so that it can depend on any variables, not just on velocity and acceleration.
(2) The interaction between the two mechanical systems is generalized to be any conservative force instead of merely Hookean springs.
(3) Mechanical models are developed for the problem that motivated Lee's study, namely that of suddenly connecting together a charged and an uncharged capacitor.

Consider a particle moving in one dimension (call it x) under the action of three forces in general: a conservative restoring force $F_c(x)$, a damping force F_d and a step-wise constant force $F_0 > 0$ which is suddenly turned on for all $t > 0$. The damping force can be of any form whatsoever provided that it has two properties: its direction is always opposite to the direction of motion of the particle, and its magnitude falls to zero when and only when the velocity of the particle is zero. This resistive force need not be viscous drag or radiation damping (proportional to velocity and to acceleration, respectively). It could for example be velocity-independent kinetic friction (allowing a slight vertical agitation of the particle to

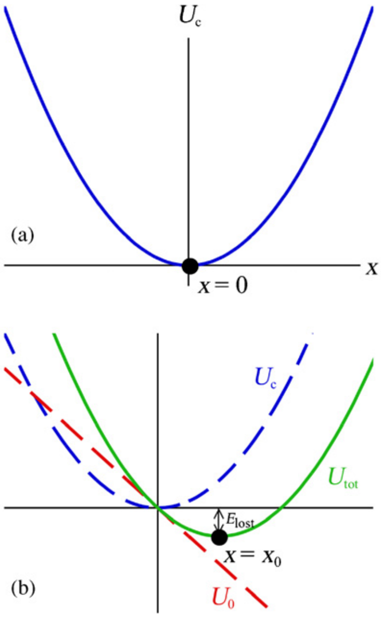

Figure 1. Graphs of the potential energy (a) before and (b) after turning on the force F_0. The equilibrium positions of the particle are indicated by the black dots.

prevent a nonzero static frictional force from arising).[1] Now suppose that at $t = 0$ the particle is at rest at position $x = 0$. That is, F_0 has not yet been switched on, $F_d = 0$ because the particle is at rest, and $F_c(0) = 0$ because the particle is in stable equilibrium. Then the particle must lie at the bottom of a minimum in the potential energy $U_c(x)$ associated with the conservative force F_c, as sketched in figure 1(a). Without loss of generality, the reference position for this potential energy can be located at $x = 0$ so that

$$U_c(x) = -\int_0^x F_c(x)\,\mathrm{d}x = -W_c(x), \tag{1}$$

where $W(x)$ denotes work done as the particle moves from the origin to position x, with a subscript (c, d or 0) on W indicating which force is doing the work.

Now turn on the force F_0. It will push the particle towards positive x. Under the combined action of the restoring force and this constant push, the particle will oscillate about some new equilibrium position $x_0 > 0$, but the amplitude of these oscillations will decay due to the damping force and eventually the particle will come to rest at x_0. It is instructive to consider the situation from an energy point of view. The potential energy associated with the constant force (adopting the same reference position as for U_c) is linear in position,

$$U_0(x) = -\int_0^x F_0\,\mathrm{d}x = -F_0 x = -W_0(x). \tag{2}$$

The total potential energy of the system is now $U_{\mathrm{tot}} \equiv U_c(x) + U_0(x)$ as sketched in figure 1(b). Initially the particle was at $x = 0$ with kinetic energy $K_i = 0$; in its final equilibrium position

[1] The standard introductory model for friction is that it has magnitude $F_d \leqslant \mu N$ where μ is the coefficient of friction (taken to be the same constant for static and kinetic friction, for simplicity) and N is the normal force between the particle and the surface with which it is in contact. The inequality becomes an equality whenever the particle is in motion. On the other hand, F_d falls to zero when the particle is at rest, assuming that all other forces acting on the particle sum to zero.

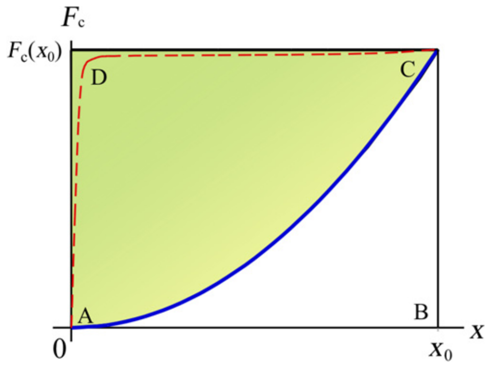

Figure 2. The solid (blue) curve graphs a typical conservative force against position, in which the upper shaded area is proportional to the mechanical energy lost. The dashed (red) curve plots a force for which the energy dissipation is minimal.

at x_0 it is again at rest so that $K_f = 0$. Consequently the system has lost the mechanical energy

$$E_{\rm lost} = U_i - U_f = 0 - U_{\rm tot}(x_0) = F_0 x_0 + \int_0^{x_0} F_c(x)\,\mathrm{d}x, \qquad (3)$$

which is equal to $-W_d(x_0)$, in agreement with the work-kinetic-energy theorem for the particle

$$W_d(x_0) + W_c(x_0) + W_0(x_0) = K_f - K_i = 0. \qquad (4)$$

The energy loss $E_{\rm lost}$ equals the vertical distance from the x-axis down to the minimum in the total potential energy in figure 1(b). Note that F_0 does net positive work on the particle as it moves from its initial to its final position; viewed as an external force, one can say it transfers energy $E_0 = F_0 x_0$ to the system. However, the conservative force field only gained an energy of $U_c(x_0)$ which in general is less than E_0. The fractional loss in mechanical energy is $f_{\rm lost} \equiv E_{\rm lost}/E_0$.

This fractional energy loss can be rewritten in terms of F_c only. When the particle is at rest in its new equilibrium position, $F_d = 0$ and therefore $F_c(x_0) = -F_0$. Equation (3) thus implies

$$f_{\rm lost} = 1 - \frac{\langle F_c\rangle}{F_c(x_0)} \qquad \text{where} \qquad \langle F_c\rangle \equiv \frac{1}{x_0}\int_0^{x_0} F_c(x)\,\mathrm{d}x. \qquad (5)$$

For example, if the conservative force is due to a Hookean spring of stiffness constant k, then $F_c(x) = -kx$ and equation (5) reproduces Lee's result that $f_{\rm lost} = 50\%$. The electrical analogue is that if a battery of emf ξ is suddenly connected to a discharged capacitor of capacitance C, then when it is fully charged up to $Q = C\xi$ the electrical energy $\frac{1}{2}Q\xi$ gained by the capacitor is only half of the electrical energy $Q\xi$ lost by the battery. The rest was dissipated as Joule heating and radiation. But the important point is that no knowledge of the exact dissipation mechanism is necessary to reach this conclusion; equation (5) predicts a 50% loss for the spring system regardless of the form of the damping force (provided only that its direction is always opposite to the direction of motion of the particle, and that its magnitude falls to zero when and only when the velocity of the particle is zero).

Equation (5) is general and also applies to non-Hookean forces, as long as they are restoring and conservative. As an example, for the anharmonic spring $F_c(x) = -kx^3$, it predicts that 75% of the mechanical energy will be lost. Equation (5) has a simple graphical interpretation: $f_{\rm lost}$ is the ratio of the shaded area (ACD) in figure 2 to the area of the entire

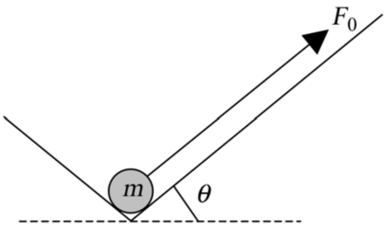

Figure 3. A ball of mass m slowly pulled up a frictionless incline θ by a constant force F_0 starting from an equilibrium position at the bottom.

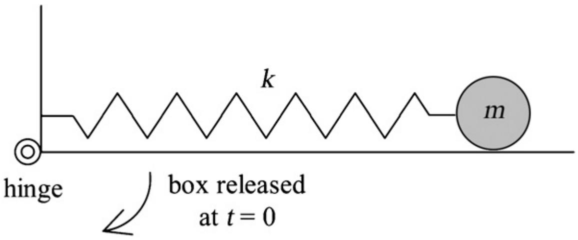

Figure 4. A ball attached to an initially relaxed spring inside a box. At $t = 0$ the box is suddenly rotated clockwise by 90° about its hinge, so that the ball's weight stretches the spring.

rectangle (ABCD). With this in mind, a force F_c needs to have the following functional form to result in minimal energy loss. Given that it has to start at point A, end at point C, and be monotonically increasing in between so that it restores everywhere (not just for some special choice of x_0), it has to hug sides AD and DC like the dashed curve in figure 2. Noting that the force is then essentially constant over most of the particle's trajectory, a simple way to realize such a system in practice is sketched in figure 3. A ball lies at the bottom (defining the initial equilibrium position) of a V-shaped groove with smooth walls inclined at angle θ to the horizontal. A string is tied to the ball. Starting the instant after $t = 0$, a constant tension of $F_0 = mg\ \sin\theta$ is applied to quasistatically drag the ball any desired final distance x_0 along the incline. The conservative restoring force is gravity and its component[2] along the direction of motion is the constant $F_c(x) = -mg\ \sin\theta$. The work done by the tension all goes into increasing the gravitational potential energy of the ball without loss because it is pulled up the frictionless incline so slowly that air resistance is negligible.

Note that gravity can be 'turned off' in figure 3 by making the angle of incline θ equal to zero. This idea suggests a simple method (sketched in figure 4) for constructing the mechanical system discussed by Lee (i.e., a particle on the end of a Hookean spring to which a constant force is suddenly applied). A mass rests on the smooth bottom of a box at the free end of a spring whose other end is fixed to the side of the box. The box is initially horizontal but is hinged at its corner. At $t = 0$ the box is suddenly tipped vertically. The mass will subsequently oscillate until air drag brings it to rest at a new equilibrium position with the spring stretched by $x_0 = mg/k$.

So far these mechanical models have treated one particle experiencing a step-wise constant external force. Similar mechanical analogues can be constructed for the circuit problem in which a charged capacitor is suddenly connected to an uncharged capacitor. Panel

[2] The introduction of a *component* is justified either by properly defining the work in equation (5) as the integrated *dot product* of force and displacement, or by computing the work done by the *sum of the gravitational and normal forces*.

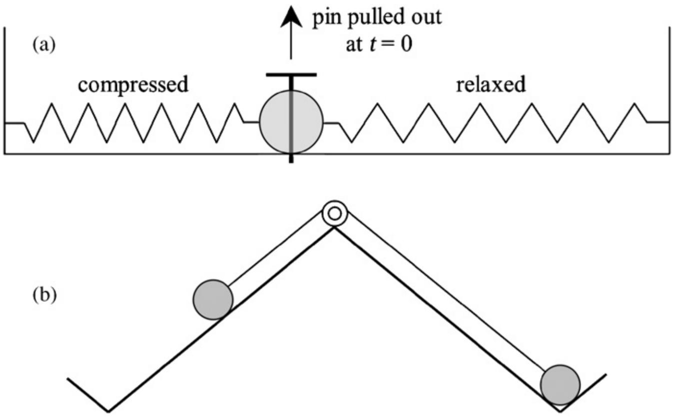

Figure 5. Interacting mechanical systems: (a) two springs of identical stiffness constant connected to a ball that experiences air drag when moving and (b) a pair of equal-mass balls on smooth inclines of equal tilt connected by a light string around a frictionless pulley. In both cases, the left system starts out with positive potential energy and the right system with none.

(a) of figure 5 shows a Hookean example, while panel (b) presents a two-particle version of figure 3. In case (a), the left spring starts out compressed by x_0 (so that it has potential energy $U_{\mathrm{Li}} = \frac{1}{2}kx_0^2$), while the right one is initially relaxed (and thus $U_{\mathrm{Ri}} = 0$). After the pin is pulled out and damping has brought the system back to rest, both springs have equal compressions of $\frac{1}{2}x_0$. Therefore the final elastic energies are $U_{\mathrm{Lf}} = U_{\mathrm{Rf}} = \frac{1}{2}k\left(\frac{1}{2}x_0\right)^2$ so that

$$f_{\mathrm{lost}} = \frac{(U_{\mathrm{Li}} + U_{\mathrm{Ri}}) - (U_{\mathrm{Lf}} + U_{\mathrm{Rf}})}{(U_{\mathrm{Li}} + U_{\mathrm{Ri}})} = 50\% \tag{6}$$

just as for the single-spring example in figure 4. On the other hand, for the system in figure 5(b), where the left particle begins with positive gravitational potential energy and the right particle with none, the system can be quasistatically brought to a final state in which both particles are at equal height with no overall change in mechanical energy.

In conclusion, a more general treatment of energy principles, illustrated with several concrete mechanical models, has been presented here to expand upon concepts introduced in Lee's paper [1]. In particular, connecting a compressed and an uncompressed spring together is an effective analogy to the well-studied problem of wiring a charged and an uncharged capacitor together. This two-spring system can be more readily visualized by students, helping them to make sense of the energy loss in such problems. In addition, it underscores the critical role of the damping mechanism in attaining a final state of equilibrium.

References

[1] Lee K 2009 The two-capacitor problem revisited: a mechanical harmonic oscillator model approach *Eur. J. Phys.* **30** 69–74