

Internal Energy of a Nuclear Gas—C.E. Mungan, Spring 2000

Consider a gas of nucleons comprising a nucleus of mass number A . Given the ratio T / T_F where T is the nuclear temperature and T_F is its Fermi temperature, estimate $\bar{\epsilon} / \epsilon_F$ where $\bar{\epsilon}$ is the internal energy per nucleon and $\epsilon_F \equiv kT_F$ is the Fermi energy.

For simplicity, I will assume that the nucleus is self-conjugate, i.e., that it is composed of equal numbers of protons and neutrons, so that $A = 2Z$ where Z is the atomic number. I will also approximate the mass of either a proton or a neutron to be

$$m = 1 \text{ amu} = \frac{1 \text{ g / mol}}{6.022 \times 10^{23} / \text{mol}}. \quad (1)$$

Finally, I will take the nucleus to be an infinite spherical well potential with a radius of

$$R = A^{1/3} R_0 \quad (2)$$

where $R_0 = 1.2$ fm, as given for example by Cutnell & Johnson Eq. (31.2). The volume of the nucleus is thus

$$V = \frac{4}{3} \pi A R_0^3. \quad (3)$$

With these simplifying assumptions, we see that we have two independent Fermi gases (that of the protons and that of the neutrons) which have identical thermodynamic properties.

Introducing the nonrelativistic density of states and the Fermi-Dirac distribution function,

$$g(\epsilon) = AC\sqrt{\epsilon} \quad \text{and} \quad f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} \quad \text{where} \quad C = \frac{16\pi^2}{3h^3} (2mR_0^2)^{3/2}, \quad (4)$$

where μ is the chemical potential, enables us to write down two equations. The total number of protons is

$$Z = \int_0^{\infty} g(\epsilon) f(\epsilon) d\epsilon. \quad (5)$$

Evaluating this at $T = 0$ gives the Fermi energy as

$$\epsilon_F = \frac{h^2}{32mR_0^2} \left(\frac{9}{\pi^2} \right)^{2/3} = 34 \text{ MeV} \quad \Rightarrow \quad T_F = 3.9 \times 10^{11} \text{ K} \quad (6)$$

for the nucleus of ANY element! This is the maximum kinetic energy of a nucleon because we have implicitly set the ground state energy to zero, subtracting off any zero-point or rest energy. We can thus compute the maximum speed of a nucleon relative to that of light, v/c , either from special relativity as

$$\varepsilon_F + mc^2 = \frac{mc^2}{\sqrt{1 - (v/c)^2}} \Rightarrow v = 0.26c \quad (7a)$$

or classically as

$$\varepsilon_F = \frac{1}{2}mv^2 \Rightarrow v = 0.27c, \quad (7b)$$

thereby justifying our use of nonrelativistic dynamics in Eq. (4). To find the chemical potential at nonzero temperatures, we need to evaluate Eq. (5) numerically, i.e., solve

$$\frac{2}{3t^{3/2}} = \int_0^\infty \frac{x^{1/2} dx}{e^{x-u/t} + 1}. \quad (8)$$

for $u \equiv \mu / \varepsilon_F$ given $t \equiv T / T_F$. (Here I changed variables of integration to $x \equiv \beta\varepsilon$.) For example, putting this into Mathcad, I find by trial and error that $u = 0.743$ when $t = 0.5$, and $u = -0.021$ when $t = 1$. (Of course, $u = 1$ when $t = 0$.) That is, the chemical potential falls to zero at very nearly the Fermi temperature. The total internal energy of the protons is

$$Z\bar{\varepsilon} = \int_0^\infty \varepsilon g(\varepsilon) f(\varepsilon) d\varepsilon \Rightarrow \frac{\bar{\varepsilon}}{\varepsilon_F} = 1.5t^{5/2} \int_0^\infty \frac{x^{3/2} dx}{e^{x-u/t} + 1}. \quad (9)$$

For example, I find $\bar{\varepsilon} / \varepsilon_F = 1.021$ when $t = 0.5$, and $\bar{\varepsilon} / \varepsilon_F = 1.697$ when $t = 1$. (One can also easily show that $\bar{\varepsilon} / \varepsilon_F = 0.6$ when $t = 0$.) That is, when the nuclear temperature is raised from room temperature to the Fermi temperature, the internal energy per nucleon increases by 180% from 20 to 57 MeV.

Incidentally, we see here the limited validity of the nuclear potential introduced in connection with Eq. (2). Even the low-temperature energy per nucleon of 20 MeV is larger than the experimental binding energies of all known atoms! That is, most of the nucleons have so much kinetic energy that the nuclei would fly apart.