Suddenly Changing the Mass of the Sun—C.E. Mungan, Fall 2008

Suppose the mass of the sun is suddenly doubled. What happens to the length of earth's year?

At the instant of the solar doubling, earth's position, velocity, and mass will not change and thus neither will its angular momentum. (Denote earth's mass by m and sun's initial mass by M_S.) But the gravitational force on the earth doubles. Thus the earth will smoothly transfer from an initially (very nearly) circular orbit of radius R equal to 1 astronomical unit and speed $V = 2\pi R / T$ (where $T = 365$ days) to an elliptical orbit of semi-major axis a. The instant of the mass doubling of the sun defines the location of aphelion. At perihelion, the earth will be at some distance r from the sun and have some speed υ. Since $r < R$, Kepler's second law implies that $\upsilon > V$. We know the distance R and speed V at aphelion, and have two unknowns: the distance r and speed υ at perihelion. We can use two conservation laws to find these two unknowns. First, conservation of angular momentum implies that

$$mVR = m\upsilon r \quad \Rightarrow \quad \upsilon = \frac{VR}{r}, \tag{1}$$

since earth's position vector (relative to the sun) and velocity vector are mutually perpendicular at aphelion and again at perihelion. Secondly, mechanical energy is conserved so that

$$\frac{1}{2}mV^2 - \frac{GMm}{R} = \frac{1}{2}m\upsilon^2 - \frac{GMm}{r} \quad \Rightarrow \quad V^2\left(1 - \frac{R^2}{r^2}\right) = \frac{2GM}{R}\left(1 - \frac{R}{r}\right) \tag{2}$$

after substituting Eq. (1) into the kinetic energy at perihelion, where M is the new mass of the sun. Defining the dimensionless constant $\alpha \equiv 2GM / RV^2$, whose value we can compute, and noting the difference of squares on the left-hand side of the second equality in Eq. (2), we obtain the perihelion distance and speed as

$$r = R/(\alpha - 1) \quad \Rightarrow \quad \upsilon = (\alpha - 1)V \tag{3}$$

using Eq. (1) for the latter. As a check, in the original circular orbit we have $M = M_S$ where

$$F = ma \quad \Rightarrow \quad \frac{GMm}{R^2} = m\frac{V^2}{R} \quad \Rightarrow \quad \alpha = 2 \tag{4}$$

and thus we correctly obtain $r = R$ and $\upsilon = V$.

In constrast, for the new orbit $M = 2M_S$ and thus $\alpha = 4$ so that $r = R/3$. Consequently, since the semi-major axis is the average of the perihelion and aphelion distances, $a = 2R/3$. Finally Kepler's third law implies

$$T'^2 = \frac{4\pi^2 a^3}{GM} = \frac{1}{2}\left(\frac{2}{3}\right)^3 \frac{4\pi^2 R^3}{GM_S} = \frac{1}{2}\left(\frac{2}{3}\right)^3 T^2 \tag{5}$$

so that $T' = 2/3^{3/2}$ yr $= 140$ days.