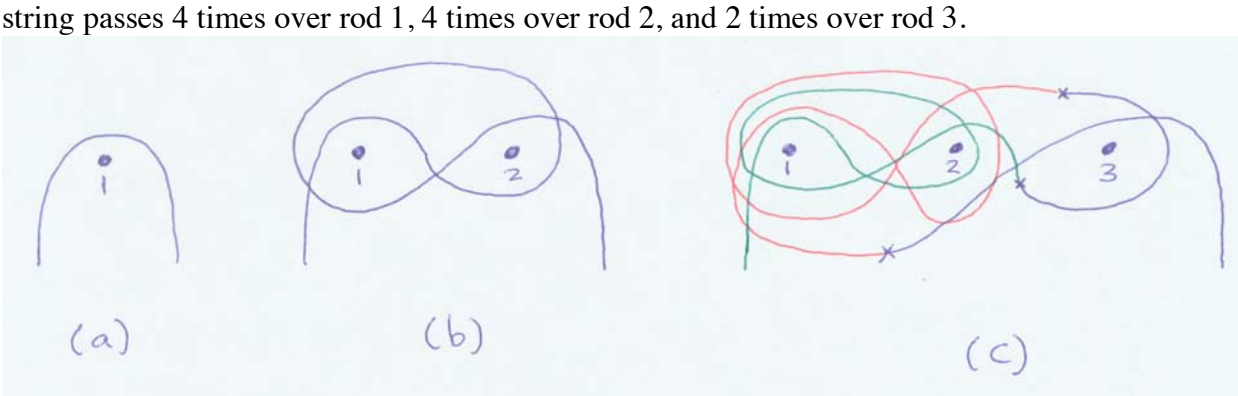
Looped String Puzzle—C.E. Mungan, Fall 2005

A set of n rods are arranged in a horizontal row with their axes parallel. A loaded loop of string is cleverly suspended from the rods in such a manner that no matter which rod is pulled out of the row, the loop falls to the floor. How is the string wound around the rods?

The solution for $n = 1$ is obvious and sketched in panel (a) below. One simply makes one half-winding forward over rod 1, i.e., the solution requires $N(1) = 1$ moves.

Suppose that we can find a solution for $n = 2$ for which the ends of the rope do not hang between the rods but before rod 1 and after rod 2. Then the two rods must be equivalent and the string pattern must be symmetric. Each half-winding must either go over one or both rods. Without too much trial and error, one discovers the pattern in panel (b). This can be wound by passing the loop entirely over rod 1 and hooking it on rod 2. Then take one of the two dangling loop strings and hook it under rod 1 onto rod 2. One might describe the sequence using the end of an unlooped string as follows: (1) pass over rod 1 in the forward direction, (2) under rod 2 forward, (3) over 2 and 1 backward, (4) under 1 forward, and (5) over 2 forward. In other words, this requires $N(2) = 5$ half-windings, defining a "half-winding" every time the string starts and ends on the horizontal plane in which the rods are located.

Armed with the result for two rods, one can build up the pattern for any number of rods recursively. For example, suppose we know how to do it for n–1 rods and want to do it for n rods. We proceed as follows. Let group 1 refer to the first n–1 rods, and group 2 to the last rod. Simply build the 2-rod pattern around these two groups, except that each of the two times that the string needs to go over top of the group 1 rods you have to build the n–1 pattern so that any one rod out of group 1 will release the string. By way of a concrete example, consider the solution for $n = 3$ sketched in panel (c). The first pass over group 1 is the green pattern (requiring n–1 moves), then we make a half-loop over group 2 (one move in blue), then we make a backward pass over group 1 in red (another n–1 moves), and finally we make the last two moves of the original 2-rod pattern (under group 1 and over group 2 in the forward direction in blue) to finish. If you are trying to actually wind pattern (c), it is a useful check to note that the string passes 4 times over rod 1, 4 times over rod 2, and 2 times over rod 3.



Consequently the number of half-windings required for n rods is

$$N(n) = N(n-1) + 1 + N(n-1) + 2 \,. \tag{1}$$

Initializing the recursion using $N(1) = 1$, one finds

$$N(n) = 2^{n-1} + 3\sum_{m=0}^{n-2} 2^m = 2^{n+1} - 3. \tag{2}$$

For example the pattern in panel (c) above requires 13 moves.

Adam Lewandowski has suggested a more formal and elegant way to describe the windings as follows. Let the operation corresponding to passing the string forward over the top of the first rod be denoted by matrix operator A, forward over the top of the second rod by B, and so on. Then the inverse matrix A^{-1} corresponds to passing the end of the string backward over the top of the first rod, and similarly for the other rods. We do not need to separately label half-windings of the string under a rod because these are uniquely specified by how the string passes over the rods. For example, AC^{-1} means forward over rod 1, forward under rods 2 and 3, and then backward over rod 3; the free ends of the string hang before the first rod, and between the second and third rods in this case.

Now my windings in panel (b) above can be represented as $AB^{-1}A^{-1}B$. The effect of removing a rod is simply to substitute the identity matrix I for every occurrence of the matrix corresponding to passage over that rod. For example, if we remove the second rod, we get

$$AI^{-1}A^{-1}I = AA^{-1} = I \tag{3}$$

and this unit result implies that the loop falls free. As a second example, my solution for 3 rods in panel (c) is

$$\left(AB^{-1}A^{-1}B\right)C^{-1}\left(AB^{-1}A^{-1}B\right)^{-1}C \tag{4}$$

where I've grouped the green windings in the first parentheses, the red windings in the second, and the blue windings outside them. Grouping the terms in this way not only makes the logic of the backward pass (in red) over the first pair of rods evident, but it makes it easy to see at a glance that Eq. (4) reduces to the identity matrix whenever any one of the three matrices A, B, or C is replaced by a unit matrix.

Another advantage of the matrix representation is that it makes it clear that there are other possible solutions. For example, $ABA^{-1}B^{-1}$ is another possible set of windings for the case of two rods, although it lacks the pleasing symmetry of my solution in panel (b) above. The reader is invited to try actually winding this alternative solution to see that it does in fact work.

Note that the minimum number of matrices required to describe a solution for n rods is given by Pascal's n-th diamond number. For example, Pascal's 2nd diamond (obtained by reflecting his 2nd triangle about its bottom edge) is drawn to the right and the sum of all numbers in it is 10, which is the number of matrices in Eq. (4). Thanks to Tim Royappa for pointing out this interesting tidbit.

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{array}$$