# Legendre Transforms for Dummies—C.E. Mungan, Fall 2009

Suppose that we have a function of two independent variables, call it $f(x,y)$. Its differential is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \,. \tag{1}$$

Defining $u \equiv \partial f / \partial x$ and $w \equiv \partial f / \partial y$, Eq. (1) can be rewritten as

$$df = u\,dx + w\,dy \,. \tag{2}$$

We call *u* and *x* a *conjugate* pair of variables, and likewise *w* and *y*. Note that we can tell that *f* has independent variables *x* and *y* because the right-hand side of Eq. (2) is written in terms of differentials of those two variables.

Proceeding, use the product rule to compute the differential

$$d(wy) = y\,dw + w\,dy \tag{3}$$

and subtract this equation from Eq. (2) to get

$$dg = u\,dx - y\,dw \tag{4}$$

where I have introduced the *Legendre-transformed* function $g \equiv f - wy$. Noting that since we are taking differentials of *x* and *w*, this new function has those two as its independent variables so that we have ended up with $g(x,w)$.

To summarize, we have done a Legendre transformation from an original function $f(x,y)$ to a new function $g(x,w)$ by switching from variable *y* to its conjugate variable *w*. Of course, one could instead switch *x* to *u* to obtain $h(u,y)$ or one could switch both independent variables to get $j(u,w)$. We see therefore that for two variables, there are 4 possible variants on the function. To make contact with thermodynamics, we might call these various functions the *potentials*. If instead we have 3 independent variables, there are 8 different potentials, or in general there are $2^n$ potentials for a function of *n* variables, since each variable can be either one of a conjugate pair.

## Example 1: Legendre transform from internal energy *U* to enthalpy *H*

Suppose we have a system (such as a fixed quantity of a gas) whose independent variables are entropy *S* and volume *V*. Then according to the thermodynamic identity,

$$dU = T\,dS - P\,dV \tag{5}$$

where the temperature $T$ and pressure $P$ are therefore the variables conjugate to the entropy and volume, respectively. We wish to transform from $U(S,V)$ to a new thermodynamic potential $H(S,P)$. To apply the formalism developed above, we merely have to make a table of equivalences:

$f \equiv U$ (the original function)
$x \equiv S$ (the variable we are not switching)
$y \equiv V$ (the variable to be switched)
$w \equiv \left(\frac{\partial f}{\partial y}\right)_x = \left(\frac{\partial U}{\partial V}\right)_S = -P$ (the conjugate of the variable to be switched)

where the final partial derivative of $U$ was calculated from Eq. (5). The transformed function is

$$g = f - wy \equiv (U) - (-P)(V) = U + PV \equiv H(S,P)\,. \tag{6}$$

In accord with Eq. (4), its differential is

$$dH = T\,dS + V\,dP\,. \tag{7}$$

Formulas for the Gibbs free energy $G(T,P)$ and the Helmholtz free energy $F(T,V)$ can be similarly obtained.

### Example 2: Legendre transform from the Lagrangian *L* to the Hamiltonian *H*

Suppose we have a mechanical system with a single generalized coordinate $q$ and corresponding velocity $\dot{q}$. Then the Lagrangian is defined as the difference between the kinetic and potential energies, $L(q,\dot{q}) \equiv K - U$. We wish to transform to a new function $H(q,p)$ where $p$ is the canonical momentum. We again construct a table of equivalences:

$f \equiv L$ (the original function)
$x \equiv q$ (the variable we are not switching)
$y \equiv \dot{q}$ (the variable to be switched)
$w \equiv \left(\frac{\partial f}{\partial y}\right)_x = \left(\frac{\partial L}{\partial \dot{q}}\right)_q = p$ (the conjugate of the variable to be switched)

where the last equality is the *definition* of the canonical momentum. For example, if $q$ is the ordinary one-dimensional position $x$ of a particle of mass $m$, so that $\dot{q} = \upsilon$ is the velocity and $K = \frac{1}{2}m\upsilon^2$ is the kinetic energy, then

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial K}{\partial \upsilon} = m\upsilon = p\,, \tag{8}$$

noting that since potential energy $U$ is conservative, it cannot be a function of velocity (but only of position). Anyhow, returning to our table of equivalences, the transformed function is

$$g = f - wy \equiv L - p\dot{q} \tag{9}$$

which *defines* the negative of the Hamiltonian $H(q,p)$. The minus sign is an inconsequential historical accident. For the simple example above,

$$L - p\dot{q} = \left(\tfrac{1}{2}m\upsilon^2 - U\right) - (m\upsilon)(\upsilon) = -(K+U) = -H \tag{10}$$

since the Hamiltonian in this simple case is the total energy of the system.

*Endnote:* An alternative way to introduce the Legendre transform uses a graphical method. For a recent review of it, see R.K.P. Zia, E.F. Redish, and S.R. McKay, “Making sense of the Legendre transform,” Am. J. Phys. **77**, 614–622 (July 2009).