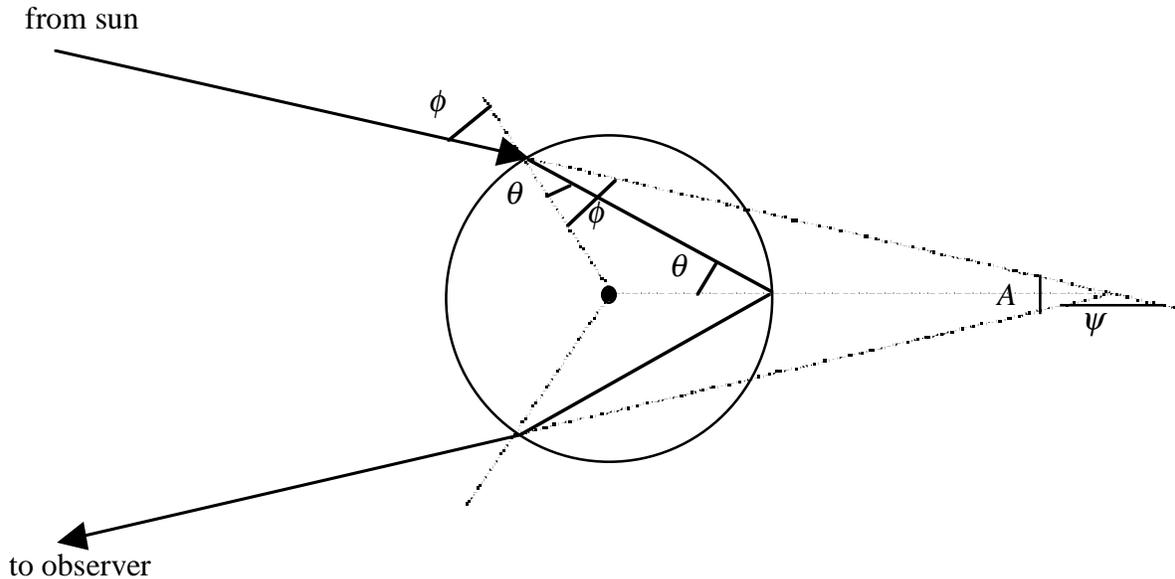


Rainbows—C.E. Mungan, Fall 1999

Introductory textbooks often mention that a rainbow can only be seen with the sun to your back. More specifically, a primary rainbow is seen by looking at an angle $A = 42^\circ$ to the line from the sun to the water droplets, and a secondary rainbow occurs just outside of this primary rainbow at 51° . The order of the colors in the secondary rainbow is reversed compared to that of the primary. The purpose of this note is to prove these claims.

For the primary rainbow, the light refracts as it enters the water, reflects once off the back surface, and then refracts again as it leaves the spherical droplet. Hence, there is a vertical symmetry to the geometry, as sketched below.



From this figure it is clear that

$$A = 4\theta - 2\phi. \quad (1)$$

But from Snell's law,

$$\theta = \sin^{-1}\left(\frac{\sin\phi}{n}\right) \quad (2)$$

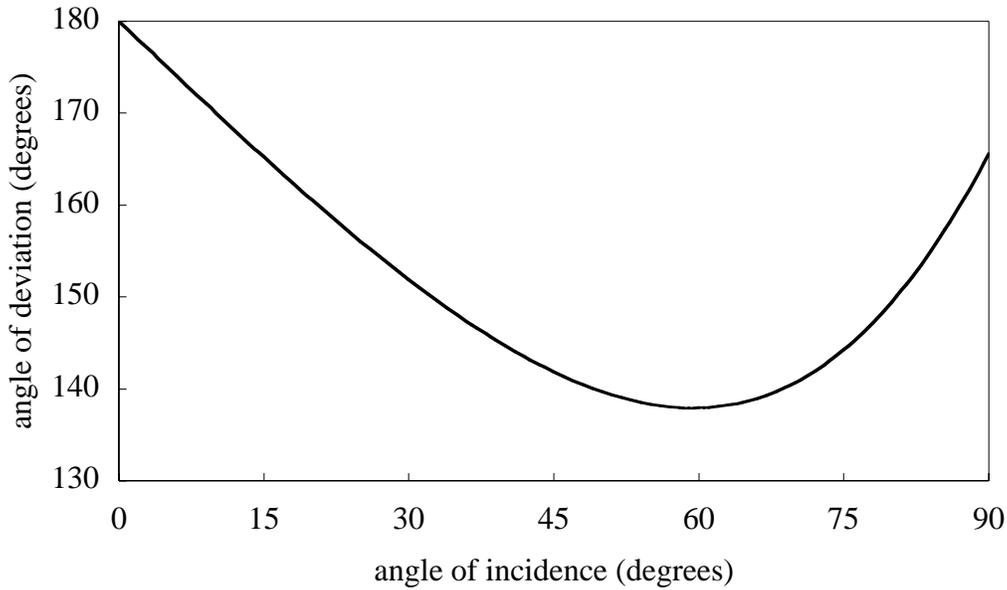
where $n = 1.333$ is the index of refraction of water. Substituting Eq. (2) into (1) gives an equation for the angle A in terms of the incident angle ϕ alone. The angle of deviation of the light (also known as the “rainbow angle”) is given by

$$\psi = 180^\circ - A, \quad (3)$$

which is plotted on the top of the following page.

We see that there is an angle of minimum deviation; at this angle, the intensity of the refracted rays is a maximum (because adjacent rays entering a drop all come out at nearly the same angle) and the rainbow will be seen. Mathematically, this corresponds to maximizing A . Differentiating A with respect to ϕ and setting the result to zero gives

$$\phi = \cos^{-1}\sqrt{\frac{n^2 - 1}{3}} = 59.4^\circ \Rightarrow \theta = 40.2^\circ \Rightarrow A = 42^\circ. \quad (4)$$



Note that the angle of reflection off the back face, θ , is less than the critical angle which for water is 48.8° . Careful examination of the diagram on the first page indicates that this had to be true because otherwise the sunlight could not have penetrated the water droplet at all! Thus, some light escapes through the back face and the rainbow is not as bright as one might have hoped.

For the secondary rainbow, the light makes two reflections off the inner surface of a water drop before escaping. A similar geometric construction as above reveals that

$$A = 6\theta - 2\phi - 180^\circ \quad (5)$$

where θ continues to be given by Eq. (2). Maximizing this gives

$$\phi = \cos^{-1} \sqrt{\frac{n^2 - 1}{8}} = 71.8^\circ \Rightarrow \theta = 45.5^\circ \Rightarrow A = -51^\circ. \quad (6)$$

The negative sign means that the exiting ray crosses over the incident ray outside of the drop. Hence the secondary rainbow is “upside down” and its color order is reversed. Owing to the fact that each ray reflects twice from the water surface, even more light escapes from the drop (despite the increased value of θ and hence increased reflectance) and thus the secondary rainbow is considerably fainter and not always seen.

In looking these results back over, notice that one only gets a primary rainbow if $1 < n < 2$ where n is the index of the drops relative to the surrounding air (eg. 1.333 for the usual case of water). What happens if $n > 2$? Then the secondary rainbow is dominant, provided $2 < n < 3$. This suggests a general rule: The N^{th} rainbow is dominant (lowest order and therefore brightest) provided $N < n < N + 1$. In fact this is correct, as shown by J.D. Walker in her detailed article “Multiple rainbows from single drops of water and other liquids,” *Am. J. Phys.* **44**, 421 (1976). The general formula for the viewing angle of the N^{th} rainbow is found from geometry to be

$$A = 2(N + 1)\theta - 2\phi - (N - 1)180^\circ, \quad (7)$$

which is easily shown to be a maximum for an angle of incidence of

$$\phi = \cos^{-1} \sqrt{\frac{n^2 - 1}{N(N + 2)}}. \quad (8)$$

Notice that there is no solution for ϕ when $n > N + 1$.