Enhanced Summary of G.P. Agrawal *Nonlinear Fiber Optics* (3rd ed) Chapter 9 on SBS

Stimulated Brillouin scattering is a nonlinear three-wave interaction between a forward-going pump beam P, a forward-going acoustic wave, and a backward-traveling Stokes beam S. The pump field generates the acoustic wave via electrostriction. That is, the electric field of the pump wave exerts oppositely directed forces on the positive and negative ions or domains in the silica, inducing a strain which generates the sound wave. In turn, that acoustic field modulates the refractive index, resulting in a grating that Bragg scatters the pump into the Stokes beam. The Stokes scattered light is downshifted in frequency by the Doppler shift, because the grating is moving forward at the acoustic (sound) speed, which for fused silica is $\upsilon_A = 5960$ m/s . (Quantum mechanically, a pump photon is annihilated and a Stokes photon and an acoustic phonon are simultaneously generated.) Energy conservation requires that the Brillouin frequency shift be $(\omega_P - \omega_S)/2\pi = \omega_B/2\pi \equiv \nu_B$, while momentum conservation says that the acoustic wavevector must be $\mathbf{k}_B = \mathbf{k}_P - \mathbf{k}_S \Rightarrow k_B = k_P + k_S$ since the Stokes wave travels backward along the optical fiber. But $\omega_B = \upsilon_A k_B = 2n\upsilon_A 2\pi/\lambda_P$ because $k_S \cong k_P$. Here the refractive index of silica is $n = 1.45$. Thus for a vacuum pump wavelength of $\lambda_P = 1.55$ μm , the shift is $\nu_B = 2n\upsilon_A/\lambda_P = 11$ GHz . Acoustic waves are assumed to decay as $\exp(-\Gamma_B t)$ where the acoustic phonon lifetime in silica is $T_B \equiv 1/\Gamma_B \approx 10$ ns . The SBS gain spectrum is assumed to be Lorentzian with angular frequency dependence

$$\frac{g_B(\Gamma_B/2)^2}{(\omega-\omega_B)^2 + (\Gamma_B/2)^2} \tag{1}$$

and thus its FWHM is $\Delta\nu_B = \Gamma_B/2\pi \approx 15$ MHz . Here the peak Brillouin gain coefficient is

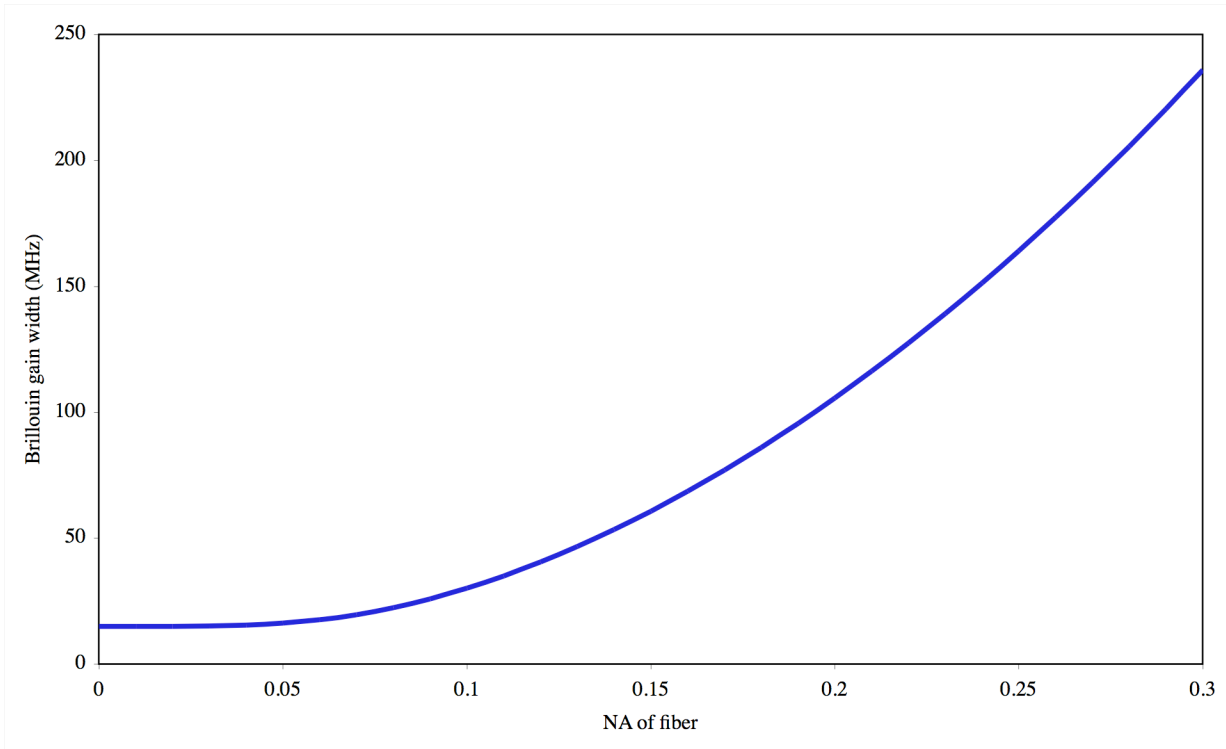
$$g_B = \frac{\pi n^7 p_{12}^2}{c\lambda_P^2 \rho \upsilon_A \Delta\nu_B} \tag{2}$$

where the vacuum speed of light is $c = 3\times10^8$ m/s , the density of silica is $\rho = 2200$ kg / m^3 , and the measured longitudinal elasto-optic coefficient of silica is $p_{12} = 0.286$. [The Lorentz-Lorenz relation predicts that $p_{12} = (n^2-1)(n^2+2)/(3n^4) = 0.34$ in rough agreement.] In bulk silica, values of $\Delta\nu_B$ between 10 and 20 MHz have been measured at a pump wavelength of 1.5 μm; I will the split the difference and use $\Delta\nu_{B,bulk} = 15$ MHz in this review. In a fiber, the width depends on the numerical aperture (NA) according to Kovalev & Harrison, Opt. Lett. **27**:2022 (2002) as

$$(\Delta\nu_{B,fiber})^2 = (\Delta\nu_{B,bulk})^2 + \left(\frac{\nu_B}{2}\right)^2 \left(\frac{\mathrm{NA}}{n}\right)^4 \tag{2.5}$$

because large numerical apertures imply that Stokes beams can travel at angles to the exact backward direction, thus slightly relaxing the momentum conservation condition stipulated above. Equation (2.5) is plotted in Fig. 1. Inhomogeneities in the fiber cross section can similarly relax the wavevector condition and thereby increase the Brillouin bandwidth.

Fig. 1. Variation in the Brillouin gain FWHM with the numerical aperture of a silica fiber at 1.5 μm.



Substituting the preceding values for bulk silica at 1.5 μm into Eq. (2) gives $g_B \approx 2 \times 10^{-11}$ m/W . Theory predicts that $\Delta\nu_B \propto \nu_B^2 \propto \lambda_P^{-2}$, which implies that g_B should be independent of the pump wavelength. Note that if the polarization angle between the pump and Stokes beams varies randomly (as in a non-polarization-maintaining fiber), then the value of g_B needs to be reduced by a factor of 1.5.

Steady-state conditions

Suppose that the pump is at least quasi-CW. Then the coupled intensity equations are

$$\frac{dI_P}{dz} = -g_B I_P I_S - \alpha I_P \tag{3}$$

for the pump as a function of length z along the fiber ranging from 0 to L, and

$$\frac{dI_S}{dz} = -g_B I_P I_S + \alpha I_S \tag{4}$$

for the Stokes beam. If the absorption coefficient (assumed to be the same at the pump and Stokes wavelengths) is negligible, then $\alpha \approx 0$ and it immediately follows from Eqs. (3) and (4) that $I_P - I_S = C$, some constant intensity difference.

Brillouin threshold in the absence of losses

If we neglect both pump depletion and losses (the latter implying $\alpha \approx 0$), then I_P can be taken to be constant along the length L of the fiber, and Eq. (4) can be separated and integrated to get

$$\int_{I_S(L)}^{I_S(0)} \frac{dI_S}{I_S} = -g_B I_P \int_L^0 dz \quad \Rightarrow \quad I_S(0) = I_S(L)\exp(g_B PL / A) \tag{5}$$

where $P = I_P A$ is the (input) pump power and A is the (effective) core area of the fiber. Equation (5) applies to Brillouin amplification whereby a Stokes signal is input at $z = L$. If instead the Stokes beam grows from noise, we model it by injecting one (fictitious) photon per mode at a distance where the gain and loss balance; the method of steepest descent can then be used to show that the critical pump power P_c defining the Brillouin threshold is

$$g_B P_c L / A \approx 21 . \tag{6}$$

Assuming telecom fiber values of $A = 25\ \mu\text{m}^2$, $L = 25$ km, and $g_B = 2 \times 10^{-11}$ m/W then Eq. (6) implies $P_c = 1$ mW. It is precisely because this threshold is so low that SBS is significant in fibers.

Gain saturation in the absence of losses

Now turn on pump depletion but continue to neglect losses, so that $I_P = I_S + C$. Then Eq. (4) becomes

$$\int_{I_S(z)}^{I_S(0)} \frac{dI_S}{C}\left(\frac{1}{I_S} - \frac{1}{I_S + C}\right) = -g_B \int_z^0 dz \quad \Rightarrow \quad \frac{I_S(z)}{I_S(0)} = \frac{I_S(z) + C}{I_S(0) + C}\exp(-g_B C z) . \tag{7}$$

Replacing $I_S(0)$ by $I_P(0) - C$, Eq. (7) rearranges into

$$I_S(z) = \frac{b_0(1 - b_0)}{G(z) - b_0} I_P(0) \tag{8}$$

where $G(z) = \exp[(1 - b_0) g_0 z]$ with

$$b_0 \equiv \frac{I_S(0)}{I_P(0)} \text{ and } g_0 \equiv g_B I_P(0) . \tag{9}$$

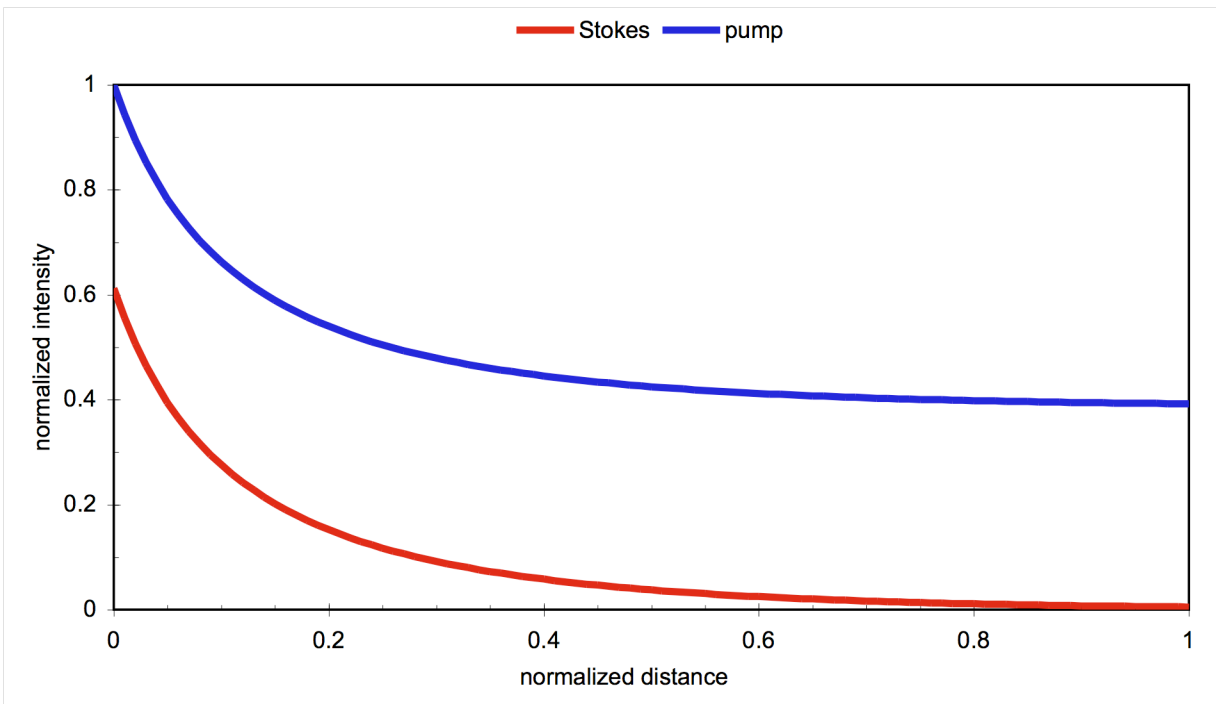
Here b_0 measures the SBS efficiency (fraction of the input pump power converted to output Stokes power) and g_0 is the small-signal SBS gain (in units of inverse distance).

Defining the ratio of input signals as $b_{in} \equiv I_S(L) / I_P(0)$, the pump and Stokes powers (normalized to the input pump power) are plotted in Fig. 2 for $b_{in} = 0.5\%$ and $g_0 = 10 / L$. For this purpose, one needs to compute the value of b_0. Evaluating Eq. (8) at $z = L$ and solving for b_0 in the exponential gives

$$b_0 = 1 - \frac{1}{g_0 L} \ln \frac{b_0(1+b_{\text{in}}) - b_0^2}{b_{\text{in}}} \tag{10}$$

or $b_0 = 1 - 0.1\ln(201 b_0 - 200 b_0^2)$ for our values of b_{in} and g_0. Iterating this formula starting from $b_0 = 0.5$, it quickly settles down to $b_0 = 0.61273$, i.e., 61% of the input pump is transformed into output Stokes power, and thus $C / I_{\text{P}}(0) = 1 - b_0 = 0.38727$. Notice from the graph that most of the power transfer occurs in the first quarter of the fiber's length.

Fig. 2. Lossless intensities of the Stokes and pump beams as a function of distance along the fiber's length in dimensionless form.



The value $g_0 L = 10$ used here corresponds to an unsaturated gain of $G_{\text{u}} = \exp(g_0 L) = 22\,000$ or $10\log(22\,000) = 43$ dB, although the actual saturated gain is only $G_{\text{s}} \equiv I_{\text{S}}(0) / I_{\text{S}}(L) = b_0 / b_{\text{in}} = 120$ due to pump depletion. Equation (8) evaluated at $z = L$ implies

$$b_{\text{in}} = \frac{b_0(1-b_0)}{\exp[(1-b_0)g_0 L] - b_0}. \tag{11}$$

If b_0 is small then $b_{\text{in}} \approx b_0 \exp(-g_0 L)$ is also small and $G_{\text{s}} \approx G_{\text{u}}$. But for larger values of b_{in} the gain rolls off, falling to about half at a saturation input Stokes power comparable to the threshold pump power of about 1 mW.

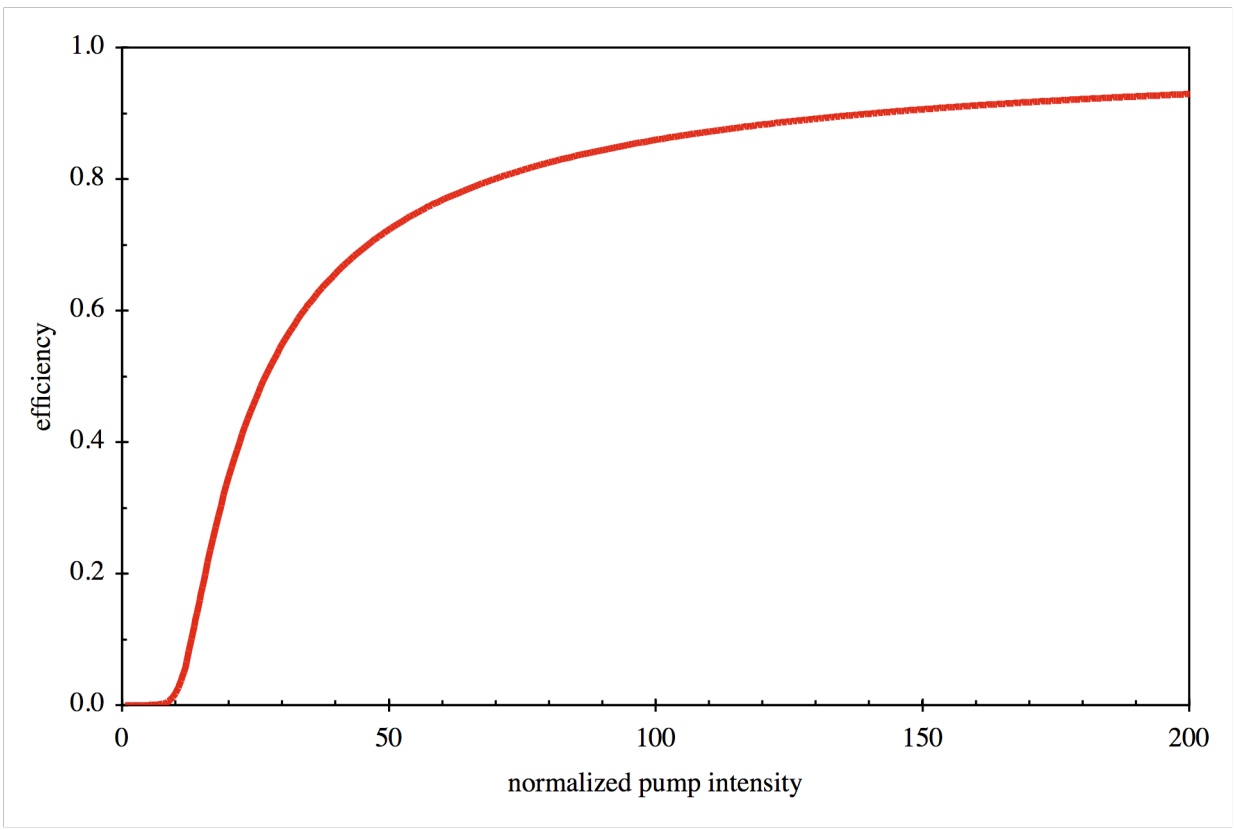
To find the threshold exactly, suppose the Stokes input (seed) intensity is fixed, say at a normalized value of $g_{\text{B}} I_{\text{S}}(0) L = 10^{-5} \equiv k$. Then $g_0 L = g_{\text{B}} I_{\text{P}}(0) L = k / b_{\text{in}}$ where we now treat

$I_P(0)$ —and hence b_{in}—as a variable rather than as a constant, and plot the Stokes efficiency b_0 against it. For this purpose, define $u \equiv 1/b_{in}$ and rewrite Eq. (11) as

$$u = \frac{\exp[k(1-b_0)u] - b_0}{b_0(1-b_0)} . \tag{12}$$

A numerical root-finder in MatLAB was used to solve this equation for b_0 as a function of u for fixed $k = 10^{-5}$. The results are plotted in Fig. 3 where ku is on the horizontal axis, i.e., the normalized pump intensity $g_B I_P(0)L$, and b_0 is on the vertical axis, i.e., the efficiency or "Stokes reflectivity" equal to the ratio of the Stokes output intensity to the pump input intensity. The efficiency saturates at value 1 in the limit of very strong pumping. If we define the pump threshold value to occur at an efficiency of $1/e = 0.3679$, we see from the graph that it occurs at a value of $g_0 L = 21$, in exact agreement with Eq. (6).

Fig. 3. Stokes efficiency as a function of the pump intensity in dimensionless form.



Although it is not evident on the scale of the above plot, the curve actually turns around for very low pump intensities and the efficiency then grows without bound. The reason for this behavior is that for very weak pumping, the (fixed) Stokes input simply propagates straight through the fiber and equals the Stokes output. Thus the efficiency becomes a constant divided by the pump, which diverges in the limit of vanishing pump! This behavior becomes obvious if one starts with a stronger seed, such as $k = 0.1$. For that value, in fact, the turn around occurs

near threshold, so that the efficiency merely shows a dip rather than a zero plateau at low pump intensities.

*Experimental results of Cotter in Electronics Letters **18**:495 (1982)*

An isolated Nd:YAG cw laser beam at 1.32 μm with a linewidth much smaller than the Brillouin bandwidth $\Delta\nu_B$ was sent into a 13.6 km fiber having losses of only 0.41 dB/km [so that $\alpha = 0.41\ln(10)/10 = 0.094\ \text{km}^{-1}$], corresponding to an effective fiber length of

$$\int_0^L e^{-\alpha z}\,dz = \frac{1-\exp(-\alpha L)}{\alpha} = 7.66\ \text{km}\,. \tag{13}$$

(Note that the effective length equals L for a lossless fiber, but it decreases to $1/\alpha$ if $\alpha >> 1/L$.) The effective fiber area was $A = 47\ \mu\text{m}^2$. At low input powers, the back-reflected signal was 4% due to the air-fiber interface. The Brillouin threshold was reached at about 5 mW, manifested as a substantial increase in backward-going power. Simultaneously, the transmitted power dropped, saturating at about 2 mW for inputs exceeding 10 mW, corresponding to an SBS conversion efficiency of about 65%. The Brillouin frequency shift was measured using a Fabry-Perot to be 12.7 GHz.

Time-dependent amplitude equations

Neglecting group velocity dispersion, as well as self and cross phase modulation, the coupled equations for the complex pump and Stokes amplitudes A_P and A_s are

$$\frac{\partial A_P}{\partial z} + \frac{n}{c}\frac{\partial A_P}{\partial t} + \frac{\alpha}{2}A_P = -\frac{1}{2}g_B A_S Q \tag{14}$$

and

$$-\frac{\partial A_S}{\partial z} + \frac{n}{c}\frac{\partial A_S}{\partial t} + \frac{\alpha}{2}A_S = \frac{1}{2}g_B A_P Q^* \tag{15}$$

where Q is the acoustic power density (in W/m^2) resulting from density variations of the silica due to the sound wave, described by

$$T_B\frac{\partial Q}{\partial t} + Q = A_P A_S^*\,. \tag{16}$$

For pump pulses of temporal width $T_P >> T_B \approx 10$ ns, we can neglect $\partial Q/\partial t$. Defining $I_P = |A_P|^2$ and $I_S = |A_S|^2$, substitute $Q = A_P A_S^*$ into Eq. (14), multiply that equation through by A_P^*, and add the result to its complex conjugate to get

$$\frac{\partial I_P}{\partial z} + \frac{n}{c}\frac{\partial I_P}{\partial t} = -g_B I_P I_S - \alpha I_P \tag{17}$$

which reduces to Eq. (3) under steady-state conditions. In a similar way, Eq. (15) becomes

$$-\frac{\partial I_S}{\partial z}+\frac{n}{c}\frac{\partial I_S}{\partial t}=g_B I_P I_S-\alpha I_S\,. \tag{18}$$

Equations (17) and (18) exhibit relaxation oscillations with a period of twice the fiber transit time, $T_r = nL/c$. (Physically, the Stokes power rapidly grows near the input end of the fiber, thereby depleting the pump. That reduces the gain until the depleted portion of the pump exits the far end of the fiber. The gain then builds back up and the process repeats, resulting in oscillations.) In the presence of external feedback (for example due to back-reflections into the fiber), these relaxation oscillations can develop into stable oscillations via self-induced intensity modulation.

The graphs below show an example of relaxation oscillations in the absence of feedback, i.e., the pump and Stokes beams pass into and completely out of the fiber starting from opposite ends. These were computed numerically using the Method of Lines (MOL) by modifying the MatLAB code available online at http://www.scholarpedia.org/article/Method_of_lines. [Another technique is to decouple the PDEs by defining $u \equiv z + ct/n$ and $w \equiv z - ct/n$ so that Eq. (17) becomes $\partial I_P/\partial u = -2g_B I_P I_S - 2\alpha I_P$ and (18) becomes $\partial I_S/\partial w = -2g_B I_P I_S + 2\alpha I_S$. This is called the Method of Characteristics, but it has the disadvantage of mixing together the initial conditions and boundary values.] It is convenient to rewrite Eqs. (17) and (18) in normalized form by multiplying every term in them by $L/I_P(0)$ to get

$$\frac{\partial P}{\partial \tilde{z}}+\frac{1}{N}\frac{\partial P}{\partial \tilde{t}}=-gPS-\tilde{\alpha}P \tag{19}$$

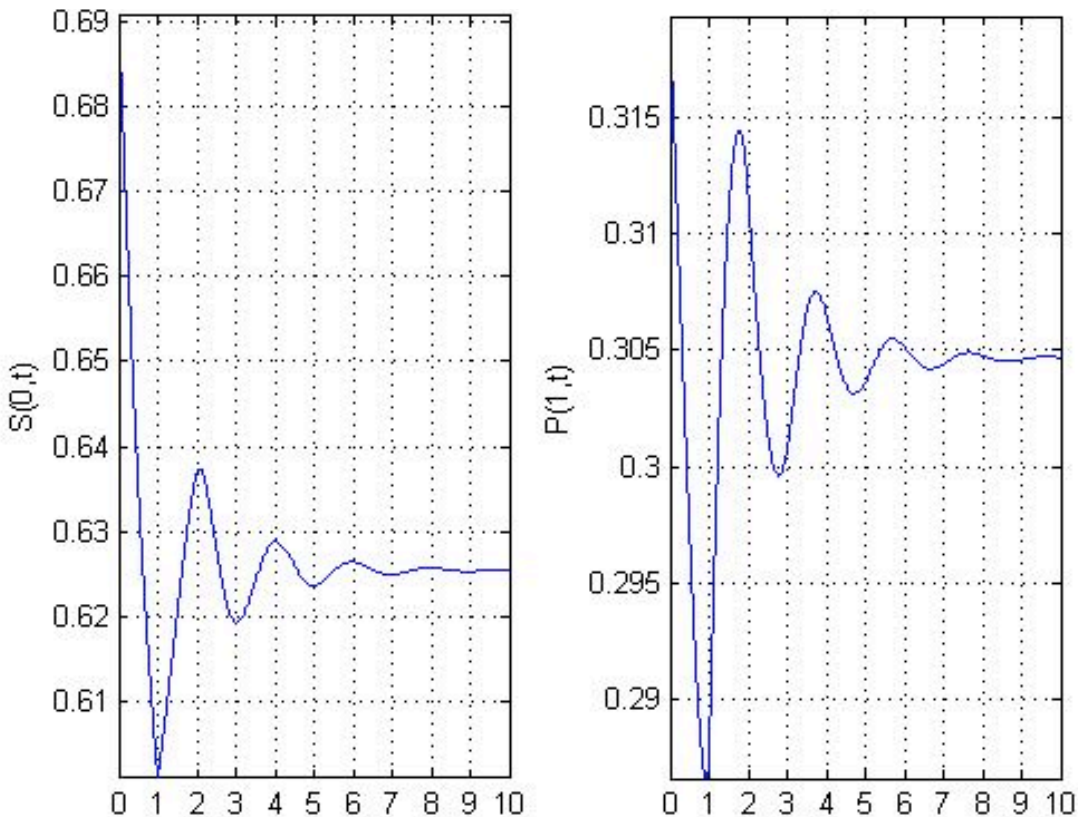
and

$$-\frac{\partial S}{\partial \tilde{z}}+\frac{1}{N}\frac{\partial S}{\partial \tilde{t}}=gPS-\tilde{\alpha}S \tag{20}$$

where $P \equiv I_P(z)/I_P(0)$, $S \equiv I_S(z)/I_P(0)$, $\tilde{z} \equiv z/L$, $\tilde{t} \equiv t/(NT_r)$, $g \equiv g_B I_P(0)L$, and $\tilde{\alpha} \equiv \alpha L$. Here $\tilde{z}$ and $\tilde{t}$ both numerically evolve on grids running from 0 to 1, and N is the number of transit times over which the simulation is permitted to run. In Fig. 4, I used a normalized gain of $g = 10$, normalized absorption of $\tilde{\alpha} = 0.15$, $N = 10$ periods, 800 grid points in the $\tilde{z}$ direction, and 20 000 grid points in the $\tilde{t}$ direction. (I found it is important to use finer time than spatial discretization, so that the oscillations have time to settle down.) The spatial derivatives are computed using five-point, fourth-order finite-difference approximations in subroutine "dss004.m" (where some "for" loops were replaced with "parfor" to take advantage of modern dual-core computer processing). The time integration is then performed using subroutine "ode23" which implements the Bogacki-Shampine four-stage third-order adaptive Runge-Kutta method. (Higher order methods were found to be too unstable.) The relative and absolute tolerances were both set to 10^{-8}. The Stokes input relative to the pump input, b_{in}, was chosen to be 1%. To increase stability, I used the lossless steady-state profiles (corresponding to $b_0 = 0.69064$) for the pump and Stokes beams (similar to the curves in Fig. 2) as initial conditions. Due to the loss, the normalized Stokes signal (left-hand graph) evolves away from its

initial output value (at the front face $\tilde{z} = 0$ of the fiber) of b_0 to a final value of 0.62548. Meanwhile, the normalized pump signal (at the rear face $\tilde{z} = 1$) evolves from an initial value of $1 - b_0 + b_{\text{in}} = 0.31936$ to a final value of 0.30465.

Fig. 4. Time evolution of the output Stokes and pump beams, showing relaxation oscillations over the course of 10 single-pass transits through the fiber. The horizontal axis is in units of t / T_{r}.



If optical communication pulses of "0" and "1" are sent into a fiber at a repetition rate of say 1 GHz with pulse widths of about 100 ps (much shorter than T_{B}), they can still be treated using the quasi-cw analysis presented above because the time interval between successive "1" bits is usually short enough that they can pump an acoustic wave coherently. On average, the only effect is to increase the Brillouin threshold by a factor of approximately 2. Further increase in the threshold can be achieved by phase modulating the carrier wave at 100 MHz (or faster) to increase its spectral bandwidth.

Brillouin fiber lasers

The Brillouin gain can be used to make lasers by placing the fiber in a ring or mirrored cavity. For a ring geometry, the boundary condition $I_{\text{S}}(L) = RI_{\text{S}}(0)$ implies that Eq. (5) becomes

$$R\exp(g_{\text{B}}PL / A) = 1 . \tag{21}$$

Consequently the factor of 21 in Eq. (6) is replaced with a value between 0.1 and 1 (depending on R) if L is less than about 100 m. For short fibers, such that the longitudinal mode spacing $\Delta\nu_{\mathrm{L}} = c / nL$ is larger than the Brillouin gain bandwidth $\Delta\nu_{\mathrm{B}}$, the laser operates stably in a single longitudinal mode.