

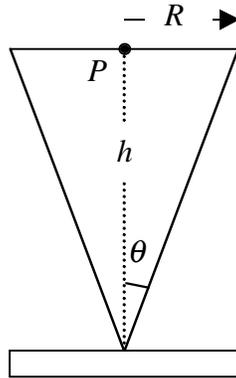
## Alternate Solution to “The Cone Artist”—C.E. Mungan, Spring 2002

An inverted glass cone with refractive index  $n$  and apex half-angle  $\theta$  has base radius  $R$ . The tip of the cone is centered on a plate of radius  $R$ . Find the minimum radius  $r$  of the uninverted image of the plate as viewed through the base of the cone. (Reference: *The Physics Teacher*, May 2002, p. 312.)

Note that in general there is also an inverted image if you look more or less straight down through the base, as the right-hand edge of the plate reflects off the left-hand sloped edge of the cone. But in this document, I treat the more complicated case of an uninverted image. Also note that since there is not a well-defined focal plane, different rays from a point on the object do not in general intersect. That is, as you tilt your head towards glancing angles relative to the base, the image of the plate appears to rotate. To get around this limitation, I specifically am treating the smallest uninverted image. Starting with your eye looking down along the axis of the cone and rotating your head around point  $P$  below, the minimum radius corresponds to the first image you see of the right-hand edge of the plate. Try it with an equilateral prism if you want to see what I mean!

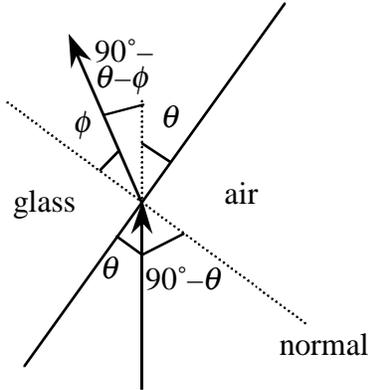
Let  $h$  be the height of the cone. From the geometry it follows that

$$h = R \cot \theta. \tag{1}$$



Now consider two rays beginning at the right-hand edge of the plate: ray 1 travels vertically upward, while ray 2 heads up and leftward at a small angle  $\delta$  to ray 1. Both rays strike the cone and are refracted twice. After exiting the glass, they extrapolate backward to intersect at the right-hand edge of the virtual image of the plate, thus defining the size of the image of interest. Hence, one method of solution is do a graphical ray tracing. Instead I will actually calculate the intersection point algebraically. Note that both rays travel through such a thin amount of glass that we can neglect their lateral displacements while inside the cone.

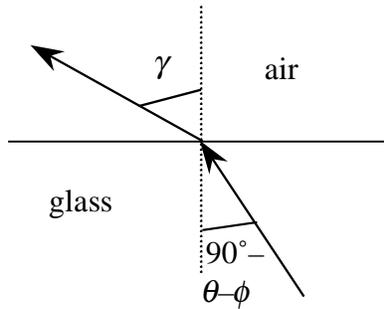
Ray 1 just barely strikes the sloped surface of the glass and bends toward the normal as sketched at the top of the next page.



From Snell's law we obtain

$$n \sin \phi = \sin(90^\circ - \theta) = \cos \theta. \quad (2)$$

The ray in the glass is then immediately incident on the base and bends away from the normal, i.e., further to the left.



Again from Snell's law we find

$$\sin \gamma = n \sin(90^\circ - \theta - \phi) = n \cos(\theta + \phi), \quad (3)$$

provided that the incident angle is sub-critical, i.e., that  $\gamma < 90^\circ$ . By substituting Eq. (2) for  $\phi$  into Eq. (3) and simplifying, one finds that the requirement for this to be true is that

$$n < \sqrt{1 + (1 + \tan \theta)^2}. \quad (4)$$

For example, if  $\theta = 30^\circ$  we must have  $n < 1.868$ .

In the first ray diagram above, we replace the incident beam for ray 2 by an arrow tilted leftward at angle  $\delta$  to the vertical. Hence, the incident angle becomes  $90^\circ - \theta - \delta$  (instead of  $90^\circ - \theta$ ) and the transmitted angle becomes say  $\phi - \Delta\phi$  (rather than  $\phi$ ). Thus Eq. (2) is now

$$n \sin(\phi - \Delta\phi) = \cos(\theta + \delta).$$

Expand both sides using the double-angle formulae, apply the small-angle approximations  $\sin \alpha \cong \alpha$  and  $\cos \alpha \cong 1$  to the small angles  $\delta$  and  $\Delta\phi$ , and simplify the result using Eq. (2) to obtain

$$\Delta\phi = \frac{\delta \sin \theta}{n \cos \phi}. \quad (5)$$

Similarly the second ray diagram applied to ray 2 involves changing the incident angle to  $90^\circ - \theta - \phi + \Delta\phi$  and the transmitted angle to say  $\gamma + \Delta\gamma$ . Therefore Eq. (3) becomes

$$\sin(\gamma + \Delta\gamma) = n \cos(\theta + \phi - \Delta\phi).$$

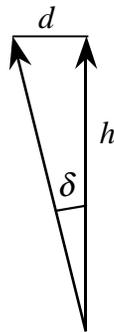
Running through the same sequence of operations used to get Eq. (5) gives

$$\Delta\gamma = \frac{n\Delta\phi \sin(\theta + \phi)}{\cos\gamma}.$$

Substituting Eq. (5) then results in

$$\Delta\gamma = \frac{\delta \sin\theta \sin(\theta + \phi)}{\cos\gamma \cos\phi}. \quad (6)$$

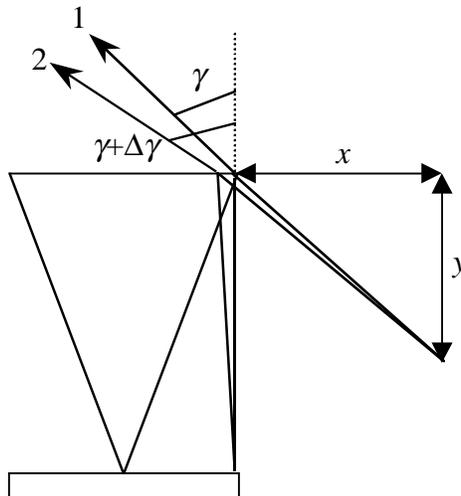
While we neglect the leftward displacements of the two rays inside the glass due to the refraction at the sloped surface of the cone, we cannot ignore the displacement of beam 2 due to its angular deviation  $\delta$ . (Otherwise, rays 1 and 2 after transmission through the cone would intersect at the corner rather than at the image!) Specifically, we compute the horizontal leftward displacement of ray 2 as if the prism were not there.



Using the small-angle approximation, this diagram implies that

$$d = h\delta. \quad (7)$$

Now we are ready to extrapolate backward the two rays exiting from the prism to find the location of the image of the right-hand edge of the plate.



We see from this diagram that

$$\tan\gamma = \frac{x}{y} \quad \text{and} \quad \tan(\gamma + \Delta\gamma) = \frac{x + d}{y}.$$

Solve the first equation for  $y$  and substitute it into the second equation along with Eq. (7) to obtain

$$\frac{\sin\gamma + \Delta\gamma\cos\gamma}{\cos\gamma - \Delta\gamma\sin\gamma} = \frac{x + h\delta}{x\cot\gamma}$$

after application of the double-angle formulae and small-angle approximations. Next, substitute Eq. (1) and rearrange to get

$$\frac{x}{R} = \frac{\delta\cot\theta(1 - \Delta\gamma\tan\gamma)}{\Delta\gamma(\cot\gamma + \tan\gamma)}.$$

Drop the second-order term involving the product of  $\delta$  and  $\Delta\gamma$  and substitute Eq. (6) to obtain the final result

$$\frac{x}{R} = \frac{\cot\theta\cos\gamma\cos\phi}{\sin\theta\sin(\theta + \phi)(\cot\gamma + \tan\gamma)}. \quad (8)$$

Together with Eqs. (2) and (3) this gives the solution. Algebra fanatics may wish to use these two equations to explicitly express the solution in terms of the given data only,

$$\frac{x}{R} = \frac{\csc\theta\left(n^2 - \cos^2\theta - \sin\theta\sqrt{n^2 - \cos^2\theta}\right)\left[1 - \cos^2\theta\left(\sqrt{n^2 - \cos^2\theta} - \sin\theta\right)^2\right]}{\sin\theta + \tan^2\theta\sqrt{n^2 - \cos^2\theta}}. \quad (9)$$

For example, using the values  $\theta = 30^\circ$  and  $n = 1.5$  implies that  $r \equiv R + x = 2.18R$ .

It is always valuable to check the limiting cases of a solution:

A. As  $\theta \rightarrow 90^\circ$ , Eq. (9) implies that  $x \rightarrow 0$ . This is expected because the cone became a thin flat plate with no lensing power.

B. As  $n \rightarrow 1$ , Eq. (9) implies that  $x \rightarrow 0$ . This is expected because the cone is index matched.

C. Finally as  $\theta \rightarrow 0^\circ$  while  $1 < n < \sqrt{2}$  so that Eq. (4) remains satisfied, Eq. (9) implies that  $x/R \rightarrow \infty$ . This is expected because the cylinder has become a slightly tapered optical fiber of infinitesimal radius.