Volume of a Hypersphere—C.E. Mungan, Summer 2000

Problem: Find the volume, V_n, of an n-dimensional hypersphere of radius R. The three lowest values of n are well known. In one dimension, we have a line segment extending a distance R in each direction, so that its length is $V_1 = 2R$. The case of $n = 2$ corresponds to a circle, whose area is $V_2 = \pi R^2$. Finally, $n = 3$ corresponds to a sphere of volume $V_3 = 4\pi R^3/3$. Derive a compact formula for the general case.

Hint: (Courtesy of Bob Sciamanda.) Evidently we can write the answer as $V_n(R) = R^n \upsilon_n$ where $\upsilon_n \equiv V_n(1)$ is the volume of a hypersphere of unit radius. Using Cavalieri's principle, we can calculate the volume of any closed solid as

$$V_3 = \int_{z_i}^{z_f} A(z)dz \tag{1}$$

where $A(z)$ is the cross-sectional area of a slab of thickness dz cut through the solid like a loaf of bread, and we integrate from z_i to z_f along any arbitrary axis z. In the present case, it is convenient to choose $z = R\cos\theta$ to be the conventional polar axis, where we integrate upward from $\theta_i = \pi$ to $\theta_f = 0$, and where the area of a slice through the sphere is $V_2(r)$ with $r = R\sin\theta$. Making these substitutions in Eq. (1) gives

$$V_3(R) = R^3 V_2(1)\int_0^{\pi} \sin^3\theta\, d\theta. \tag{2}$$

The reader is invited to perform this simple integral (by using the identity $\sin^2\theta = 1 - \cos^2\theta$) and check that it correctly gives 4/3, as implied in the problem statement above. Generalizing Eq. (2) to n dimensions immediately gives

$$\upsilon_n = \upsilon_{n-1}\int_0^{\pi} \sin^n\theta\, d\theta. \tag{3}$$

The problem is thus reduced to performing this definite integral and then finding a non-recursive formula for υ_n.

Solution: The integral is recognized as a beta function, which is easily recast in terms of gamma functions as

$$I_n \equiv \int_0^{\pi} \sin^n\theta\, d\theta = B\left(\frac{n+1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}. \tag{4}$$

Recall that the gamma function is a generalization of the factorial function, $\Gamma(n+1) = n\Gamma(n)$, where $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$. First consider the case where n is even. We then have

$$\begin{aligned}
\Gamma\left(\frac{n+1}{2}\right) &= \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \\
&= \frac{(n-1)(n-3)\cdots(1)}{2^{n/2}}\sqrt{\pi} \\
&= \frac{(n)(n-1)(n-2)(n-3)\cdots(2)(1)}{2^{n/2}(n)(n-2)\cdots(2)}\sqrt{\pi} \\
&= \frac{n!\sqrt{\pi}}{2^{n/2}2^{n/2}\left(\frac{n}{2}\right)!}
\end{aligned} \tag{5}$$

so that

$$I_n = \frac{n!\pi}{2^n\left[\left(\frac{n}{2}\right)!\right]^2} \quad \text{(if } n \text{ even).} \tag{6}$$

Similarly, if n is odd we have

$$\begin{aligned}
\Gamma\left(\frac{n+2}{2}\right) &= \left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \\
&= \frac{(n)(n-2)\cdots(1)}{2^{(n+1)/2}}\sqrt{\pi} \\
&= \frac{(n)(n-1)(n-2)(n-3)\cdots(2)(1)}{2^{(n+1)/2}(n-1)(n-3)\cdots(2)}\sqrt{\pi} \\
&= \frac{n!\sqrt{\pi}}{2^{(n+1)/2}2^{(n-1)/2}\left(\frac{n-1}{2}\right)!}
\end{aligned} \tag{7}$$

which implies that

$$I_n = \frac{2^n\left[\left(\frac{n-1}{2}\right)!\right]^2}{n!} \quad \text{(if } n \text{ odd).} \tag{8}$$

Notice from this last equation that $I_1 = 2$. Equation (3) will now correctly reproduce the one-dimensional value $\upsilon_1 = 2$ if we define $\upsilon_0 \equiv 1$. Next, it is left as a straightforward exercise for the reader to verify from Eqs. (6) and (8) that

$$I_n = \frac{2\pi}{n} \cdot \frac{1}{I_{n-1}} \tag{9}$$

regardless of whether n is even or odd. Substituting this into Eq. (3) results in a near-miraculous simplification,

$$\begin{aligned} \upsilon_n &= \upsilon_{n-1} I_n = \upsilon_{n-2} I_n I_{n-1} = \upsilon_{n-2} \frac{2\pi}{n} \\ &= \begin{cases} \dfrac{2\pi}{n} \dfrac{2\pi}{n-2} \cdots \dfrac{2\pi}{2} 1 & \text{if } n \text{ even} \\ \dfrac{2\pi}{n} \dfrac{2\pi}{n-2} \cdots \dfrac{2\pi}{3} 2 & \text{if } n \text{ odd} \end{cases} \end{aligned} \tag{10}$$

where I made use of the facts that $\upsilon_0 = 1$ and $\upsilon_1 = 2$ to terminate the even and odd recursions, respectively. It is easy to show that both of these cases can be written in the single compact way,

$$\upsilon_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \tag{11}$$

which completes the exercise. The first six values are tabulated below.

n	I_n	$\Gamma\left(\frac{n}{2}+1\right)$	υ_n
1	2	$\frac{1}{2}\sqrt{\pi}$	2
2	$\frac{1}{2}\pi$	1	π
3	$\frac{4}{3}$	$\frac{3}{4}\sqrt{\pi}$	$\frac{4}{3}\pi$
4	$\frac{3}{8}\pi$	2	$\frac{1}{2}\pi^2$
5	$\frac{16}{15}$	$\frac{15}{8}\sqrt{\pi}$	$\frac{8}{15}\pi^2$
6	$\frac{5}{16}\pi$	6	$\frac{1}{6}\pi^3$