ORBITS ON A CONCAVE FRICTIONLESS SURFACE*

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ABSTRACT

The equations of motion of a puck sliding frictionlessly inside a parabolic bowl can be straightforwardly deduced using the conservation laws of mechanical energy and angular momentum. But the solution of these equations requires that they be recast into the form of Newton's second law. The simple example of a ball in vertical freefall illustrates why this is necessary and how to perform the conversion. The method is then applied to the richer problem of a puck gliding on a paraboloidal surface for which the nonlinear equations require numerical solution. A rich variety of orbital patterns of the puck is found.

Introductory Example of One-Dimensional Freefall

Consider a ball thrown straight upward (which will be designated as the +*z* direction) from the origin with an initial velocity of υ_{0z}. Let's find its resulting path of motion $z(t)$ in the absence of air resistance. Because mechanical energy is conserved (for the system of ball and earth), the sum of the kinetic (*K*) and gravitational potential (*U*) energies at any point on the ball's path can be written as

$$K + U = K_0 + U_0 , \tag{1}$$

where the subscript "0" throughout this article denotes the initial instant $t = 0$. Choosing the gravitational reference position to be at the origin and assuming the ball's altitude never gets large compared to Earth's radius, Eq. (1) becomes

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$$\tfrac{1}{2}m\upsilon_z^2 + mgz = \tfrac{1}{2}m\upsilon_{0z}^2 + 0 \tag{2}$$

where m is the mass of the ball, $g = 9.80$ N/kg is Earth's surface gravitational field, and $\upsilon_z \equiv dz/dt$ is the velocity of the ball. Equation (2) can be rearranged as

$$\left(\frac{dz}{dt}\right)^2 = \upsilon_{0z}^2 - 2gz\,. \tag{3}$$

Unfortunately this equation is double-valued and cannot be uniquely solved as written. At any given height z, there are two solutions, one corresponding to the ball traveling upward with a positive velocity and the other to the ball descending with an equal-magnitude negative velocity. In order to circumvent this ambiguity, the time derivative of Eq. (3) can be taken to produce the readily solvable form

$$2\left(\frac{dz}{dt}\right)\left(\frac{d^2z}{dt^2}\right) = -2g\left(\frac{dz}{dt}\right) \quad \Rightarrow \quad a_z = -g \tag{4}$$

where $a_z \equiv d^2z/dt^2$ is the acceleration of the ball. The final equation is simply Newton's second law with the ball's mass divided out of both sides. Integrating it twice with respect to time gives the expected solution $z = \upsilon_{0z}t - \tfrac{1}{2}gt^2$.

In this easy example, one could alternatively solve Eq. (3) by manually changing the sign of the square root of the right-hand side of the equation after the topmost point of the trajectory is reached by the ball. But this procedure becomes cumbersome if the orbit has a large number of turning points. In such a case, it is easier to differentiate the energy equation with respect to time and then solve the resulting second-order equation, as was done above.[1] Let's now apply this method to the richer problem of interest in this paper.

Orbiting On a Frictionless Parabolic Surface

Suppose that a puck is sliding frictionlessly about the bottom of a concave bowl which has cylindrical symmetry around the vertical axis z, described by the parabolic cross-sectional profile

$$z = \tfrac{1}{2} k\rho^2 \tag{5}$$

using cylindrical coordinates, ρ, ϕ, z, as illustrated in Fig. 1. The origin of the coordinate system is at the vertex of the bowl, and a factor of ½ has been included in Eq. (5) to avoid factors of 2 that otherwise arise.

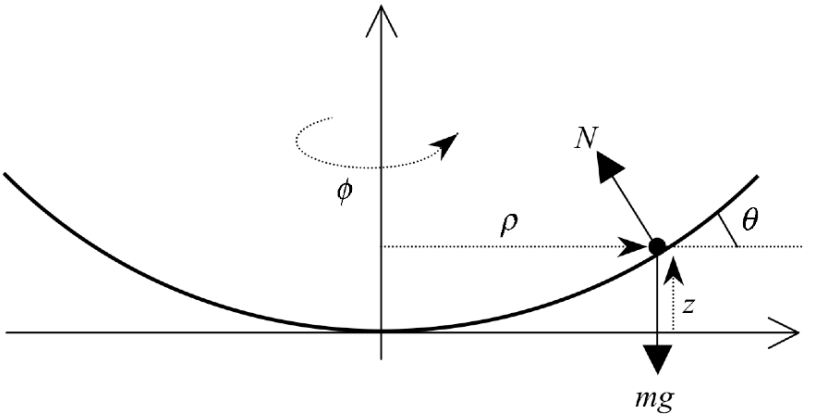

Fig. 1. Free-body diagram indicating the normal (N) and gravitational forces (mg) acting on the puck (indicated by the dot) when it is located at arbitrary position (ρ, ϕ, z). The paraboloidal surface has slope $\tan\theta$ in the radial direction.

Energy conservation implies that

$$\tfrac{1}{2} m\upsilon^2 + mgz = \text{constant} \quad \Rightarrow \quad \upsilon^2 + gk\rho^2 = \text{constant} \tag{6}$$

where $\upsilon^2 = \upsilon_\rho^2 + \upsilon_\phi^2 + \upsilon_z^2$ and the first constant has been divided by a factor of ½ m to get the second constant. Here $\upsilon_\rho \equiv d\rho / dt$, $\upsilon_\phi = \rho\omega$ (where $\omega \equiv d\phi / dt$ is the puck's angular velocity about the axis of symmetry), and $\upsilon_z \equiv dz / dt = k\rho\, d\rho / dt$. Since neither gravity nor the normal force exerts a vertical torque on the puck about the origin, the z-component of the angular momentum is constant and therefore equals its initial value,

$$L_z = m\rho^2\omega = m\rho_0^2\omega_0 \quad \Rightarrow \quad \omega = \left(\frac{\rho_0}{\rho}\right)^2 \omega_0 . \tag{7}$$

Inserting this expression into the speed squared in Eq. (6) and taking the time derivative to eliminate the constant yields

$$\frac{d}{dt}\left[\left(1+k^2\rho^2\right)\upsilon_\rho^2+\rho_0^4\omega_0^2\rho^{-2}+gk\rho^2\right]=0\,. \tag{8}$$

The derivative can be performed and a factor of $2\upsilon_\rho$ divided out of every term, in analogy to how Eq. (4) was obtained from Eq. (3), to get

$$a_\rho=\frac{\rho_0^4\omega_0^2\rho^{-3}-k\rho\left(g+k\upsilon_\rho^2\right)}{1+k^2\rho^2} \tag{9}$$

where $a_\rho \equiv d^2\rho/dt^2$. This equation can also be obtained (but with considerably more effort) by finding the two orthogonal surface tangential components (to avoid the unknown normal force) of Newton's second law in cylindrical coordinates.

One final step is helpful before proceeding to a computer solution. Equation (9) can be rewritten in terms of the dimensionless variables $R \equiv k\rho$ and $T \equiv \omega_0 t$ as

$$\frac{d^2R}{dT^2}=\frac{R_0^4-R^4\left[C+(dR/dT)^2\right]}{R^3\left(1+R^2\right)} \tag{10}$$

where $C \equiv gk/\omega_0^2$ is a dimensionless constant. This is a second-order differential equation to be solved with the initial conditions $R(0) \equiv R_0 = k\rho_0$ and $V(0) \equiv V_0 = k\upsilon_{\rho 0}/\omega_0$ where $V \equiv dR/dT$. Suppose the initial angular velocity is chosen so that the puck travels in a stable counter-clockwise circular orbit around the vertex of the bowl. The puck is then given a quick push toward the rim of the dish. The push provides a radial impulse to the puck. (Note that a radial impulse does not change the value of L_z.) Prior to the push, R must have the constant value R_0 so that dR/dT and d^2R/dT^2 are both zero, and Eq. (10) therefore implies that $C=1$. In turn this result requires that $\omega_0=(gk)^{1/2}$ *regardless* of the puck's position on the surface. This is a special property of a parabolic dish and is the reason that the surface of a rotating liquid settles into a paraboloidal shape, a property that can be exploited to make the primary collecting mirror of a reflecting telescope.[2]

Once Eq. (10) is solved for $R(T)$, it can be substituted into Eq. (7) written in the dimensionless form $d\phi/dT=(R_0/R)^2$. That result can

then be integrated to obtain $\phi(T)$ with the initial condition $\phi(0)=0$ (by choosing the *x*-axis to point to the puck's position at the instant of application of the radial impulse). The results can then be plotted parametrically to give an overhead view of the *xy*-coordinates of the puck in the dimensionless form

$$X = R\cos\phi \quad \text{and} \quad Y = R\sin\phi\,. \tag{11}$$

Here is the complete code we wrote to solve and plot the motion of the puck using the commercial software program Maple™ for the case $R_0 = 1 = V_0$, as graphed in Fig. 2(a):

```
R0:=1; V0:=1;
eqR:=diff(R(T),T,T)=(R0^4-R(T)^4*(1+diff(R(T),T)^2))/R(T)^3/(1+R(T)^2);
eqphi:=diff(phi(T),T)=(R0/R(T))^2;
sol:=dsolve({eqR,eqphi,R(0)=R0,phi(0)=0,D(R)(0)=V0},{R(T),phi(T)},numeric);
r:=T–>rhs(sol(T)[2]); p:=T–>rhs(sol(T)[4]);
X:=T–>r(T)*cos(p(T)); Y:=T–>r(T)*sin(p(T));
plot(['X(T)','Y(T)',T=0..50*Pi],scaling=constrained);
```

By varying the initial values R_0 and V_0 in the first line, a rich variety of orbital patterns result; two further examples are plotted in panels (b) and (c) of Fig. 2, chosen to illustrate some common patterns. Our school has a site license for Maple™ and students are introduced to its use in their introductory calculus sequence and could be given the above code with which to experiment. At other schools, Mathematica™ or implementation of Euler's method in a spreadsheet such as Excel™ might be a better choice.[3] However the comparative simplicity of the code above makes this a good example with which to introduce students to algorithmic software packages.

Further insight into the puck's motion is obtained by making the radial impulse very weak, so that the circular orbit is only slightly perturbed.[4] In that case, it is easier to see the resulting small effect by jumping into a frame of reference that rotates with the puck's initial angular speed of ω_0. The *xy*-coordinates of the puck in this rotating frame can be computed using Eq. (11) provided we replace ϕ by $\phi - \omega_0 t \equiv \phi - T$. An example is plotted in Fig. 2(d). The puck starts on the *x*-axis at $(R_0, 0)$ and travels[5]

clockwise with very nearly uniform circular motion of dimensionless diameter V_0 at an angular frequency of $2\omega_0$. That is, the puck performs one clockwise orbit in the rotating frame during the time that the puck rotates counter-clockwise halfway around the bowl in the lab frame. This trajectory is a result of the Coriolis force which produces a rightward deflection of the puck in the rotating frame,[6] analogous to the rotation of hurricanes in the northern hemisphere of the earth. The radially outward centrifugal force is almost perfectly canceled by the inward component of the normal force.

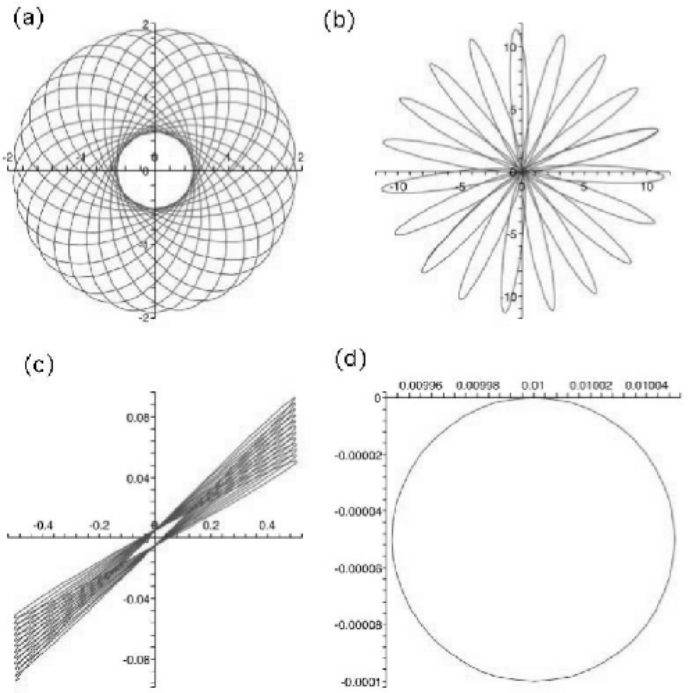

Fig. 2. Overhead views of the trajectory of the puck **(a)** in the lab frame for $R_0 = 1$ and $V_0 = 1$ over the interval $0 \le T \le 50\pi$; **(b)** in the lab frame for $R_0 = 1$ and $V_0 = 8$ over the interval $0 \le T \le 150\pi$; **(c)** in the lab frame for $R_0 = 0.05$ and $V_0 = 0.5$ over the interval $0 \le T \le 25\pi$; **(d)** in the rotating frame for $R_0 = 0.01$ and $V_0 = 0.0001$ over the interval $0 \le T \le \pi$ (in a highly magnified view).

For Further Investigation

At least two interesting lines of inquiry are left for future work, the first theoretical and the second experimental:

(1) Under what circumstances is the orbital motion closed? Close inspection of Fig. 2(b) indicates that the orbit appears to repeat after tracing out 19 lobes. In contrast, the pattern in Fig. 2(a) is starting over (after 25 time periods of $2\pi / \omega_0$) at a slightly shifted angular position. By writing $V \equiv dR / dT = (R_0 / R)^2 (dR / d\phi)$ and equating it to the positive square root of V from Eq. (6) as the puck travels from closest to farthest approach from the bowl's vertex, one can integrate to find an expression for $\Delta\phi$ along this path. The orbit is closed if $\pi / \Delta\phi$ is a rational number. (In particular if that number is an integer, then the orbit never crosses itself.) Similarly, Eq. (10) can be recast into an orbital differential equation for $R(\phi)$ rather than $R(T)$.

(2) To investigate experimentally the trajectories described in this paper, one could construct a parabolic "air hockey" table by drilling holes in a suitable dish and blowing air through them. Alternatively one could roll a marble on an old parabolic mirror or satellite television dish and modify the present theory to include frictional forces. (One could even spin the dish to keep the marble from slowing down.) For comparison, interesting effects occur when a ball rolls without slipping on the surface of a rotating flat plate,[7] on the inner surface of a vertical cylinder such as a golf cup,[8] on the surface of an elastic membrane,[9] or on the inner surface of a sphere.[10]

Acknowledgments

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References

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