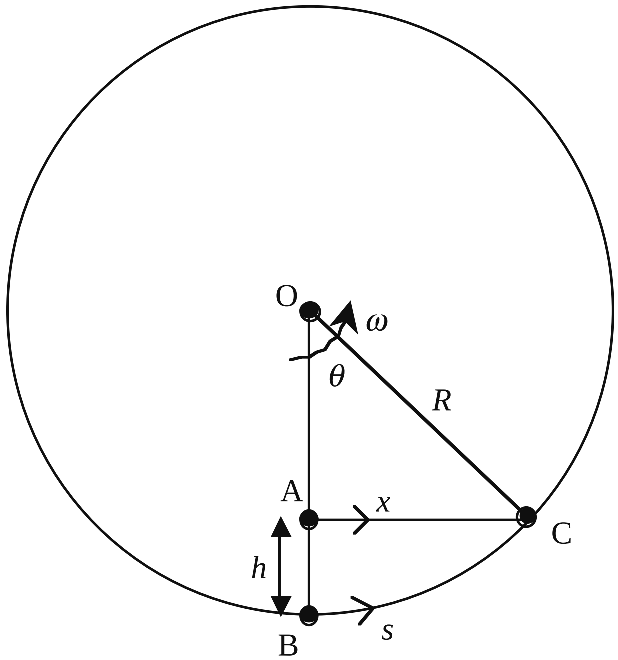
**Artificial Gravity in a Rotating Space Station—C.E. Mungan, Spring 2001**

You are standing on the inner surface of a rotating cylindrical space station of radius $R$. You drop a ball from height $h$. Does the ball land right at your feet? This is the gist of question 6 at the end of chapter 5 of Serway.




Let O denote the center of the cylinder and A the position at which the ball is released, at the same instant that your feet are located at position B. As viewed from an external inertial reference frame, no forces act on the ball after its release, and hence it travels along a straight line a distance $x$ until it strikes the station floor at position C. If the station is rotating at constant angular speed $\omega$, then the time the ball needs to travel this distance is

$$t_{ball} = \frac{x}{\upsilon_{ball}} = \frac{x}{\omega(R-h)} = \frac{\tan\theta}{\omega}. \tag{1}$$

Meanwhile, the time required for your feet to travel along the circular arc of length $s$ to the same point C is

$$t_{feet} = \frac{s}{\upsilon_{feet}} = \frac{\theta R}{\omega R} = \frac{\theta}{\omega}. \tag{2}$$

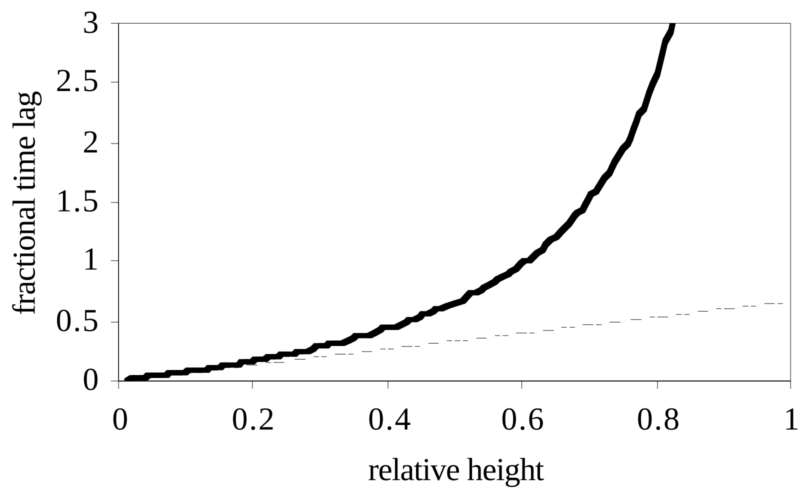
Comparing Eqs. (1) and (2) in radians, we conclude that $t_{ball} > t_{feet}$ and hence the ball falls behind you. Specifically, its fractional time lag is

$$\frac{t_{ball} - t_{feet}}{t_{feet}} = \frac{1}{\theta}\left[\frac{\sqrt{1-\cos^2\theta}}{\cos\theta} - \theta\right] = \frac{\sqrt{\dfrac{1}{(1-h/R)^2} - 1}}{\cos^{-1}(1-h/R)} - 1, \tag{3}$$

since $\cos\theta = (R-h)/R = 1 - h/R$. This is *independent* of the angular speed of the space station and is plotted below (bold curve) as a function of the relative height, $h/R$. For (realistically) small angles, $\cos\theta \approx 1-\theta^2/2$ and $\tan\theta \approx \theta+\theta^3/3$, so that

$$\frac{t_{ball}-t_{feet}}{t_{feet}} \approx \frac{2h}{3R}. \tag{4}$$

This is plotted as the straight line below. On the other hand, the time lag diverges to infinity as $h \to R$ because in this limit the released ball has zero translational speed and hence never reaches position C!




A Quicktime movie of the motion of balls released from various heights can be found on John Mallinckrodt's web site at http://www.csupomona.edu/~ajm/special/inertial.qt.

Note that the implications of this discussion is that a ball dropped from a height $h$ above the Earth's surface will also not quite fall at an observer's feet, owing to Earth's rotation at angular speed $\omega$. A complete analysis of this situation is rather complicated, but an approximate treatment proves the point. Let's compare the time required for the ball and your feet to travel through the same angle $\theta$ about Earth's center. As above, your feet require a time described by Eq. (2). In contrast, the ball's angular speed increases as it falls in order to conserve angular momentum,

$$t_{ball} = \int \frac{d\theta}{\omega_{ball}} = \frac{\int r^2 d\theta}{(R+h)^2\omega} \tag{5}$$

since $r^2\omega_{ball} = \text{constant} = r_i^2\omega_i$. In principle, this can be solved numerically by invoking conservation of mechanical energy, as given in the Appendix. However, a quick approximation is to replace $r$ in the integral by its average value $\bar{r}$. If the ball hits the ground after traveling through angle $\theta$, then $\bar{r} \approx (r_i + r_f)/2 = (R + h/2)$ assuming $h << R$, and therefore

$$t_{ball} \approx \left(1-\frac{h}{R}\right)t_{feet}. \tag{6}$$

Now the ball lands ahead of you and your feet lag by a fractional time comparable to Eq. (4).

## Appendix—Integral Solution near Earth's Surface

Conservation of mechanical energy, $E$, for the ball of mass $m$ is

$$\upsilon_\theta^2 + \upsilon_r^2 - \frac{2GM}{r} = \frac{2E_i}{m} = (R+h)^2\omega^2 - \frac{2GM}{R+h} \tag{7}$$

where $M$ is the mass of the Earth. The angular and radial components of the ball's velocity are

$$\upsilon_\theta = \frac{L_i}{mr} = \frac{(R+h)^2\omega}{r} \tag{8}$$

from conservation of angular momentum, $L$, and

$$\upsilon_r = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{\upsilon_\theta}{r} = \frac{dr}{d\theta}\frac{(R+h)^2\omega}{r^2} \tag{9}$$

using Eq. (8) in the last step. Substituting Eqs. (8) and (9) into (7) gives after some algebra,

$$d\theta = \frac{dx}{\sqrt{1 + A(x-1) - x^2}} \tag{10}$$

where the negative square root was chosen (since $dr < 0$ while the ball falls) and where I defined the dimensionless variables $x \equiv (R+h)/r$ and

$$A \equiv \frac{2GM}{(R+h)^3\omega^2} \approx \frac{gT^2}{2\pi^2 R} = 582 \tag{11}$$

with $g = 9.80\ \mathrm{m/s^2}$, $T = 24$ hours, and $R = 6370$ km. Substituting Eq. (10) into (2) gives

$$t_{feet} = \frac{T}{2\pi}\int_1^{1+h/R} \frac{dx}{\sqrt{1 + A(x-1) - x^2}}, \tag{12}$$

whereas substitution of Eq. (10) into (5) results in

$$t_{ball} = \frac{T}{2\pi}\int_1^{1+h/R} \frac{x^{-2}\,dx}{\sqrt{1 + A(x-1) - x^2}}. \tag{13}$$

An approximate solution to this last equation is

$$t_{ball} \approx \frac{1}{\bar{x}^2}\frac{T}{2\pi}\int_1^{1+h/R} \frac{dx}{\sqrt{1 + A(x-1) - x^2}}. \tag{14}$$

Putting $\bar{x} \approx (x_i + x_f)/2 = 1 + h/2R$ gives Eq. (6) to linear order in the relative height. A more exact solution to Eqs. (12) and (13) can be obtained either numerically (given $h$) or from integration tables. For example, Eq. (12) can be considerably simplified by transforming variables to $y \equiv x - 1$, while a solution to Eq. (13) is probably known from orbit theory.