Irreversible Adiabatic Compression of an Ideal Gas

Carl E. Mungan, U.S. Naval Academy, Annapolis, MD

Consider the following problem.[1] A frictionless, massive piston partitions an insulated box vertically into two parts. The upper compartment is evacuated, while the lower contains an equilibrated ideal gas. A weight is suddenly placed on the piston. How much is the gas compressed when equilibrium is again restored, neglecting the heat capacity of the cylinder and piston?

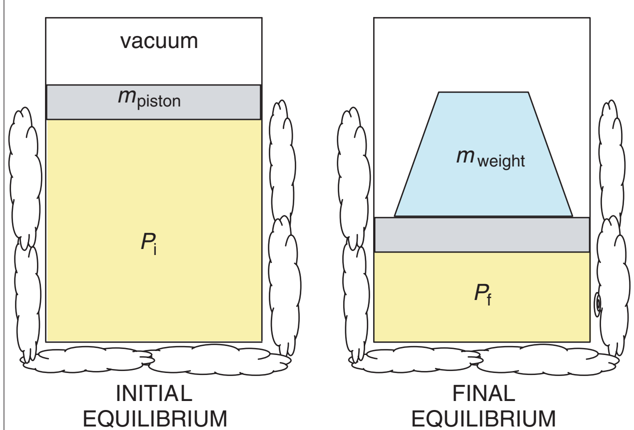

Fig. 1. An ideal gas in the lower portion of an insulated box (as indicated by the cotton wool), with the top portion evacuated. Initially the gas pressure balances the weight of the piston. A mass is then suddenly placed on the piston. Eventually a new equilibrium state is attained, in which the final pressure again balances the total weight.

Solution by Conservation of Energy

Referring to Fig. 1, suppose the piston has mass m_{piston} and the added mass is m_{weight}, for a total of $m = m_{\text{piston}} + m_{\text{weight}}$. Choose the zero level of the gravitational potential to lie at the final equilibrium height of the piston, and let its initial position be at height h above this. The internal energy of an equilibrated ideal gas is $U = PV/(\gamma - 1)$, where $\gamma \equiv C_P/C_V$ is the ratio of the isobaric and isochoric heat capacities. Label the initial equilibrium state by subscript "i" and the final equilibrium state by "f" to obtain

$$\frac{P_i V_i}{\gamma - 1} + mgh = \frac{P_f V_f}{\gamma - 1}. \tag{1}$$

But the force balance on the piston (of cross-sectional area A) in its final position implies $mg = P_f A$, and the net displacement of the piston is $h = (V_i - V_f)/A$. Substitute these two relations into the second term in Eq. (1) to obtain[2]

$$\frac{V_f}{V_i} = \frac{\gamma - 1}{\gamma} + \frac{P_i}{\gamma P_f}. \tag{2}$$

Note that the pressure ratio can be expressed in terms of the masses as $P_i/P_f = (1 + m_{\text{weight}}/m_{\text{piston}})^{-1}$. Consequently, the right-hand side of Eq. (2) correctly reduces to unity when $m_{\text{weight}} = 0$. In the opposite limit, note that the fractional compression $(V_i - V_f)/V_i$ of the gas becomes $1/\gamma$ when an infinite weight (and hence final pressure) is applied. In particular the maximum compression is 60% for a monatomic ideal gas, as graphed in blue in Fig. 2. An ideal gas cannot be compressed to zero equilibri-

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um volume no matter how heavy of a boulder is suddenly placed (or dropped) onto the piston!

An interesting variant on this problem is described in Ref. 3. A stone whose weight is equal to that of the piston is dropped from some height h above the piston. What is that height if the equilibrated volume of the gas is the same before and after the stone's fall, $V_i = V_f = AL$, where L is the distance from the bottom of the cylinder to the piston? Since the pressure of the gas has doubled from $P_i = mg/A$ to $P_f = 2mg/A$, the answer is $h = L/(\gamma - 1)$ from Eq. (1). When dropped from this height, the initial (gravitational potential) energy of the rock exactly suffices to double the absolute temperature of the gas.

Comparison with a Reversible Compression

Suppose that instead of suddenly placing the weight on the piston, it is lowered on infinitesimally slowly (using a crane say). A quasistatic adiabatic compression[4] of an ideal gas obeys the law PV^{γ} = constant. Thus, $V_f/V_i = (P_i/P_f)^{1/\gamma}$, which is plotted in red in Fig. 2. In contrast to the irreversible compression (blue curve), the final volume now tends to zero as the final pressure (due to the added weight) becomes infinite. However, the irreversible and reversible curves overlap for small fractional changes in volume $\Delta V/V_i$. This explains why Rüchhardt's method of measuring γ, in which a marble oscillates in the tightly fitting neck of a gas-filled jar, can be analyzed using this reversible law *even though the compressions and expansions of the gas are actually irreversible*.[5]

The contrast between the irreversible and reversible compressions can be made even more apparent by following up the reversible compression with a second reversible process that drives the system to the final state of the irreversible compression. As one sees from Fig. 2, this second process must be isobaric because the total weight establishing the final pressure P_f has already been added. This is a specific example of an equivalent reversible path[6] connecting the initial and final equilibrium states of the actual irreversible process, such as can be used to calculate the change in volume, entropy, or any other state function of interest.

The mathematical details are worked out in the Appendix. Although this method of solution requires

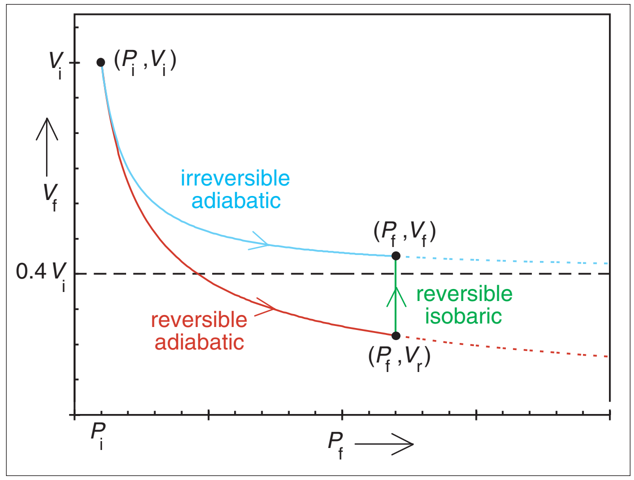

Fig. 2. The blue curve is a graph of V_f from Eq. (2) for P_f varying from $P_i = m_{piston}g/A$ to $20P_i$ in a sudden compression of a monatomic ideal gas; the dashed black line represents the value that V_f asymptotically approaches in the limit as $P_f = (m_{piston} + m_{weight})g/A \rightarrow \infty$. The red curve plots the compression that results if the extra weight m_{weight} is instead added slowly. The green line connects the final states of a reversible and irreversible compression for the same (arbitrary) weights, as discussed in the Appendix. (Note that the irreversible curve is the locus of final equilibrium volumes for different applied weights and so *the blue arrow should not be interpreted as an actual trajectory in configuration space*, unlike the portions of the reversible curves labeled with the red and green arrows.)

more computation, it shows that even though the gas volume after a quasistatic adiabatic compression can be made as small as one likes by choosing m_{weight} large enough, the equilibrium volume after a sudden adiabatic compression using the same weight is no smaller than the original volume multiplied by $(\gamma - 1)/\gamma$. In the irreversible case, all of the potential energy of the weight went into doing work[2] on the gas; but in the reversible case, some of this potential energy went into doing work on the outside world (viz. the crane). Since less work is done on the gas in the reversible case, the final internal energy (and hence volume) of the gas is smaller for the same final pressure. In thermodynamic terms, only configuration work is done in the reversible case, while both configuration and dissipative work are done on the gas in the irreversible case.[7] This reveals a shortcoming in the problem statement: Some dissipative mechanism must be provided to damp out the oscillations of the piston and enable the gas to attain a final state of equilibrium.

One might suppose that introducing friction between the piston and the cylinder walls would do the trick. However, this would heat up the cylinder, allowing energy to leak out of the gas. A better option is to enable dissipative work to be done by the relaxation of turbulent pressure gradients in the gas,[8] although this goes beyond the ideal gas model.

As an alternative to the crane, one can avoid doing work on the outside world in the reversible case if, rather than gently lowering one large weight onto the piston, infinitesimal weights are instead dribbled onto the piston bit by bit.[9] One imagines sliding tiny weights onto the piston from a multitude of shelves of gradually decreasing height as the gas compresses. Hence, the center of mass of the whole set of weights is lower than the initial height of the piston,[10] and it is once again clear that less energy is added to the gas in the reversible case than in the irreversible case, so that a larger compression is attained for a given weight.

Acknowledgments

The problem addressed in this paper arose from discussions with Tim Royappa. Posts on the PHYS-L email list, particularly by Leigh Palmer and John Denker, expanded the solutions.

Appendix: Connecting the Final States of the Reversible and Irreversible Compressions

Consider a reversible path from the initial to the final states of the system, consisting of an adiabatic compression followed by an isobaric expansion (cf. Fig. 2). First, adiabatically lower the weight onto the piston infinitesimally slowly using a crane. During this process, work W_{crane} is done on the crane. Use this work to, say, charge up a capacitor by turning the axle of a generator with the crane's cable. Disconnect the crane when the gas pressure reaches the final equilibrium value $P_f = mg/A$.

The tension T exerted on the weight by the crane must always balance the net downward force $mg - PA$ during the lowering, where P is the (varying) gas pressure. Unlike in the irreversible case, the gas pressure is always well defined during this quasistatic process.

Writing $PV^{\gamma} = k$, where k is a constant, implies

$$W_{\text{crane}} = -\int T dy = \frac{1}{A}\int_{V_i}^{V_r}(PA - mg)dV$$
$$= k\int_{V_i}^{V_r} V^{-\gamma}\,dV - P_f(V_r - V_i) = P_f V_i - \frac{\gamma P_f V_r}{\gamma - 1} + \frac{P_i V_i}{\gamma - 1}, \tag{3}$$

where V_r is the gas volume at the end of the reversible adiabatic leg of the two-step process. Next, during the isobaric step, the gas remains at pressure P_f while the energy stored in the capacitor is used to slowly add heat $Q = W_{\text{crane}}$ to the gas via an electrical heater. Consequently, the gas will expand to its final volume V_f and negative work $W = -P_f(V_f - V_r)$ will be done on it by the weight. The internal energy of the gas correspondingly changes from $U_r = P_f V_r/(\gamma - 1)$ to $U_f = P_f V_f/(\gamma - 1)$. Applying the first law of thermodynamics, $U_f - U_r = W + Q$, and substituting the above expressions finally reproduces Eq. (2). But note in the limit as $P_f \to \infty$ that $V_r \to 0$, whereas $V_f \to (\gamma - 1)V_i/\gamma$.

References

1. Similar problems are discussed in two recent anthologies: P. Gnädig, G. Honyek, and K. Riley, *200 Puzzling Physics Problems* (Cambridge University, Cambridge, 2001), Prob. P133, and S.B. Cahn, G.D. Mahan, and B.E. Nadgorny, *A Guide to Physics Problems*, Part 2 (Plenum, New York, 1997), Prob. 4.23. In such problems, "adiabatic" means thermally insulated *not isentropic.*
2. Alternatively, one may appeal to the first law of thermodynamics and calculate the work done on the gas as the product of the constant "external pressure" and the decrease in volume of the gas. As discussed in F.W. Sears and G.L. Salinger, *Thermodynamics, Kinetic Theory, and Statistical Thermodynamics*, 3rd ed. (Addison-Wesley, Reading, MA, 1986), p. 63, the external pressure is the ratio of the net external force to the cross-sectional area of the piston, $P_{\text{ext}} = mg/A \equiv P_f$. Consequently, $(P_f V_f - P_i V_i)/(\gamma - 1) = P_f(V_i - V_f)$ which rearranges into Eq. (2). This approach only gives the correct answer because P_{ext} is well defined throughout the compression (in contrast to the nonuniform internal gas pressure). Another example where this method works is a free expansion, for which $P_{\text{ext}} = 0$. But in more general problems (say, if the upper compartment were also gas

filled), the concept of external pressure is not helpful — cf. S.G. Canagaratna, "Critique of the treatment of work," *Am. J. Phys.* **46,** 1241–1244 (Dec. 1978).

3. W.G. Hoover and B. Moran, "Pressure-volume work exercises illustrating the first and second laws," *Am. J. Phys.* **47**, 851–856 (Oct. 1979).
4. As pointed out on p. 142 of R.P. Bauman, *Modern Thermodynamics with Statistical Mechanics* (Macmillan, New York, 1992), the speed of the piston must always be much less than that of sound in order to be reversible but much faster than the thermal conduction rate to the walls in order to be adiabatic.
5. Thus, the amplitude of oscillation must be small. This is also required so that the motion be simple harmonic and so that the gas remains ideal. A nice summary of Rüchhardt's method can be found in M.W. Zemansky and R.H. Dittman, *Heat and Thermodynamics*, 7th ed. (McGraw-Hill, New York, 1997), Sec. 5.6. See the papers by Clark and Katz referenced therein for a discussion of the corrections for finite amplitudes of oscillation.
6. Ibid., Sec. 8.7.
7. Sears and Salinger, op. cit., p. 71.
8. Note that the weight, piston, cylinder, and environment are implicitly assumed not to accept energy inelastically via sound, deformation, heat transfer, or the like. For a numerical hydrodynamic solution of the viscous damping of the piston, including the elastic bouncing of the weight on the piston when dropped from a height, see Ref. 3.
9. D.R. Olander, "Compression of an ideal gas as a classroom example of reversible and irreversible processes," *Int. J. Eng. Educ.* **16**, 524–528 (2000).
10. If the total weight to be added is large, its center of mass must lie not at half of the piston's initial height but instead close to the bottom of the cylinder! That is, if the tiny weights are all of equal mass, then the shelves cannot be equally spaced but must instead lie closer and closer together near the bottom. This is clear from a consideration of the shape of the red curve in Fig. 2: A small increase in pressure initially leads to a large reduction in the volume of the gas, but as the compression continues, it takes a larger and larger increase in pressure to effect even a small volume decrease.

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Carl E. Mungan *is an assistant professor and coordinates the classical mechanics course at the Naval Academy. His current research interests are in organic LEDs and solid-state laser cooling.*

Physics Department, U.S. Naval Academy, Annapolis, MD 21402-5040; mungan@usna.edu