RELATIVE SPEEDS OF INTERACTING ASTRONOMICAL BODIES

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Abstract

Simultaneous conservation of linear momentum and of mechanical energy can be used to calculate the relative speed of an isolated pair of astronomical bodies as a function of the distance separating them. An exact treatment is straightforward and has application to such contemporary topics as the launch velocities of rockets, and collisions between an asteroid and the Earth. In contrast, when these topics are discussed in introductory physics courses, an infinite-Earth-mass approximation is typically invoked. In addition to being unphysical, this denies students an opportunity for a richer exploration of the conservation laws of mechanics.

Introduction

Consider two spherically symmetric bodies 1 and 2 moving through space and interacting with each other gravitationally but not subject to any other forces (such as gravitational forces from other bodies or thrusts from propulsion systems). This configuration is depicted in Fig. 1. Object 1 has mass m_1 and velocity $\boldsymbol{\upsilon}_1$, while the second body has mass m_2 and velocity $\boldsymbol{\upsilon}_2$. The distance between the centers of the two objects is r. Then conservation of linear momentum implies that

$$m_1\boldsymbol{\upsilon}_{1\mathrm{i}} + m_2\boldsymbol{\upsilon}_{2\mathrm{i}} = m_1\boldsymbol{\upsilon}_{1\mathrm{f}} + m_2\boldsymbol{\upsilon}_{2\mathrm{f}}\,, \tag{1}$$

while conservation of energy states

$$\frac{1}{2}m_1\upsilon_{1\mathrm{i}}^2 + \frac{1}{2}m_2\upsilon_{2\mathrm{i}}^2 - \frac{Gm_1m_2}{r_\mathrm{i}} = \frac{1}{2}m_1\upsilon_{1\mathrm{f}}^2 + \frac{1}{2}m_2\upsilon_{2\mathrm{f}}^2 - \frac{Gm_1m_2}{r_\mathrm{f}}\,, \tag{2}$$

where the subscripts "i" and "f" denote initial and final instants in time, and G is the universal gravitational constant.

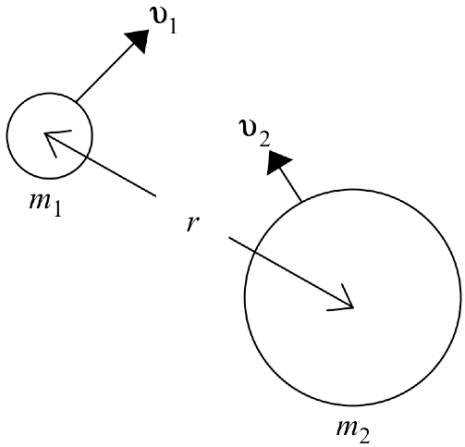

Fig. 1. Geometry of two objects moving under the influence of their mutual gravitational attraction. Object 2 is represented as being larger than object 1 because we will think of 2 as being the Earth and 1 as a meteoroid or rocket. Since object 1 is the body whose motion is of primary interest, we define the relative velocity to specify its velocity relative to that of object 2.

Define the relative speed υ of the two objects as the magnitude of the relative velocity vector $\boldsymbol{\upsilon} \equiv \boldsymbol{\upsilon}_1 - \boldsymbol{\upsilon}_2$. Then Eqs. (1) and (2) can be combined (see the Appendix) to find

$$\upsilon_{\mathrm{f}} = \sqrt{\upsilon_{\mathrm{i}}^2 + 2GM\left(\frac{1}{r_{\mathrm{f}}} - \frac{1}{r_{\mathrm{i}}}\right)} \tag{3}$$

where $M \equiv m_1 + m_2$ is the total mass of the system. It is worth emphasizing that this result is *independent* of the directions of the initial and final relative velocity vectors;[1] they need not be directed one-dimensionally along the line joining the two bodies. This angle independence is akin to the fact that we can use energy conservation to predict the landing speed of a projectile tossed off a building of known height with a known launch speed *regardless* of the launch angle.

Also note that Eq. (3) can be generalized to the motion of particles under the action of other mutual inverse-square forces. For example, it can be applied to the electrostatic interaction of two charges q_1 and q_2 if we replace G by $-k(q_1 / m_1)(q_2 / m_2)$ where k is the Coulomb constant.

Three applications of Eq. (3)

An immediate application of this result is to compute the escape speed. This is the minimum initial speed that enables the two objects to climb out of each other's gravitational potential wells, or in other words that causes their relative speed to fall to zero as they approach infinite separation. Putting $\upsilon_{\mathrm{f}} = 0$ at $r_{\mathrm{f}} = \infty$ implies that the launch speed $\upsilon_{\mathrm{i}} = \upsilon_{\mathrm{esc}}$ is

$$\upsilon_{\mathrm{esc}} = \sqrt{\frac{2GM}{R}} \tag{4}$$

where $R = r_{\mathrm{i}}$ is the distance between the centers of the two objects at launch. (In the case of a terrestrial rocket, R is the distance of the spacecraft from the center of the Earth after the engines have been shut off and the booster stages ejected. Unless one is launching off a high-orbit platform, R is essentially equal to Earth's radius in this case.) Note that Eq. (4) differs from the usual approximate textbook expression[2] in that M is the sum of both masses, rather than just m_2 alone. This difference is of negligible consequence when launching a rocket off Earth, but can be significant in the case of two astronomical bodies of more comparable mass trying to escape from each other (e.g., the Moon's original breakaway from the Earth, or the response of a pair of orbiting bodies after a third body sweeps past or collides into one of them).

Another important application of Eq. (3) is to calculate the impact speed of a meteoroid (object 1) striking Earth (object 2). In that case, the final distance is Earth's radius, $r_{\mathrm{f}} = R_{\mathrm{E}} = 6380\ \mathrm{km}$. Suppose the meteoroid is initially detected when it is far from the Earth, $r_{\mathrm{f}} / r_{\mathrm{i}} \approx 0$, and that it is then traveling at about the same speed as the Earth because of the Sun's gravitational pull, $\upsilon_{1\mathrm{i}} = \upsilon_{2\mathrm{i}} = \upsilon_{\mathrm{E}}$ where Earth's orbital speed about the Sun is $\upsilon_{\mathrm{E}} = (Gm_{\mathrm{S}} / R_{\mathrm{ES}})^{1/2} = 29.8\ \mathrm{km/s}$. (Here m_{S} is the solar mass and R_{ES} is one astronomical unit or 150 million kilometers. This expression is derived by setting the Sun-Earth gravitational force $Gm_{\mathrm{S}}m_{\mathrm{E}} / R_{\mathrm{ES}}^2$ equal to the product of Earth's mass m_{E} and centripetal acceleration $\upsilon_{\mathrm{E}}^2 / R_{\mathrm{ES}}$.) If we take the dot product of the expression $\boldsymbol{\upsilon}_{\mathrm{i}} = \boldsymbol{\upsilon}_{1\mathrm{i}} - \boldsymbol{\upsilon}_{2\mathrm{i}}$ with itself, we get $\upsilon_{\mathrm{i}}^2 = 2\upsilon_{\mathrm{E}}^2(1-\cos\theta)$ where θ is the angle between the initial directions of travel of the meteoroid and the Earth (so that $\boldsymbol{\upsilon}_{1\mathrm{i}} \bullet \boldsymbol{\upsilon}_{2\mathrm{i}} = \upsilon_{1\mathrm{i}}\upsilon_{2\mathrm{i}}\cos\theta = \upsilon_{\mathrm{E}}^2\cos\theta$). Equation (3) now becomes

$$\upsilon_{\mathrm{f}} = \sqrt{2\upsilon_{\mathrm{E}}^2\left(1-\cos\theta\right) + \upsilon_{\mathrm{esc,E}}^2} \tag{5}$$

where $\upsilon_{\mathrm{esc,E}} = (2Gm_{\mathrm{E}} / R_{\mathrm{E}})^{1/2} = 11.2$ km/s from Eq. (4), assuming the meteoroid is small. This impact speed υ_{f} is plotted in Fig. 2 as a function of the angle θ. The results are in good agreement with astronomical data collected for actual meteoroid arrival speeds at Earth's upper atmosphere.[3]

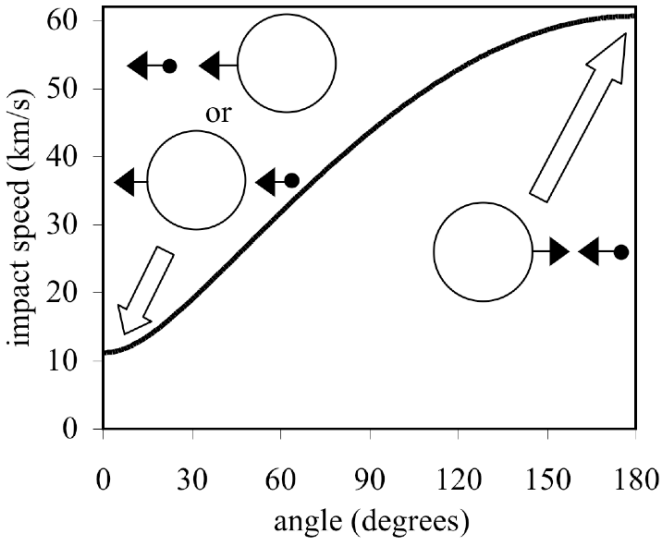

Fig. 2. Speed relative to Earth with which a meteoroid strikes our atmosphere (assuming the meteoroid is much smaller in size and mass than the Earth). The abscissa is the angle between Earth's orbital velocity (assumed fixed in direction) and the meteoroid's initial velocity. Large angles imply a head-on collision (so that the relative impact speed is approximately $2\upsilon_{\mathrm{E}}$), while small angles imply that either the asteroid strikes Earth from behind or vice-versa (so that the intercept along the ordinate is $\upsilon_{\mathrm{esc,E}}$), as the inset diagrams suggest.

A third important application of Eq. (3) is Solar System escape: How should a rocket be launched from Earth's surface so that it escapes both the Earth and Sun? A solution can be obtained by separately considering the escape from each of these bodies. This is called the "independent escape" approximation and its validity has been confirmed by numerical solution of the exact three-body problem.[4] Substitute into Eq. (3) the values $M \approx m_{\mathrm{E}}$, $r_{\mathrm{i}} = R_{\mathrm{E}}$, $r_{\mathrm{f}} = \infty$, launch speed $\upsilon_{\mathrm{i}} = \upsilon_{\mathrm{esc,SS}}$ relative to the Earth in order to escape from the Solar System, and final velocity $\boldsymbol{\upsilon}_{\mathrm{esc,S}}$ relative to the Sun [in order to escape from it with speed $\upsilon_{\mathrm{esc,S}} \equiv (2Gm_{\mathrm{S}} / R_{\mathrm{ES}})^{1/2} = 42.1$ km/s] which implies a final speed relative to Earth of $\upsilon_{\mathrm{f}} = \upsilon_{\mathrm{esc,S}} - \upsilon_{\mathrm{E}}$, assuming the rocket is launched in the

direction of Earth's orbital velocity $\boldsymbol{\upsilon}_{\mathrm{E}}$. (Earth's axial velocity can also be included if the rocket is launched eastward from the equator, as is often done for deep-space satellites.) Rearranging, one thereby obtains

$$\upsilon_{\mathrm{esc,SS}} = \sqrt{\left(\upsilon_{\mathrm{esc,S}} - \upsilon_{\mathrm{E}}\right)^2 + \upsilon_{\mathrm{esc,E}}^2} = 16.7 \text{ km/s}, \tag{6}$$

which is only a little larger than the escape speed from Earth alone! In particular, this speed is much smaller than the 42.1 km/s escape speed from the Sun starting *at rest* relative to the Sun at Earth's distance. Taking advantage of Earth's motion by launching in the direction of its orbital velocity confers a huge assist. (Additional boosts are possible using the gravitational slingshot effect as the spacecraft passes other planets on its way out of the Solar System.)

It is important to note that Eq. (6) *cannot* be obtained by assuming that the sum of the kinetic energy of the rocket (in Sun's frame of reference) and the potential energy of the rocket relative to the Sun and Earth is conserved, i.e., by letting *m* be the rocket's mass and writing

$$\frac{1}{2}m\left(\upsilon_{\mathrm{esc,SS}} + \upsilon_{\mathrm{E}}\right)^2 - \frac{Gm_{\mathrm{E}}m}{R_{\mathrm{E}}} - \frac{Gm_{\mathrm{S}}m}{R_{\mathrm{ES}}} \stackrel{?}{=} 0 \Rightarrow \upsilon_{\mathrm{esc,SS}} \stackrel{?}{=} \sqrt{\upsilon_{\mathrm{esc,S}}^2 + \upsilon_{\mathrm{esc,E}}^2} - \upsilon_{\mathrm{E}} \tag{7}$$

which does not agree with Eq. (6). The error is that the change in Earth's kinetic energy (in Sun's frame of reference) has been neglected. In the solar frame the Earth is moving, and the rocket is exerting a gravitational force on it in its direction of motion. Therefore work is done on the Earth, so that Earth's kinetic energy *must* increase. To put it another way, work (and hence the change in kinetic energy) are dependent on the reference frame of the observer. (In the terrestial frame, no work is done on the Earth.) It is only the *sum* of the work that the Earth and rocket do on each other that is frame independent (namely it equals the decrease in gravitational potential energy of the Earth-rocket system), as can be seen from Eq. (13) in the Appendix.

Conclusions

In summary, calculation of the relative speed between two astronomical bodies resulting from their mutual gravitational interaction (or between two point charges interacting electrostatically) is an elegant and useful application of the conservation laws of energy and momentum. The math is considerably simplified by measuring the positions and

velocities of the bodies in the center-of-mass reference frame, so that an exact derivation is within the scope of an introductory physics course. In contrast, standard treatments such as Eq. (7) only consider the mechanical energy of a single body. The latter approach not only violates conservation of linear momentum, it is not even properly defined because potential energy is actually a property of the *system* of interacting bodies and not of one body alone. That standard approach only gives the correct answer, such as Eq. (4), when one body is much more massive than the other *and* the velocities are measured in the rest frame of the heavy body, as required by the work-kinetic-energy theorem.

One application of the exact result given by Eq. (3) is to compute the escape speed of one body relative to another. It is given by Eq. (4) regardless of the sizes of the two objects (unlike the usual textbook expression). That explains why the formula is symmetric in the radii and masses of the two bodies. The escape speed for object 1 to escape from 2 must be the same as for body 2 to escape from 1.

A second application is the calculation of the speeds of meteoroids impacting the Earth. Most of the variation in speed here is due to the large range of angles between the meteoroid's and Earth's velocities, as can be seen from Eq. (5). A head-on collision approximately doubles the impact speed (ignoring the small boost due to Earth's gravity described by the $v_{\mathrm{esc,E}}$ term), while a rearward collision almost cancels it, assuming the Earth and meteoroid have similar initial speeds relative to the Sun.

Finally Eq. (6), describing escape from the Solar System, depends on three separate speeds: the escape speed from Earth's surface, the escape speed from the Sun at Earth's distance, and the orbital speed of Earth about the Sun. The two escape terms are added in quadrature because kinetic energy depends on speed squared. Meanwhile, the orbital speed is subtracted from the solar escape speed because Earth's motion about the Sun boosts the rocket toward escape, provided one launches in the direction that takes advantage of this assist. In fact, Earth's orbital speed is 71% $(2^{-1/2})$ of the required escape speed from the Sun, which explains why Solar System escape is actually dominated by escape from the Earth.

Appendix—Derivation of Eq. (3)

The simultaneous solution of Eqs. (1) and (2) is simplified by wise choice of coordinate system. Since the two bodies 1 and 2 are isolated from external forces, the total linear momentum of the system is

conserved, and hence the center of mass has constant velocity. We can thus choose the origin to be fixed at the center of mass and to move with it, which properly defines an inertial reference frame. In that case, the total linear momentum of the system is always zero, and Eq. (1) implies that

$$m_1\vec{\upsilon}_{1\mathrm{i}} = -m_2\vec{\upsilon}_{2\mathrm{i}} = -m_2\left(\vec{\upsilon}_{1\mathrm{i}} - \vec{\upsilon}_{\mathrm{i}}\right) \tag{8}$$

since $\vec{\upsilon}_{\mathrm{i}}$ is the initial velocity of object 1 relative to 2. This equation can be rearranged to obtain

$$\vec{\upsilon}_{1\mathrm{i}} = \frac{m_2}{M}\vec{\upsilon}_{\mathrm{i}} \tag{9}$$

where M is the total mass of the system. Similar reasoning for the second body gives

$$\vec{\upsilon}_{2\mathrm{i}} = -\frac{m_1}{M}\vec{\upsilon}_{\mathrm{i}}\,. \tag{10}$$

(The minus sign here reflects the one in the definition of the relative velocity.) Equations (9) and (10) imply that the initial kinetic energy of the system is

$$\frac{1}{2}m_1\upsilon_{1\mathrm{i}}^2 + \frac{1}{2}m_2\upsilon_{2\mathrm{i}}^2 = \frac{1}{2}\mu\upsilon_{\mathrm{i}}^2 \tag{11}$$

where $\mu \equiv m_1 m_2 / M$ is called the *reduced mass* of the system. (The reason for this name is that it is a quantity with units of mass and is smaller than both m_1 and m_2. One can think of the total mass as the "series" sum of the individual masses, $M = m_1 + m_2$, while the reduced mass is the "parallel" sum, $1/\mu = 1/m_1 + 1/m_2$.) In like fashion, the final kinetic energy is

$$\frac{1}{2}m_1\upsilon_{1\mathrm{f}}^2 + \frac{1}{2}m_2\upsilon_{2\mathrm{f}}^2 = \frac{1}{2}\mu\upsilon_{\mathrm{f}}^2\,. \tag{12}$$

Equation (2) in the main text can therefore be compactly expressed in terms of the relative speeds and distances as

$$\frac{1}{2}\mu\upsilon_{\mathrm{i}}^{2} - \frac{Gm_1m_2}{r_{\mathrm{i}}} = \frac{1}{2}\mu\upsilon_{\mathrm{f}}^{2} - \frac{Gm_1m_2}{r_{\mathrm{f}}} \,. \tag{13}$$

Finally, noting that $m_1m_2 = M\mu$, Eq. (13) can be immediately rearranged to give Eq. (3).

References

1. In contrast, if we wished to determine the final velocities, $\boldsymbol{\upsilon}_{1\mathrm{f}}$ and $\boldsymbol{\upsilon}_{2\mathrm{f}}$, and not merely the relative speed, then we would have 6 unknowns (i.e., three components of each velocity). Equations (1) and (2) only provide 4 independent relationships (3 components of linear momentum plus 1 scalar energy expression). One would then need to invoke conservation of angular momentum to get 2 more relations. The resulting analysis is no longer introductory level but instead invokes non collinear scattering theory, treated in texts such as S.T. Thornton and J.B. Marion, *Classical Dynamics of Particles and Systems*, 5th ed. (Thomson, Belmont CA, 2004).

2. See for example R.A. Serway and J.W. Jewett, Jr., *Principles of Physics*, 4th ed. (Thomson, Belmont CA, 2006), p. 348.

3. A. Díaz-Jiménez and A.P. French, “A note on ‘Solar escape revisited’,” *Am. J. Phys.* **56**, 85–86 (1988).

4. N.J. Harmon, C. Leidel, and J.F. Lindner, “Optimal exit: Solar escape as a restricted three-body problem,” *Am. J. Phys.* **71**, 871–877 (2003). Also see A.Z. Hendel and M.J. Longo, “Comparing solutions for the solar escape problem,” *Am. J. Phys.* **56**, 82–85 (1988).