Relativistic Effects on Clocks Aboard GPS Satellites

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Consider a clock aboard a satellite orbiting the Earth, such as a Global Positioning System (GPS) transmitter. There are two major relativistic influences upon its rate of timekeeping: a special relativistic correction for its orbital speed and a general relativistic correction for its orbital altitude. Both of these effects can be treated at an introductory level, making for an appealing application of relativity to everyday life.

First, as observed by an earthbound receiver, the transmitting clock is subject to time dilation due to its orbital speed. A clock aboard a spaceship traveling at speed v runs slow (compared to a stationary clock) by a factor of[1]

$$\frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}} \approx 1 - \frac{v^2}{2c^2}, \tag{1}$$

provided $v << c$, as would be the case for a satellite. Thus when one second of proper time elapses, the moving clock loses $v^2/2c^2 = K/E_0$ seconds, where K and E_0 are the kinetic and rest energies of the clock, respectively.

Second, a clock at the higher gravitational potential of orbit runs faster than a surface clock. The gravitational potential energy of a body of mass m in Earth's gravity is $U = mV$, where $V = -Gm_E/r$ is Earth's gravitational potential (at distance r from the center of the Earth of mass m_E). In the case of a photon, we replace m by E/c^2, where $E = hf$ is the photon's energy. If the photon travels downward in Earth's gravitational field, it therefore loses potential energy of $(hf/c^2)\Delta V$ and gains an equal amount of kinetic[2] energy $h\Delta f$. We thereby deduce that the falling photon is gravitationally blue-shifted by

$$\Delta f = f\frac{\Delta V}{c^2}. \tag{2}$$

(This expression can also be straightforwardly deduced[3] using the equivalence principle to treat Earth's downward gravitational field as an upward accelerating frame, and then calculating the Doppler shift in the light[1] between emission high up and observation low down in this moving frame.) If the clock's ticking is synchronized to a light wave, the orbiting clock will be observed at Earth's surface to be ticking faster due to this gravitational frequency shift. Therefore, when one second of Earth time elapses, the clock at high altitude gains $\Delta V/c^2 = \Delta U/E_0$ seconds, where U is the gravitational potential energy of the clock.

The sum of the two relativistic effects can be compactly expressed as

$$\frac{\Delta t}{\tau} = \frac{K - U}{E_0}, \tag{3}$$

where Δt is the time lost by the orbiting clock when a time interval τ elapses on the surface-bound clock. Here $K - U$ is the Lagrangian[4] of the orbiting clock where the reference level for the gravitational potential energy is chosen to lie at Earth's surface.

As a concrete example, let's calculate the size of these two effects for a GPS satellite, located at an altitude of $r = 26{,}580$ km, about four times Earth's radius of $r_E = 6380$ km. From Newton's second law, we have

DOI: 10.1119/1.2353579

$$a = \frac{F}{m} \quad \Rightarrow \quad \frac{v^2}{r} = \frac{Gm_E}{r^2} \quad \Rightarrow \quad v^2 = \frac{gr_E^2}{r}, \qquad (4)$$

where Earth's surface gravitational field is $g \equiv Gm_E/r_E^2 = 9.8\ \text{m/s}^2$. Hence the fractional time loss due to the satellite's orbital speed is $-gr_E^2/2rc^2$ per second, or –7.2 μs/day. Meanwhile, the general relativistic fractional time gain due to the satellite's altitude is

$$\frac{\Delta V}{c^2} = \frac{1}{c^2}\left[-\frac{Gm_E}{r} + \frac{Gm_E}{r_E}\right] = \frac{gr_E}{c^2}\left(1 - \frac{r_E}{r}\right), \qquad (5)$$

which works out to be +45.6 μs/day. Notice that the gravitational effect is more than six times larger than the speed effect: the dominant GPS correction is general, not special, relativistic!

If we instead consider satellites in progressively lower altitude orbits, their speeds will increase according to Eq. (4), while the gravitational potential difference in Eq. (5) will decrease. Eventually we will reach an altitude at which the two corrections exactly cancel, so that the satellite's clock will run synchronously with an earthbound clock.[5] This occurs when

$$\frac{v^2}{2c^2} = \frac{\Delta V}{c^2} \Rightarrow \frac{gr_E^2}{2r} = gr_E\left(1 - \frac{r_E}{r}\right) \Rightarrow r = 1.5r_E, \qquad (6)$$

i.e., at an altitude of half an Earth radius.

An interesting postscript: One might assume, as Einstein did[6] in 1905, that if we set the altitude equal to zero (i.e., sea level), a clock fixed to Earth's surface would run slower at the equator than at the poles, owing to Earth's rotational speed. But in fact that is incorrect because sea level is an equipotential surface.[7] To put it another way, the Earth bulges at the equator, thereby gravitationally raising a clock located there by exactly the amount required to cancel the time dilation due to Earth's spin.

References

1. For example, see D.C. Giancoli, *Physics for Scientists and Engineers*, 3rd ed. (Prentice Hall, Upper Saddle River, NJ, 2000), Chap. 37.
2. The relativistic kinetic energy of a body is defined as $K \equiv E - E_0$, where E_0 is the body's rest energy. In the case of a photon, $E_0 = 0$ so that $E = K$.
3. I.R. Kenyon, *General Relativity* (Oxford Univ. Press, Oxford, 1990), Chap. 2. These approximate methods of deriving Eq. (2) agree with an exact general relativistic treatment, presented in an accessible manner in E.F. Taylor and J.A. Wheeler, "Project A: Global Positioning System," in *Exploring Black Holes: Introduction to General Relativity* (Addison-Wesley, San Francisco, 2000), pp. A-1–A-9.
4. E. Huggins, "GPS satellites and Lagrangians," presented at the AAPT Summer Meeting, Madison, WI, 2003. Equation (3) implies that the action per unit mass for an orbiting clock is equal to the time lost multiplied by the speed of light squared.
5. N. Ashby, "Relativity and the Global Positioning System," *Phys. Today* **55**, 41–47 (May 2002). Also see N. Ashby, "Relativity in the Global Positioning System," *Living Reviews in Relativity* **6** (Jan. 2003), online at http://www.livingreviews.org/lrr-2003-1.
6. A. Harvey and E. Schucking, "A small puzzle from 1905," *Phys. Today* **58**, 34–36 (March 2005). Also see S.P. Drake, "The equivalence principle as a stepping stone from special to general relativity: A Socratic dialog," *Am. J. Phys.* **74**, 22–25 (Jan. 2006).
7. Specifically, in Earth's rotating frame of reference the total surface potential is computed as follows. The gravitational potential difference in Eq. (5) is gy for small altitudes y above polar sea level. The centripetal acceleration of an object revolving with angular speed ω at a distance r from the axis is $\omega^2 r$. (For the Earth, $\omega = 2\pi/24$ h.) In a rotating frame, this can be treated as an outward centrifugal acceleration. Its integral corresponds to a centrifugal potential of $-\frac{1}{2}\omega^2 r^2 = -\frac{1}{2}v^2$. If we now require that the total potential $gy - v^2/2 = 0$ everywhere on Earth's surface, then Eqs. (1) and (5) cancel each other's effects. [Note, however, that $\omega^2 r_E^2/2g = 11$ km is half the height of Earth's actual equatorial bulge, as discussed in standard texts such as D.L. Turcotte and G. Schubert, *Geodynamics,* 2nd ed. (Cambridge Univ. Press, Cambridge, 2002), Chap. 5. In fact, an equatorial clock needs to be raised beyond 11 km to compensate for the gravitational field from the mass of the bulge.]

PACS codes: 04.00.00, 03.30.+p, 95.40.+s

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