

Density of Air down a Bore Hole—C.E. Mungan, Summer 2000

Multiple Choice: (A) A deep shaft is drilled into the earth starting at sea level and room temperature. Neglecting both the geothermal gradient and the weakening of the gravitational force with depth, and assuming air behaves like an ideal gas under all conditions, at about what depth will the density of air equal that of water?

- (i) 10 km (ii) 50 km (iii) 100 km (iv) 250 km (v) 500 km

(B) Same choices, but this time account for a typical geothermal profile for the rise in the ambient temperature with increasing depth.

Solution: Consider a pancake-shaped slab of air in the shaft having height dz . There are three forces per unit cross-sectional area on the slab: pressure P downward on the top surface, a larger pressure $P+dP$ upward on the bottom surface, and the downward force of gravity $\rho g dz$ where ρ is the density of the air at depth z and $g = 9.8 \text{ m/s}^2$ is the assumed constant acceleration due to gravity. Note that the third term is the usual hydrostatic gauge pressure which arises, for instance, in the derivation of the height of a barometric column. Since the air slab is in mechanical equilibrium, the upward force must balance the downward forces, so that

$$dP = \rho g dz. \quad (1)$$

On the other hand, the ideal gas law can be written in terms of mass density as

$$\rho \propto \frac{P}{T} \quad (2)$$

where T is the absolute temperature. (The constant of proportionality is in fact the molar mass of the gas divided by the universal gas constant, but that is irrelevant for the present purpose.)

In part (A) where the temperature is also taken to be constant, Eq. (2) becomes

$$\frac{P}{P_0} = \frac{\rho}{\rho_0} \quad (3)$$

where $P_0 = 101 \text{ kPa}$ and $\rho_0 = 1.29 \text{ kg/m}^3$ are the standard sea-level pressure and density of air at room temperature. (Unit consistency is obtained if z is expressed in km.) Taking the differential of both sides of Eq. (3) and substituting the result into the left-hand side of Eq. (1) gives

$$\frac{d\rho}{\rho} = \frac{\rho_0 g}{P_0} dz \Rightarrow \rho = \rho_0 \exp\left(\frac{\rho_0 g z}{P_0}\right). \quad (4)$$

This can easily be inverted to deduce that $\rho = 1000 \text{ kg/m}^3$ when $z = 53 \text{ km}$. Equation (4) gives the well-known exponential decrease in pressure with increasing altitude in an isothermal atmosphere.

With respect to part (B), a typical assumption is that the earth's temperature increases linearly with depth—denote this linear gradient by k in $^\circ\text{C/km}$. Hence, instead of Eq. (3) we must now write

$$\rho = \frac{\rho_0 T_0}{P_0} \cdot \frac{P}{T_0 + kz} \quad (5)$$

where room temperature is $T_0 = 300$ K. Again taking the differential of both sides and substituting the right-hand side of Eq. (1) for dP gives

$$\frac{d\rho}{\rho} = \left(\frac{\rho_0 g T_0}{P_0 k} - 1 \right) \frac{dz}{z + T_0/k} \Rightarrow \rho = \rho_0 \left(1 + \frac{kz}{T_0} \right)^\gamma \quad (6)$$

where $\gamma \equiv \rho_0 g T_0 / P_0 k - 1$. An approximate value for the continental geothermal gradient is

$$k = \begin{cases} 11 \text{ K/km} & \text{for 0 - 100 km depth} \\ 1 \text{ K/km} & \text{thereafter.} \end{cases}$$

(See <http://www.geol.umd.edu/~kaufman/ppt/lithosphere/sld016.htm> and <http://bors.tamu.edu/Lec6.htm>.) Consequently, using $k = 11$ K/km, we find the temperature, density, and pressure at 100 km depth to be $T_{100} = 1400$ K, $\rho_{100} = 53.2$ kg/m³ from Eq. (6), and $P_{100} = 19400$ kPa from Eq. (5). Notice that the large rise in temperature is keeping the density comparatively low despite the large increase in pressure. Equation (6) can now be inverted to obtain

$$z = \frac{T_{100}}{k'} \left[\left(\frac{\rho}{\rho_{100}} \right)^{1/\gamma'} - 1 \right] = 120 \text{ km} \quad (7)$$

where $\gamma' \equiv \rho_{100} g T_{100} / P_{100} k' - 1$ with $k' = 1$ K/km. With the thermal gradient now small, the density rises very rapidly to $\rho = 1000$ kg/m³; if instead k had remained at the value of 11 K/km, the density of water would not be attained until a depth of 400 km.

Comments: At a depth of $z = 120$ km $\equiv 6380$ km $- r$, $g = 4\pi G \rho_E r / 3$ still has $6260 / 6380 = 98\%$ of its sea-level value, assuming the density of earth, ρ_E , to be uniform. Hence, the neglect of the weakening of the gravitational force with depth is reasonable. In contrast, air could not be expected to continue to behave as an ideal gas at these high densities. A nice Mathematica exercise might be to replace Eq. (2) with the Van der Waals equation of state. This multiple-choice problem was motivated by a submission to PHYS-L by John Mallinckrodt.