

Approximation for the Rayleigh Resolution of a Circular Aperture

Carl E. Mungan, U.S. Naval Academy, Annapolis, MD

Rayleigh's criterion states that a pair of point sources are barely resolved by an optical instrument when the central maximum of the diffraction pattern due to one source coincides with the first minimum of the pattern of the other source. As derived in standard introductory physics textbooks,¹ the first minimum for a rectangular slit of width a is located at angular position $\theta = \sin^{-1}(\lambda/a)$ for light of wavelength λ . If the angular separation of the two sources is small, we can use the small-angle approximation $\sin \theta \approx \theta$ to conclude that the resolution is $\theta_{\min} = \lambda/a$ for a rectangular aperture. On the other hand, for a circular aperture of diameter D , the limiting angle is shown in optics texts² to be $\theta_{\min} = 1.22 \lambda/D$. The derivation of the numerical prefactor of 1.22 involves finding the zero of a Bessel function and is beyond the reach of introductory physics students. Consequently, elementary texts simply pull that prefactor out of thin air. The purpose of the present paper is to briefly explain why we expect a prefactor larger than unity and to make simple estimates of its value, using only algebra.

Divide a circular aperture into a set of rectangular strips, as illustrated in Fig. 1. Any arbitrary strip, located at vertical position y with respect to the center of the circle, has width a and height Δy . It is a rectangular aperture, resulting in a Fraunhofer diffraction pattern having a central maximum of horizontal half-width $\theta = \lambda/a$. That pattern is superposed with those of all the other strips making up the circle. Consequently we get an overall pattern having an average width θ_{ave} . But one cannot simply arithmetically average the half-widths due to all the strips, because strips of larger area will let more light through (than

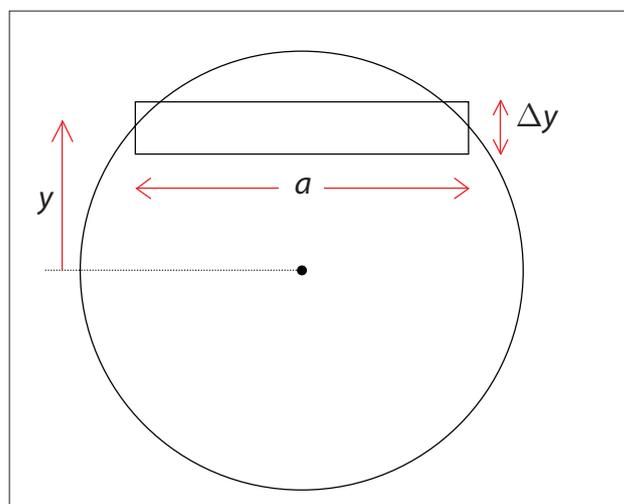


Fig. 1. A circular aperture can be divided into thin strips, one of which is shown.

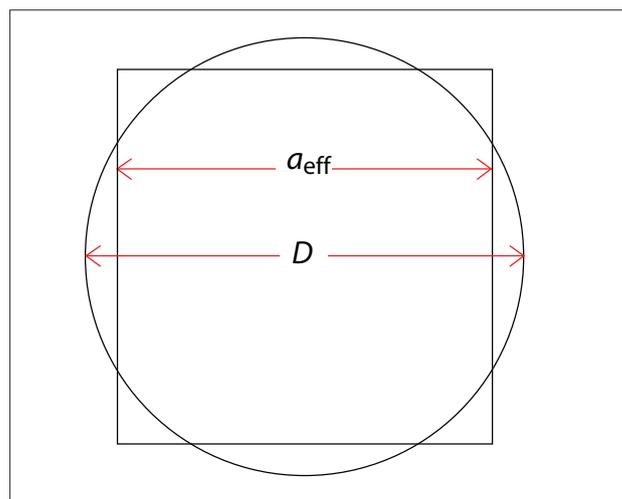


Fig. 2. The circular hole is replaced with a square one having the same area.

those of smaller area) and will thereby dominate the overall pattern. We need to weight the averaging by the amount of light that passes through a given strip, which is proportional to its area $\Delta A = a\Delta y$. We thus obtain

$$\theta_{\text{ave}} = \frac{\sum \theta_{\text{min}} \Delta A}{\sum \Delta A} = \frac{\sum \frac{\lambda}{a} a \Delta y}{\sum \Delta A} = \lambda \frac{\sum \Delta y}{\sum \Delta A}.$$

Notice that a , which is a function of y , cancels out of the numerator. To continue, the sum of the heights of the strips over the entire circle equals its diameter D . On the other hand, the sum of the areas of the strips is the area $\frac{1}{4}\pi D^2$ of the circle. Substituting these two values into the above formula, we obtain

$$\theta_{\text{ave}} = \frac{4}{\pi} \frac{\lambda}{D} \approx 1.27\lambda/D,$$

whose numerical prefactor is within 5% of the exact value.

In all fairness, while the preceding model predicts a reasonable numerical prefactor for diffraction in the horizontal direction, strips of narrow vertical height would greatly exaggerate the diffraction in the vertical direction. To treat the horizontal and vertical direc-

tions on an equal footing, we can consider an even simpler model, by “squaring the circle” as depicted in Fig. 2. Requiring the square and circular apertures to again have the same areas, we conclude that $a_{\text{eff}}^2 = \frac{1}{4}\pi D^2$. We then get diffraction of angular half-width $\theta_{\text{min}} = \lambda/a_{\text{eff}} \approx 1.13 \lambda/D$.

One does not get detailed numerical agreement from these simplistic models. But they do point up the fact that the average horizontal width of a circle of diameter D is less than D , so that the reciprocal “effective width” of a circular hole must be somewhat larger than $1/D$.

References

1. For example, see R.A. Serway and J.W. Jewett, *Physics for Scientists and Engineers*, 7th ed. (Thomson, Belmont CA, 2008), Chap. 38.
2. For example, see E. Hecht, *Optics*, 3rd ed. (Addison-Wesley, Reading, MA, 1998), Sec. 10.2.5.

PACS codes: 01.55.+b, 42.00.00

Carl Mungan is an associate professor of physics with research interests in optics at the U.S. Naval Academy in Annapolis, MD.
mungan@usna.edu
